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D.A.Alexandrova*, J.N.Kotzev

ON THE PROBLEM OF THE EQUIVALENCE OF P-TYPE PERMUTATIONAL COLOUR GROUPS

*Faculty of Physics, Sofia University

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1. INTRODUCTION

The colour crystallographic groups are considered as symmetry groups of scalar, vector or tensor functions representing crystal physical properties[1-4]. Alternatively, the colour groups might be used as the symmetry groups of the Hamiltonian of a concrete model system[5]. The interpretation of the Landau theory of continuous phase transitions in terms of permutational colour groups has shown their effectiveness in prediction, analysis and generalized classification of the possible phase transitions[6-12].

The group-equivalence definition is one of the main problems in the classification and tabulation of any kind of groups. This problem is rather complicated. The definition of the colour-group equivalence depends on the presumable applications of these groups. That is why the published results and lists of colour groups differ in conformity with the respective equivalence criterion choice. On the one hand, the comparing of these results is too complicated. On the other hand, the application of the published lists and tables for symmetry describing of a wide range of physical phenomena has almost been impossible due to the inappropriate equivalence definitions.

Because of these reasons the problem of colour-group equivalence as well as the relation between the equivalence criteria given in the literature are rather significant[1,3,13,14]. Some aspects of this subject have been discussed by Shwarzenberger[14] and Roth[15]. Part of the introduced equivalence definitions have been summarized by Opechowski[16].

In the present work a general systematization scheme of the possible equivalence criteria of P-type permutational colour groups is given. It is based on the concept of "isomorphism of groups as groups of transformations" (see Ref[17]). It is reformulated from the point of view of two other approaches to the classification of colour groups of this kind.

2. PERMUTATIONAL COLOUR GROUPS $G^{(P)}$ OF P-TYPE

The theory of P-type permutational colour groups has been clarified in details in[1-4]. For the sake of convenience we shell give some basic principles of this theory.

Let G be a crystallographic group with elements $g \in G$. The P-type colour crystallographic groups $G^{(P)}$ to be disused in this paper are subdirect product of G and P defined by the homomorphism $\pi: G \longrightarrow P$:

$$G^{(P)} = \{ <p;g> \mid \text{all } g \in G,\ \pi: G \longrightarrow P,\ \pi(g) = p \in P \leq S_n \} \subset P \times G \quad (1)$$

Here P is a transitive subgroup of S_n, the symmetric group permutations of n objects (colours) $F = \{f_1, \ldots, f_n\}$.

The colour groups $G^{(P)}$ are isomorphic to G by an isomorphism ξ,

$$\xi: G^{(P)} \longrightarrow G,\ \xi(<p;g>) = g \in G . \quad (2)$$

They are called permutational colour groups of the family PG of G and P, isostructural to G. For the sake of brevity we shall call them "colour groups".

Obviously, the question which colour groups $G^{(P)}$ to be considered as equivalent is related to the classification of the groups G and P, as well as the homomorphisms π. As it will be shown later, the choice of equivalence criteria for $G^{(P)}$ is related to appropriated equivalence one for G and P, i.e. for the family PG.

Each colour group $G^{(P)}$ contains two specific subgroups: the colour-preserving subgroup

$$H^{(1)} = \{<e_p;h> \mid \text{identity } e_p \in P,\ h \in H \lhd G \} \lhd G^{(P)}$$
$$G^{(P)}/H^{(1)} \cong G/H \cong P \quad (3)$$

and a subgroup $H'^{(p')}$, the maximal one in $G^{(P)}$, preserving the colour $f_1 \in F$

$$H'^{(p')} = \{<p';h'> \mid h' \in H' \leq G,\ p' \in P' = \pi(H') \leq P\} \leq G^{(P)} \quad (4)$$
$$H'^{(p')} \cong H' \leq G.$$

