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SUPERFLUIDITY OF ^{4}He AND QUANTUM FLUCTUATIONS OF CONDENSED STATE

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PREFACE

Last autumn I was lucky to spend two weeks together with Professor Nicolas N. Bogolubov in Italy. We had long conversations every day. During one of our discussions I told him about my view on the problem of quantum fluctuations in the superfluid ^{4}He and how they could be described within the framework of the famous Bogolubov Model. He told me that it is now expedient to examine quantum fluctuations and that he did not consider them in 1946 since he believed in their smallness, and the neutron scattering measurements were performed much later. He recommended me to put off other problems and to concentrate efforts in this field.

On the 3d of February I had to tell him about my results. The day before in the evening he broke his thigh and few days later (13th of February) he passed away.

I dedicate this paper to N.N.Bogolubov to pay last tribute to his memory. I believe that ideas of this really Great Scientist will stimulate many generations of scientists in the World.

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In 1946 in his report[1] of the General Meeting of Section of Physical and Mathematical Sciences of the USSR Academy of Sciences N.N.Bogolubov has formulated the microscopic theory of superfluidity of liquid ^{4}He. For the aims of the present paper we will formulate these ideas as the following statements.

1. Below some temperature atoms in condensate state with momentum $k = 0$ are weakly coupled with the over-condensate excitations (the same idea underlies the semiphenomenological two-liquid theories[2-4]). So the interaction Hamiltonian H may be replaced by an effective Hamiltonian H_{eff} which contains the high-order terms with respect to the condensate atom operators a_0^+ and a_0.

2. Every condensate atom operator in H_{eff} is supposed to be replaced by C-number according to the rule: $a_0 \propto \sqrt{N_0}$ where N_0 is a number of atoms in the condensate state (N_0 is supposed to be much greater than the number of atoms in the over-condensate excitations N_{exc}).

3. Hamiltonian H_{eff} is a quadratic form with respect to the operators of excited atoms and thus it can be diagonalized with the aid of the Bogolubov Canonical Transformation (BCT) which describes the connection between the processes of annihilation of an atom with momentum k and creation of an atom with momentum -k. This procedure leads to the famous expression for the excitation spectrum with the phonon and roton parts.

All subsequent development of the theories of superfluidity and superconductivity is in close connection with these ideas. Their paraphrase is given in a lot of books on Statistical Mechanics and Condensed Matter Physics (e.g.[5-10]).

The neutron scattering experiments[11] have confirmed the form of the spectrum of excited atoms predicted by the microscopic theory[1]. At the same time, the attempts to measure the quota of condensed atoms in superfluid ^{4}He have given the following main result[12-15]: N_0 is much smaller than N_{exc} ($N_0 \le 0.02\, N_{exc}$).

This discrepancy between the principal statement of the microscopic theory and experimental results is usually assigned to the short-range correlations in the atomic system. On the other hand, it is important to take into account quantum fluctuations of the number of atoms in condensate state.

Let us consider for this aim the standard Bogolubov model of weak coupling with Hamiltonian[1] $H_{eff}=$

$$\frac{1}{2}\sum_k \frac{k^2}{2m}(a^+_k a_k + a^+_{-k}a_{-k}) + \frac{\lambda}{2}a_0^{+2}a_0^2 +$$

$$\frac{\lambda}{2}\sum_{k\neq 0}\{2(a_k^+a_k+a_{-k}^+a_{-k}^+)a_0^+a_0+a_k^+a_{-k}^+a_0^2+a_0^{+2}a_{-k}a_k\}. \quad (1)$$

where k is the momentum, m is the mass and λ is the coupling constant. By virtue of the supposition 2 we should change now every operator with index "0" by a C-number. This procedure can be considered as an averaging of (1) with respect to some state of condensed atoms[16-19]. Thus, instead of (1) we get

$$\bar{H}_{eff}=\frac{1}{2}\sum_{k\neq 0}\{C_k(a_k^+a_k+a_{-k}^+a_{-k})+\lambda a_k^+a_{-k}^+\langle a_0^2\rangle+\lambda\langle a_0^{+2}\rangle a_{-k}a_k\}$$

$$+\frac{\lambda}{2}\langle a_0^{+2}a_0^2\rangle-\mu\langle a_0^+a_0\rangle, \quad (2)$$

where

$$C_k=k^2/2m-\mu+2\lambda\langle a_0^+a_0\rangle,$$

and μ is the chemical potential that has been introduced to take into account the violation of conservation of the number of particles in (2).

