объединенный
институт
ядерных
исследований
дубна

E17-91-305

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PHASE PROPERTIES OF DISPLACED NUMBER STATES

Submitted to "Quantum Optics"

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1991

1 Introduction

Properties of displaced number states have recently been studied by de Oliveira et al [1], and their interaction with two-level atoms by Kim, de Oliveira and Knight [2]. It has been shown that such states have interesting and unusual physical properties. Since the displaced number state is obtained from a number state by adding a nonzero value to the field amplitude, the state becomes phase dependent because of the phase of the displacement. The quasiprobability functions such as the Q function and the Wigner function for displaced number states have simple analytical forms which allow for clear interpretations of the oscillations in the photon number statistics [1] in terms of interference in phase space [3]. The fact that the states are phase dependent makes it interesting to study their phase properties which, to our knowledge, has not been studied so far.

This is the aim of this Letter to study the phase properties of the displaced number states. We use the Pegg-Barnett [4]– [6] Hermitian phase formalism to find the phase distribution function $P(\theta)$ for the displaced number states. It is shown that this distribution has a multi-peak structure. The Pegg-Barnett phase distribution is compared to the phase distributions obtained from the Wigner and Q functions by integrating them over the "radius". It is shown that the structures of the latter two distributions differ essentially when the number of photons is greater than one. It is also shown that the phase distribution obtained from the Wigner function reproduces quite well the Pegg-Barnett phase distribution, although it is not identical to it. This is in agreement with the area-of-overlap in phase space concept [7]. The distribution obtained from the Q function is smoother and some structure is lost.

2 Phase distributions

The displaced number states are defined by acting with the displacement operator $D(\alpha)$ on the number state $|N\rangle$, that is

$$|\psi\rangle = |\alpha, N\rangle = D(\alpha)|N\rangle, \tag{1}$$

where

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{2}$$

The number state decomposition of the displaced number state (1) can be written as

$$\begin{aligned} |\psi\rangle &= \sum_n |n\rangle\langle n|\psi\rangle = \sum_n |n\rangle\langle n|D(\alpha)|N\rangle \\ &= \sum_n b_n e^{i\varphi_n}|n\rangle, \end{aligned} \tag{3}$$

where for $n \geq N$

$$b_n = \left(\frac{N!}{n!}\right)^{1/2} |\alpha|^{n-N} e^{-|\alpha|^2/2} L_N^{n-N}(|\alpha|^2), \tag{4}$$

and

$$\varphi_n = (n - N)\varphi \tag{5}$$

with φ being the phase of $\alpha = |\alpha|\exp(i\varphi)$, and $L_N^{n-N}(|\alpha|^2)$ are associated Laguerre polynomials. For $n < N$, we have

$$b_n = \left(\frac{n!}{N!}\right)^{1/2} (-1)^{N-n}|\alpha|^{N-n} e^{-|\alpha|^2/2} L_n^{N-n}(|\alpha|^2), \tag{6}$$

and the phase φ_n remains the same as (5). The above amplitudes are obtained from the well known matrix elements of the displacement operator [8].

Knowing the number state decomposition (3) of the displaced number states, we can employ the Pegg-Barnett [4]- [6] Hermitian phase formalism to find the phase distribution function for such states. The Pegg-Barnett formalism is based on introducing a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, ..., |s\rangle$. The

Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $(s+1)$ phase states is defined on Ψ as

$$|\theta_m\rangle \equiv \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{7}$$

where

$$\theta_m \equiv \theta_0 + \frac{2\pi m}{s+1}, \qquad (m = 0, 1, ..., s). \tag{8}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{9}$$

The phase states (7) are eigenstates of the phase operator (9) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$.

The expectation value of the phase operator (9) in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s} \theta_m |\langle\theta_m|\psi\rangle|^2, \tag{10}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of being in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit, as s tends to infinity, we can write Eq. (10) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta) d\theta, \tag{11}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m|\psi\rangle|^2, \tag{12}$$

where θ_m has been replaced by the continuous phase variable θ. When the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner by integrating over θ. The choice of θ_0 defines the particular window of phase values.

In the case of the displaced number states we have

$$\begin{aligned}\langle\theta_m|\psi\rangle &= \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s} b_n \exp[-i(n\theta_m - \varphi_n)] \\ &= \frac{e^{-iN\varphi}}{\sqrt{s+1}}\sum_{n=0}^{s} b_n \exp[-in(\theta_m - \varphi)]. \end{aligned} \tag{13}$$

We choose θ_0 as to

$$\theta_0 = \varphi - \frac{\pi s}{s+1}, \tag{14}$$

that is, we symmetrize the phase window with respect to the phase φ. On inserting (14) into (13), taking the modulus squared of (13) and taking the continuum limit,we arrive at the continuous phase probability distribution $P(\theta)$ which has the form

$$P(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{\substack{n,k=0\\ n>k}}^{\infty} b_n b_k \cos[(n-k)\theta]\right\}, \tag{15}$$

where b_n are given by Eqs. (4) and (6), and the phase window is now from $-\pi$ to π. This form of the phase distribution is common for the partial phase states [5, 6]. However, due to the particular choice of b_n this phase distribution shows some interesting features that characterize the displaced number states.