If H' is an invariant subgroup in G then $H'^{(p')}$ preserves all the colours, i.e. it coincides with $H^{(1)}$ $G^{(P)}$. When H' is not invariant subgroup of G, then the maximal invariant subgroup of G contained in H' is[17]

$$\text{Core } H' = : \bigcap_{g \in G} g H' g^{-1} . \quad (5)$$

In this case

$$\text{Ker } \pi = H = \text{Core } H' \quad (6)$$
$$H \cong H^{(1)} \lhd G^{(P)} .$$

According to Van der Waerden and Burckhardt[13] each transitive permutation representation π of G defines a colour group $G^{(P)}$, isostructural to G and vice versa. All the colour groups isostructural to a given G with the same subgroup H', but acting on different sets of colours F can be constructed by the same transitive representation $\pi_G^{H'}$. It can be constructed as a set of permutations of the left cosets $\{g_i H'\}$ of the coset decomposition of G with respect to the subgroup H' of index n:

$$\pi_G^{H'}(g) = \begin{bmatrix} g_1 H' \ldots g_i H' \ldots g_n H' \\ g g_1 H' \ldots g g_i H' \ldots g g_n H' \end{bmatrix} = p \in P,\ g_1 = e \in G . \quad (7)$$

There is one-one correspondence between the colours f_i and the left cosets $g_i H'$:

$$f_i \longrightarrow g_i H', \quad \forall\ i = 1, \ldots, n . \quad (8)$$

Hence, the properties and the classification of the colour groups are reduced to investigation of the permutation representations $\pi_G^{H'}$, generated by the definite subgroups H' of G. That was the reason Schwarzenberger[7] had defined the colour groups as group-subgroup pairs $H' \subset G$. In most of the definitions of colour-group equivalence regarded in the literature such a consideration of $G^{(P)}$ is presumed (see[16] for example). Generally, they should be represented in the following way:

The colour groups $G_1^{(P_1)}$ and $G_2^{(P_2)}$, isostructural to G_1 and G_2 are called equivalent if there exists an isomorphism δ of a special kind from G_1 onto G_2, which maps H_1' onto H_2':

$$\delta : G_1 \longrightarrow G_2 \qquad (9)$$
$$H_2' = \delta(H_1').$$

The isomorphism δ can be a group isomorphism, an affine or a proper affine conjugation or an inner automorphism of G (when $G_1 = G_2 = G$).

In the most popular special case, the left components p of the combined elements $<p;g> \in G^{(P)}$ can be considered as permutations of the set $\{1,2,\dots,n\}$ of the indices of the colours $f_i \in F$. In this case the images of the permutation representations π are transitive subgroups $P \le S_n$. The colour groups are viewed as subgroups of $S_n \times G \le S_n \times A$, where A is the affine group. The equivalence criteria have been formulated by Koptsik and Kotzev[2] as follows:

The colour groups $G_1^{(P_1)}$ and $G_2^{(P_2)}$ are called equivalent on a level $\Gamma = S_n \times L$ if they are subgroups of Γ conjugated by an element $\gamma \in \Gamma$, where the choice of γ may or may not be restricted by some conditions, i.e

$$G_2^{(P_2)} = \gamma\, G_1^{(P_1)} \gamma^{-1} \qquad (10)$$
$$\gamma \in S_n \times L, \quad G_1 \subset L, \; G_2 \subset L.$$

The group L might be chosen to be the affine group A, the Euclidean group E or the group G, when $G_1 = G_2 = G$. The element γ can be arbitrary or orientation preserving one in $S_n \times L$.

Note there is not one-to-one correspondence between the colour groups regarded[14] as pairs $H' \subset G$ and as subgroups[2] of $S_n \times G$. Generally, several groups $G^{(P)} \subset S_n \times G$ isostructural to a given G can exist with the same subgroup H' preserving the colour 1. These groups are conjugated by $<p;e> \in S_n \times G$, where $p \in Stab_{S_n}(1)$ and correspond to the different enumerations of the left cosets $\{g_i H'\}$ in the representation (7).