It is easy to see that the "classical" state of condensate atoms, that have been considered in statement 2, is defined to be a coherent state. In other words

$$\langle f(a_0^+,a_0)\rangle=\langle 0|D^+(\alpha)f(a_0^+,a_0)D(\alpha)|0\rangle. \quad (3)$$

where f(.) is any operator constructed from a_0^+ and a_0, $|0\rangle$ is the vacuum state of the condensate atoms and

$$D(\alpha)=exp\ (\alpha a_0^+-\alpha^* a_0)\ .$$

By virtue of the commutation relations

$$D^+(\alpha)a_0D(\alpha)=a_0+\alpha\ ,$$

$$D^+(\alpha)a_0^+D(\alpha)=a_0^++\alpha^*$$

we obtain

$$\langle a_0^+a_0\rangle=|\alpha|^2\ ,$$

$$\langle a^2\rangle_0=\alpha^2\ , \quad (4)$$

$$\langle a_0^{+2}a_0^2\rangle=|\alpha|^4\ .$$

Here $|\alpha|^2=N_0$. This is the standard Bogolubov parameterization of the condensate state[1].

Hamiltonian (2) can be diagonalized with the aid of BCT of the form

$$a_k=u_k(b_k+\xi_k b_{-k}^+), \quad (5)$$

where $u\in\mathbb{C}$ and $\xi\in\mathbb{R}$, and

$$\xi_k=-\eta_k+\surd(\eta_k^2-1),\ |u_k|^2(1-\xi_k^2)=1\ , \quad (5a)$$

$$\eta_k=C_k/\lambda|\langle a_0^2\rangle|\ .$$

In the diagonal representation

$$\bar{H}_{eff}=\frac{1}{2}\sum_{k\neq 0}E_k(b_k^+b_k+b_{-k}^+b_{-k})+\frac{\lambda}{2}\langle a_0^{+2}a_0^2\rangle-\mu\langle a_0^+a_0\rangle$$

$$+\sum_{k\neq 0}|u_k|^2\xi_k^2\ (C_k+\lambda|\langle a_0^2\rangle|). \quad (6)$$

where

$$E_k=\surd(C_k^2-\lambda^2|\langle a_0^2\rangle|^2). \quad (7)$$

In the case of parameterization (4) expression (7) describes the famous Bogolubov spectrum of excitations.

One can see that the eigenstates of (6) are the Fock number states of quasi-particles described by the transformation (5). In terms of the Bose field describing ^{4}He atoms this is the squeezed number state[20]

$$b_k^+b_k\ \psi_k=n_k\psi_k\ ,\quad n_k=0,1,2,...$$

which can be constructed from the "two-mode squeezed vacuum state" [21] .

$$\Psi_k = S(\zeta_k)|0>_k|0>_{-k} \qquad (8)$$

with the aid of quasi particle creation operator b_k^+. Here $|0>_k$ is the vacuum state of k-th mode of the field described by atomic operators a_k^+ and a_k. The unitary two-mode squeezing operator is

$$S(\zeta_k) = exp\ (\zeta_k^* a_k a_{-k} - \zeta_k a_k^+ a_{-k})$$

and ζ_k is the known function of u_k and ξ_k. This wave-function (8) describes a state of an atomic system containing any number of pairs with the opposite momenta. The state of the system under consideration at zero temperature is defined by the wavefunction

$$\Psi = \prod_{k\neq 0} \Psi_k \ . \qquad (9)$$

Let us calculate the variance of a number of atoms in the excited state at zero temperature

$$V(N_{exc}) = <N_{exc}^2> - <N_{exc}>^2, \quad N_{exc} = \sum_{k\neq 0} (b_k^+ b_k + b_{-k}^+ b_{-k})/2 ,$$

where <.> is the expectation value with respect to the state (9). Employing expressions (5), (8) and (9) than gives

$$V(N_{exc}) = \sum_{k\neq 0} |u_k|^4 \xi_k^2 \ . \qquad (10)$$

It follows from the definition that $|u_k|^2>1$. Hence

$$V(N_{exc}) > <N_{exc}> = \sum_{k\neq 0} |u_k|^2 \xi_k^2 \ .$$

Thus we have strong enough super-Poissonian[21] quantum

fluctuations of the number of excited atoms at zero temperature.

It should be noted that the initial Hamiltonian (1) conserves the total number of atoms

$$N = N_0 + N_{exc} = a_0^+ a_0 + \sum_{k\neq 0} (a_k^+ a_k + a_{-k}^+ a_{-k}).$$

Therefore, the quantum fluctuation of the number of excited atoms (expression (10)) means the existence of the corresponding quantum fluctuation of the number of condensate atoms. In other words $V(N_0)$ should have also a nonzero value.