Another phase distribution can be obtained by integrating the Q function $Q(\alpha)$ over the radial variable $|\alpha|$. This phase distribution was referred to as "classical" by Braunstein and Caves [9] since the Q function applies to simultaneous measurement of two noncommuting observables, a process that inevitably introduces additional noise. It has been shown by Tanaś et al [10] that the phase distribution obtained in this way can be obtained from the Pegg-Barnett distribution by multiplying the nondiagonal elements in Eq. (15) by additional factors. Tanaś and Gantsog [11] have shown that these additional factors $F(n,k)$ can be calculated using a simple recurrence formula and that the nondiagonal elements are less than unity. This means an averaging the Pegg-Barnett phase distribution leading to the distribution which is broader than the

Pegg-Barnett distribution. We have [10, 11]

$$\begin{aligned} P_Q(\theta) &= \int_0^{\infty} Q(\beta)|\beta| d|\beta| \\ &= \frac{1}{2\pi}\left\{1 + 2\sum_{\substack{n,k=0\\ n>k}}^{\infty} b_n b_k \cos[(n-k)\theta] F(n,k)\right\}, \end{aligned} \tag{16}$$

where the coefficients $F(n,k)$ are given by [10, 11]

$$F(n,k) = \frac{\Gamma\left(\frac{n+k}{2}+1\right)}{\sqrt{n!k!}}. \tag{17}$$

Since $Q(\alpha)$ is positive definite, also $P_Q(\theta)$ is positive definite, and normalized, and it can be treated as a phase distribution.

The concept of interference in phase space introduced by Schleich and Wheeler [3] when applied to describe phase properties of the field indicates still another possibility to get the phase distribution [7] by integrating the Wigner distribution over the radial variable. This leads to the following phase distribution

$$\begin{aligned} P_W(\theta) &= \int_0^{\infty} W(\beta)|\beta| d|\beta| \\ &= \frac{1}{2\pi}\left\{1 + 2\sum_{\substack{n,k=0\\ n>k}}^{\infty} b_n b_k \cos[(n-k)\theta] G(n,k)\right\}, \end{aligned} \tag{18}$$

where the coefficients $G(n,k)$ are given by

$$\begin{aligned} G(n,k) &= \sum_{m=0}^{p}(-1)^{p-m} 2^{(|n-k|+2m)/2} \\ &\quad \times \sqrt{\binom{p}{m}\binom{q}{p-m}}\, F(m, |n-k|+m), \end{aligned} \tag{19}$$

where

$$p = \min(n,k), \qquad q = \max(n,k), \tag{20}$$

and $F(m, |n-k|+m)$ are given by Eq. (17). The coefficients $G(n,k)$ are symmetric $G(n,k) = G(k,n)$, and $G(n,n) = 1$. Relation (18) is quite general and can be applied for any states with known amplitudes b_n. Here, we use it to the displaced number states. Since the coefficients $G(n,k)$ take on the values that are smaller or larger than unity, their effect on the phase distribution is not as simple as in the case of $P_Q(\theta)$. In Fig. 1, we show the plots of the phase distributions, calculated according to the three formulas ((15),(16), and (18)), in polar coordinates for the displaced number states with $|\alpha| = 3$, and $N = 0, 1, 2$. It is seen that the Pegg-Barnett phase distribution and $P_W(\theta)$ are very similar and have the $N+1$ lobes, while $P_Q(\theta)$ is much broader and has only two lobes for $N \geq 2$. To understand this behaviour of the phase distributions we relate them to the forms of the Q function and the Wigner function for the corresponding displaced number states. The two functions have in the case of the displaced number states (1) quite simple analytical forms [1]

$$Q_{dN}(\beta) = \frac{1}{\pi} e^{-|\beta-\alpha|^2} \frac{|\beta-\alpha|^{2N}}{N!}, \tag{21}$$

and

$$W_{dN}(\beta) = \frac{2}{\pi} \exp(-2|\beta-\alpha|^2)(-1)^N L_N(4|\beta-\alpha|^2), \tag{22}$$