Up to the present work the relation between above two approaches of separating the colour groups into equivalence classes eq.(9) and eq.(10) has not been precisely clarified. In the next section of this work we propose a general systematization scheme of possible equivalence criteria for the colour groups.

3. EQUIVALENCE OF COLOUR GROUPS

Let two colour groups

$$G_i^{(P_i)} = \{<p^i;g^i>\}, \quad i = 1,2, \qquad (11)$$

act on the corresponding sets of colours

$$F^{(i)} = \{ f_1^{(i)},\dots,f_{n_i}^{(i)} \} \qquad (12)$$

In the equivalence definition for the colour groups we shall consider both an isomorphism of $\Delta: G_1^{(P_1)} \longrightarrow G_2^{(P_2)}$ and an one-to-one correspondence $\beta: F^{(1)} \longrightarrow F^{(2)}$, such that the Δ-isomorphic elements of the groups permute the β-corresponding colours of the sets $F^{(1)}$ and $F^{(2)}$

$$\Delta: G_1^{(P_1)} \longrightarrow G_2^{(P_2)}, \qquad \beta: F^{(1)} \longrightarrow F^{(2)} \qquad (13)$$
$$\Delta(<p^1:g^1>)\ \beta(f_i^{(1)}) = \beta(<p^1:g^1>\ f_i^{(1)}), \qquad i = 1,\dots,n.$$

This condition can be represented schematically by the following diagram:

$$
\begin{array}{ccc}
f_i^{(1)} & \xrightarrow{\quad <p^1:g^1> \quad} & f_j^{(1)} \\
\beta \downarrow & & \downarrow \beta \\
f_k^{(2)} & \xrightarrow{\quad <p^2:g^2> = \Delta(<p^1:g^1>) \quad} & f_l^{(2)} .
\end{array} \tag{14}
$$

The different classifications of equivalence are associated with different choices of the pair (Δ,β).

Two groups, acting on sets of elements, which satisfy the condition (13) are called "isomorphic as groups of transformations"/17/ (IGT). The permutation groups which are IGT are called "similar".

Due to the isomorphisms $\xi_i:G_i^{(P_i)} \longrightarrow G_i$, $i=1,2$, each isomorphism Δ of $G_1^{(P_1)}$ and $G_2^{(P_2)}$ unambiguously defines an isomorphism $\delta:G_1 \longrightarrow G_2$

$$\delta = \xi_2 \Delta\, \xi_1^{-1} \tag{15}$$

and vice versa. Therefore two colour groups, eq. (1), are IGT if and only if their groups G_1 and G_2 are IGT on the corresponding sets of colours, i.e.

$$\delta:G_1 \longrightarrow G_2 \quad , \quad \beta:F^{(1)} \longrightarrow F^{(2)}$$

$$\pi_2(\delta(g^1))\ \beta(f_i^{(1)}) = \beta(\pi_1(g^1) f_i^{(1)}) \quad , \quad i = 1,\dots,n. \tag{16}$$

Hence, the natural equivalence definition of $G^{(P)}$ up to the isomorphisms of different kinds Δ is reduced to classification of the groups G and the transitive representations π.

The classification of the colour groups from the point of view of IGT is appropriate for any dimension m of the Euclidean space and any number of colours n.

Here the problem of colour-group equivalence should be considered in two aspects:

i) an equivalence of colour groups isostructural to isomorphic G_1 and G_2. This aspect is appropriate in the mathematical crystallography (We shall refer to it as a General Definition (GD)).

ii) an equivalence of colour groups isostructural to the same G. This aspect is more appropriate in solid state physics: the geometrical structure of crystal and its symmetry group G are given and the colour groups isostructural to that G are to be classified and applied for describing some physical phenomenon (a Special Definition (SD)).

Hence we shall propose two general schemes of systematization of the possible equivalence criteria in terms of IGT. They are represented in the second columns "IGT" of Table 1 for GD and Table 2 for SD, respectively. The proposed schemes could be expended (further specification of β).