It is possible to calculate $V(N_0)$ using Bogolubov parameterization (4) of the condensate state. In this case we have the Poisson probability distribution for the number of atoms. Therefore

$$V(N_0) = N_0 = |\alpha|^2 \ .$$

Thus, the quantum fluctuation is small (proportional to $\sqrt{N_0}$).

It should be noted that parameterization of the condensed state as the coherent state (4) is not the only possibility. Moreover, the coherent state $|\alpha> = D(\alpha)|0>$ is not an eigenstate of the Hermitian operator. So, another possibility can be considered.

Let us suppose that the condensate atoms are also in the squeezed vacuum state (at zero temperature). This state can be constructed from the corresponding vacuum state $|0>_0$ with the aid of the squeezing operator

$$|\zeta_0>_0 = S(\zeta_0)|0>_0 \ ,$$

$$S(\zeta_0) = exp(\zeta_0^* a_0^2/2 - \zeta_0 a_0^{+2}/2) \ .$$

Here ζ_0 is some complex parameter. Then

$$<a_0^+ a_0> = sinh^2|\zeta_0| \equiv N_0,$$
$$V(N_0) = 2\ cosh^2|\zeta_0|\ sinh^2|\zeta_0| \equiv 2N_0(N_0 + 1), \qquad (11)$$

$$<a_0^{+2}a_0^2> = sinh^2|\zeta_0| \; (3sinh^2|\zeta_0| + 1) = N_0(3N_0 + 1) \; .$$

In this case we have very strong quantum fluctuations of the number of atoms in the condensate state (of the order of N_0). The probability to have exactly N_0 atoms in the condensed state is[21]

$$\mathcal{P}(N_0) = (N_0! \; cosh^2|\zeta_0|)^{-1} (\frac{1}{2} \; tanh|\zeta_0|)^{N_0} \; \mathcal{H}_{N_0}(0),$$

where $\mathcal{H}_n$ is the Hermite polynomial of degree n. Of course this value depends on the squeezing parameter ζ_0. It can be chosen in the following way. From the conservation condition of the total number of atoms in the system we have

$$sinh^2|\zeta_0| = N - \sum_{k\neq 0} |u_k|^2 \xi_k^2,$$

where N is the total number of atoms. It can be expressed also in terms of densities

$$\rho_0 = \rho - \rho_{exc}, \qquad (12)$$

where $\rho_i = N_i/V$ and V is the volume of the system. In virtue of expressions (2), (5a) and (11) ρ_{exc} is a function of $|\zeta_0|$. Thus expression (12) can be considered as the equation for the squeezing parameter of a condensate state.

We do not examine this equation here and confine ourselves to a brief remark.

It follows from our consideration that even at zero temperature the quantum fluctuations of the number of condensate atoms happen in the superfluid ^{4}He. They depend on the quantum state of condensate atoms. They are absent for the Fock number state only. They are present in Bogolubov parameterization (4). They can be strong enough in different parameterizations of the condensate state. The parameterization by the squeezed state (11) is possible. Analogies with quantum optics[20,21] show that the nonlinearity of quantum system should lead to the squeezed state.

The concept of squeezed states permits us to look at the condensation problem from a different point of view. Since the state (9) is formed by the groups of atoms with zero total momentum, it is possible to consider the superfluidity as a condensation of those groups (not only as a collective state of individual atoms with zero momentum). Their contribution can be measured by the neutron scattering with the smaller energy because their sizes (correlation lengths) are much larger than the atomic size.

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Сверхтекучесть ^{4}He и квантовые флуктуации конденсата

Рассмотрена модель слабого взаимодействия Боголюбова. Показано, что при нулевой температуре надконденсатные возбуждения описываются двухмодовым сжатым состоянием. Возможные параметризации конденсатного состояния исследованы. В случае сжатого вакуумного состояния конденсата найдено уравнение для параметра сжатия. Предсказана возможность конденсации групп с четным числом атомов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Superfluidity of ^{4}He and Quantum Fluctuations of Condensed State

The Bogolubov model of weak interaction is considered. It is shown that the over-condensate excitations are described by a two-mode squeezed state at zero temperature. The possible parametrizations of the condensate state are examined. In the case of a squeezed vacuum state of the condensate the equation for the squeezing parameter is obtained. The possibility of condensation of the groups with an even number of atoms is supposed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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