where $L_N(x)$ is the Laguerre polynomial of order N. In Fig. 2, we plot the Q functions, and in Fig. 3 the Wigner functions for the displaced number states with $|\alpha| = 3$, and $N = 0, 1, 2$. The Q function for $N \geq 1$ has the minimum for $|\beta - \alpha| = 0$ equal to zero, so there are only two maxima in the phase distribution $P_Q(\theta)$ that correspond to the two symmetrically disposed maxima of the area obtained when the Q function is intersected by the vertical plane along the radial coordinate. This is the idea of area-of-overlap in phase space [7] employed to the Q function. Since the Wigner function shows oscillations, the same idea applied to the Wigner function gives the number of peaks in the phase distribution $P_W(\theta)$ equal to $N+1$. So there is essential difference in the phase information carried by $P_Q(\theta)$ and $P_W(\theta)$. Because of the averaging procedure with the "probabilities" $F(n,k)$ some phase information is lost in $P_Q(\theta)$. The Pegg-Barnett phase distribution is very close to the distribution $P_W(\theta)$, although it is not identical to it, and at least in the case of the displaced number states, they carry

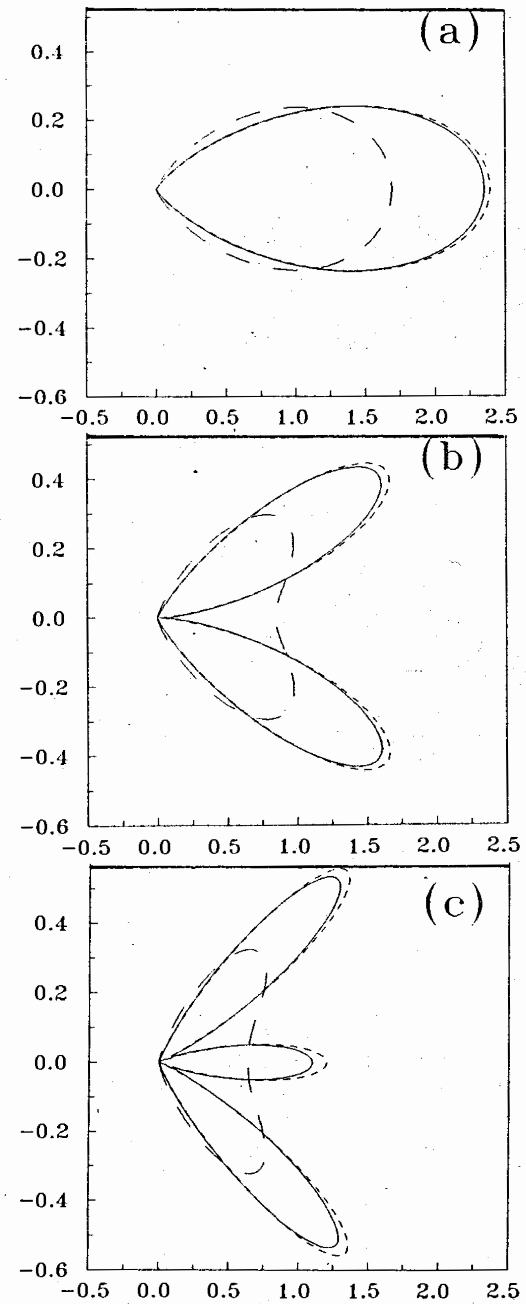

FIG. 1. Polar plots of the phase distributions $P(\theta)$ (solid line), $P_W(\theta)$ (short-dashed line), and $P_Q(\theta)$ (long-dashed line), for the displaced number states with $|\alpha| = 3$, and (a) $N = 0$, (b) $N = 1$, (c) $N = 2$.