As far as we know, equivalence criterion from the point if view of IGT has been considered only by Van der Waerden and Burckhardt/13/ for colour groups isostructural to a given G. The equivalence of colour groups is identified with the similarity of the corresponding transitive representations $\pi:G \longrightarrow P$: two transitive representations are similar if they are related by one-to-one colour substitution. This is fulfilled if and only if the corresponding subgroups H_1' and H_2' are conjugated in G/13/.

The condition (16) for IGT colour groups regarded as subgroups of $S_n \times A$ can be presented as

$$\pi_2(\delta(g^1)) = s\ \pi_1(g^1)\ s^{-1} \quad , \ \forall\ g^1 \in G_1 \ , \ s \in S_n \ , \tag{17}$$

having in mind that in this case the one-one correspondence β is a permutation $s \in S_n$ of the colour indices. Therefore the possible equivalence criterion in the frame of IGT given in tables 1 and 2 can be reformulated in the sense of (10). The results are represented in the last column "$G^{(P)} \leq S_n \times A$" of the tables.

The relationship between the equivalence in the frame of IGT and (8) can be derived on the base of the following assertion.

<u>Assertion 1</u>. Two colour groups $G_1^{(P_1)}$ and $G_2^{(P_2)}$ are isomorphic as groups of transformations by an isomorphism $\delta:G_1 \longrightarrow G_2$ if

and only if δ maps the class of conjugated subgroups $\{g H_1' g^{-1} | \quad H_1' \le \quad G_1, \quad g \in G_1\}$ onto the class $\{g H_2' g^{-1} | H_2' \le G_2, g \in G_2\}$, i.e.

$$\delta : G_1 \longrightarrow G_2$$
$$H_2' = g_o \, \delta(H_1') \, g_o^{-1} \quad , g_o \in G_2 . \tag{18}$$

This assertion is proved in details in Appendix 1. In the case, when g_o can be chosen to coincide with e G, the correspondence β maps a colour $f_1^{(1)}$ to $f_1^{(2)}$, i.e.

$$\beta \, (f_1^{(1)}) = f_1^{(2)} \quad . \tag{19}$$

By Assertion 1 the colour-group classification from the point of view of IGT is related to the classification of the corresponding group-subgroup relations $H' \le G$ and hence to the classification of the physical phenomena associated with the possible symmetry descents $G|H'$ (see Ref./18/).

Using Assertion 1 the General and the Special Definitions of equivalence can be reformulated in terms of isomorphisms between the pairs $H' \le G$. The results are given in the third columns "$H' \le G$" of Tables 1 and Table 2. From this column of Table 2 it follows that $G^{(P)}$-classifications by the criteria SD8 and SD9 are identical.

In general, the colour-group-equivalence criteria presented in the tables below (except SD8 and SD9) give rise to distinct partitions into equivalence classes of groups. However, in many special cases two or more partitions are identical. For example, if G_1 and G_2 are space groups, than partition into equivalence classes on the sense of GD1 (or SD1) and of GD2 (or SD2) are identical (as a consequence of the Bieberbach's Theorem). If the groups G are point groups in odd-dimensional Euclidean space the partition into equivalence classes in the sense of GD2, GD3, GD4 will always be identical, as well as this one by definitions from SD2 to SD7. In Table 3 the numbers of equivalence classes of 3-dimensional colour point groups are given in the sense of all

the proposed definitions. As an example, in Table 4 the equivalence classes of the colour groups isostructural to the abstract group D_2 are listed. The group symbols inside the frame of a given criterion are the class-representatives. The number of the classes is shown in parentheses under the criterion symbol.

It is worth to mention that there are two lists of colour point groups in the literature. In the first one/1-3,19/ there are 244 groups (classes of groups) according to the definition SD2 (GD2). In the second one/7/ there are 279 groups according to the definition SD8.

