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TIME OF TUNNELLING WITHIN THE COMPLEX TIME AND REAL TRAJECTORIES METHODS

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Tunnelling effect is by definition a quantum type phenomenon: the particle of energy E_o comes through the barrier of height V_o ($V_o > E_o$) [1,2]. This problem has been widely discussed, however, there still exists a question of estimation of a time interval the particle travels under the barrier (see e.g. [3-5]). In this note we discuss the tunnelling effect within the framework of path integral [6] by using a semiclassical approximation-complex time and real trajectories [7-11]. As it has been recently shown [8], the stationary phase approximation within a semiclassical approximation reproduces the WKB - results. In the case of barrier penetration the extension of time t to the complex t-plane is needed [7]. In various problems, such as probability of passing through the barrier [8], or eigenvalue problem in a case of double - well potential [9], the standard quantum mechanical [12] results or results obtained within instanton - type approximations [13] are easily reproduced.

Encouraged by satisfactory enough earlier results described above, we discuss here the problem of tunnelling time within the complex time and real trajectories method, although in this case the result is quite unexpected.

Let us consider a time evolution of a wave-packet

$$\Psi_0(x) = \int_{-\infty}^{+\infty} dk \, \exp\left[-\frac{\alpha}{2}(k-k_c)^2 + \frac{i}{\hbar} k (x - x_c)\right] = \tag{I}$$

$$= \exp\left[-\frac{(x-x_0)^2}{2\alpha}\right] \exp\left(\frac{i}{\hbar} k_0 x\right) ,$$

describing a particle of mass m moving with velocity k_c/m along the x axis in the direction of a potential barrier

$$V(x) = \begin{cases} V_c , & a \leqslant x \leqslant b \\ 0 , & x < a , \quad x > b \end{cases} \tag{2}$$

It seems reasonable to relate the coordinate of particle with the coordinate of maximum of wave-packet. Therefore, initially, t = 0, the particle is somewhere around $x = x_0$ ($x_0 \ll a$) and after time t its position is determined by the function

$$\Psi(x'', t) = \int_{-\infty}^{+\infty} dx' \, K(x'' t ; x' 0) \, \Psi_0(x') , \tag{3}$$

where K is the propagator [6]:

$$K(x'' t ; x' 0) = \langle x'' | \exp \frac{i}{\hbar} H t \, | \, x' \rangle \tag{4}$$

$$= \int_{x(0)=x'}^{x(t)=x''} \mathcal{D}[x(t)] \exp\left[\frac{i}{\hbar} S\right] ,$$

$$S = \int_0^t dt \left[\frac{m \dot{x}^2}{2} - V(x)\right] \tag{4a}$$

As we are interested in the effect of tunnelling the particle through the barrier, by assumption

$$\frac{k_c^2}{2m} \ll V_0 , \quad \left(k_c > \frac{1}{\sqrt{\alpha}}\right) \tag{5}$$

$x' < a$, $x'' > b$.

In this case the propagator K found within a semiclassical approximation – complex time and real trajectories, takes the following form (see [7-9]):

$$K(x'' t ; x' 0) \simeq K_0 \, K_1(\Delta) , \tag{6}$$

$$K_0 = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left\{\frac{i}{\hbar}\left[\varkappa (x'' - x' - \Delta) - \frac{\varkappa^2}{2m} \cdot t\right]\right\} , \tag{6a}$$

$$K_1(\Delta) = \exp\left(-\frac{1}{\hbar} \Delta \sqrt{2m V_0 - \varkappa^2}\right) , \tag{6b}$$

$$\Delta = b - a ,$$

$$\frac{\varkappa}{m} = \frac{x'' - x' - \Delta}{t} ,$$

(from the whole class of real trajectories in complex time [7] satisfying the condition

$$\delta S = 0$$

only those are considered here which pass the barrier once ; higher order additive propagators $K_c K_{2n+1}(\Delta)$ are omitted for simplicity).

Inserting the propagator (6) into Eq.(3) and using a stationary phase approximation (see also [8]) the function $\Psi(x'',t)$ is given as the following integral:

$$\Psi(x'',t) = \int dk\, \exp\left\{-\frac{\alpha}{2}(k-k_0)^2 - \frac{\Delta}{\hbar}\sqrt{2mV_0-k^2} + \frac{i}{\hbar}\left[k(x''-x_0'-\Delta) - \frac{k^2 t}{2m}\right]\right\} \qquad (7)$$

where the point of stationary phase $\bar{x}$ is such that

$$t = \left(\int_{\bar{x}}^{a} + \int_{b}^{x''}\right)\frac{dx}{(k/m)} = \frac{x''-\bar{x}-\Delta}{(k/m)} \,. \qquad (7a)$$

The integration over k in Eq.(7) may be performed, e.g., by using a saddle-point method. We are able to obtain an analytical result in the case of narrow, high barrier:

$$\frac{1}{\hbar}\frac{\Delta}{\sqrt{2mV_0-k_0^2}} \ll \alpha \;: \qquad (8)$$

Then

$$|\Psi(x'',t)| \simeq \exp\left(-\frac{\Delta}{\hbar}\sqrt{2mV_0-k_0^2}\right)\exp\left\{-\frac{1}{2\bar{\alpha}(t)}\left[x''-\left(x_0+\Delta+\frac{\bar{k}}{m}t\right)\right]^2\right\}, \qquad (9)$$

where

$$\bar{k} = k_0(1+\delta)\,, \qquad (9a)$$

$$\bar{\alpha}(t) = \hbar^2\bar{\alpha}\left[1+\frac{1}{\bar{\alpha}^2}\left(\frac{t}{\hbar m}\right)^2\right], \qquad (9b)$$

$$\bar{\alpha} = \alpha(1-\delta)\,, \qquad (9c)$$

$$\delta = \frac{1}{\alpha}\frac{\Delta}{\hbar\sqrt{2mV_0-k_0^2}} \,. \qquad (9d)$$

To make the result (9) more clear, let us remember the result of Holstein and Swift's paper [8]. Showing a consistence of the semiclassical treatment with the standard textbook result in estimation of the probability of tunnelling through a barrier, they used the approximation

$$\exp\left(-\frac{1}{\hbar}\Delta\sqrt{2mV_0-k^2}\right) \simeq \exp\left(-\frac{1}{\hbar}\Delta\sqrt{2mV_0-k_0^2}\right). \qquad (10)$$

This approximation used in Eq. (7) leads to the following result:
time of propagation from $x_0(\ll a)$ to $x''(>b)$ is the same as time of a free propagation of a particle from a point x_0 to $x''-\Delta$, thus, it may be said that the particle remains under the barrier zeroth time interval

$$\Delta t = 0$$

Our saddle point approximation is well - justified in contradiction with the condition (10). The result (9) means that the particle appears at x'' a little bit earlier than in the case of approximation (I0). The particle remains under the barrier somewhat "less" than zero time interval.

This result is obviously associated with approximations used within complex time and real trajectories – method. On the other hand, this method reproduces various, more or less, standard, quantum mechanical results. It also allows one to obtain an estimation of decay time τ_0 of a metastable state [14].

It is quite clear however that the above estimation of "time tunnelling" may eventually be justified or excluded only by further investigations. Such studies will be presented in a series of papers in the nearest future, where mathematical details omitted here (e.g. normalization factors or discussion of the used approximations of saddle point-type) will be given.

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