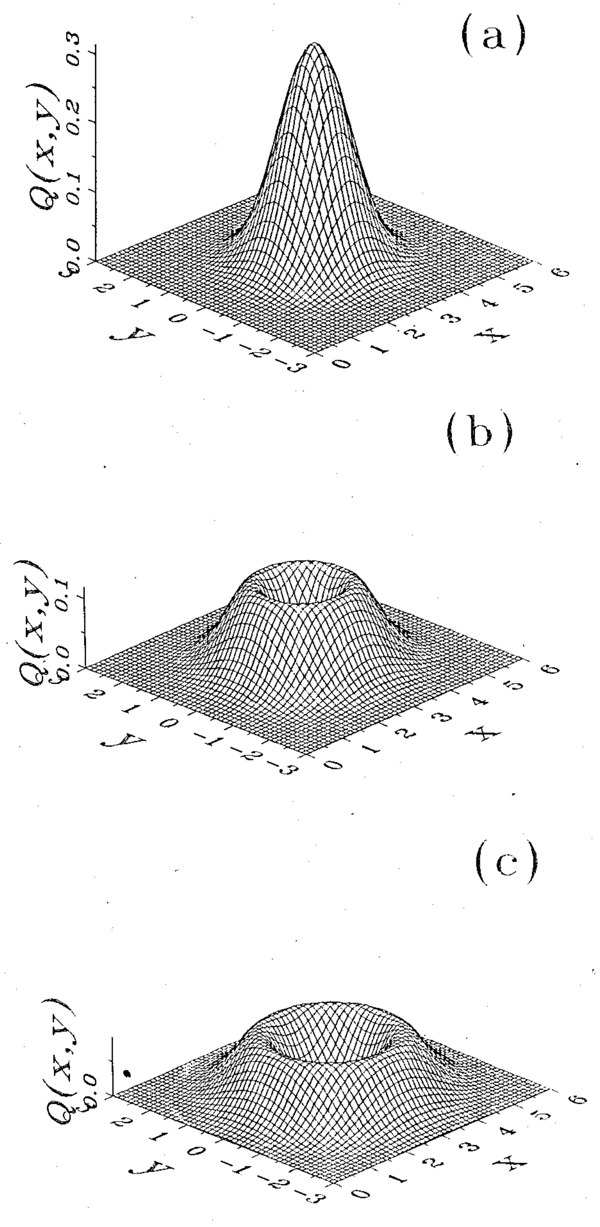

FIG. 2. Plots of the Q function for the displaced number states. The parameters are the same as in Fig. 1, x=Re$(\beta-\alpha)$, y=Im$(\beta-\alpha)$, and $\varphi=0$.

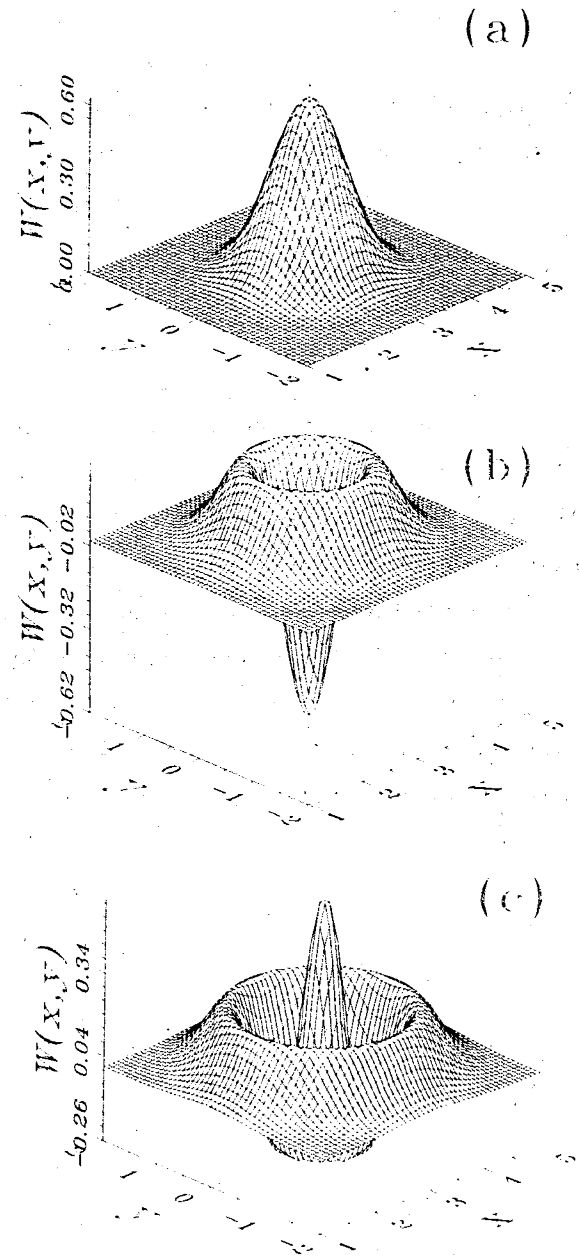

FIG. 3. Plots of the Wigner function for the displaced number states. The parameters are the same as in Fig. 2.

basically the same phase information. The phase peaks of $P_W(\theta)$ are slightly narrower than those of $P(\theta)$. This similarity is in agreement with the area-of-overlap in phase space arguments, where that is the Wigner function that represents quantum states in the phase space. However, the Wigner function can take on negative values and the positive definiteness of $P_W(\theta)$ is not automatically guaranteed, while there are not such problems with the Pegg-Barnett phase distribution.

3 Conclusion

We have discussed the phase properties of the displaced number states showing that the Pegg-Barnett phase distribution for such states exhibits the multi-peak structure with $N+1$ peaks. We have compared the Pegg-Barnett distribution to the phase distributions $P_Q(\theta)$ and $P_W(\theta)$ obtained by integrating the Q function and the Wigner function over the radial coordinate. We have shown that while the Pegg-Barnett and $P_W(\theta)$ distributions carry basically the same phase information,the distribution $P_Q(\theta)$ has lost an essential part of phase information. Since the displaced number states are the states for which the Q function and the Wigner function differ essentially for $N \geq 2$, they can serve as a good test of various phase approaches.

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Received by Publishing Department
on July 2, 1991.