It is clear that all the groups isostructural to a given G with the same H' are equivalent in the sense of any adopted equivalence criterion in our systematization schemes. It is more convenient for these colour groups to use the "full" symbol

$$G^{(P)} \equiv G/H'/H(A,A')_n \tag{20}$$

introduced by Koptsik and Kotzev/2/. It contains a comprehensive information for the structure of $G^{(P)}$, very useful for a classification and physical applications of these groups. Here the transitive permutation group $P \in S_n$ is denoted by the symbol $(A,A')_n$. The abstract groups A and A' are isomorphic to the factor-groups and the corresponding permutational subgroups of S_n:

$$A \cong G^{(P)} / \, H^{(1)} \cong G/H \cong P$$
$$A' \cong H'^{(P)} / H^{(1)} \cong H'/H \cong P' \subset P . \tag{21}$$

The groups A and $A' \subset A$ play a significant role in the chromomorphic classification/2,8/ of the colour groups with respect to Image $\pi = P$. Due to the isomorphisms (21) it is easy to show that colour groups with the same A and A' have similar permutational groups. They can be constructed by the same faithful representation $\pi_A^{A'}$ of A and are denoted by the symbol $(A,A')_n$. Usually the similar groups P are considered in the literature as identical. Hence the chromomorphic classification[2] of the colour groups implies classification into similarity of the permutational groups $P = (A,A')_n$. (Practically the introduced in[3] equivalence of two P-symmetries implies similarity of the corresponding groups P.)

Let the groups P are presented as groups of permutations of the colour indices $\{1,\dots,n\}$. Having in mind that the similar P are faithful representations of the same group A, it follows from (17) that two transitive permutational groups are similar if and only if they can be presented as conjugated subgroups of S_n.

So, the chromomorphic classification of the colour groups considered as subgroups of $S_n \times A$ implies a classification into classes of conjugated subgroups of S_n.

Because of the mentioned relation between the colour-group classification and the symmetry-descent one there is one-to-one correspondence between the chromomorphic classes $(A,A')_n$ (see Ref.[2,8] and exomorphic types of the symmetry descents, introduced by Kopsky[18].

The proposed approach to the equivalence of $G^{(P)}$ can be applied to colour groups isostructural to any groups G, not only to the crystallographic groups G. Then the different criteria should correspond to appropriate choice of the pairs (δ,β) in (16). In a similar way these criteria should be reformulated in the frame of (9) and (10) on the base of the relation (17) and Assertion 1.

4. CONCLUSIONS

In the present paper the equivalence problem for the P-type permutational colour groups has been studied. An approach aimed at classification of the colour groups has been proposed. It is based on the concept of isomorphisms of groups as groups of transformations of sets[17]. The relationships between the proposed approach and the widely used in the literature ones is clarified. The last mentioned approaches are based on a) group-subgroup relations and classification of subgroups; b) classification of colour groups $G^{(P)}$ as subgroups of a major group, for example $S_n \times A$.

Our results are summarized in Table 1 and Table 2. Keeping in mind the conventional space group classification in the crystallography and solid-state physics as well as the possible physical applications of colour groups, some recommendations for classification of 3-dimensional colour space groups can be done.

All space groups fall into 230 types (equivalence classes) with respect to proper affine conjugation by $a \in A^+$. Therefore, following the crystallographic traditions, it is appropriate to use criteria based on proper affine transformations $a \in A^+$ only: GD3, GD4, SD3, SD4, SD6, SD7, SD8 (≡ SD9), SD10. However in the general case the criteria GD4, SD4, SD7, SD10, based exclusively on the use of proper rotations, do not permit to establish an equivalence relationships between a given subgroup $H'_1 \subset G_1$ and all the subgroups of the class $\{g H'_2 g^{-1} | g \in G_2\}$ of $G_2 = a G_1 a^{-1}$. For example such a case always occurs when H'_1 and H'_2 are enantiomorphic pairs. The resulting splitting with respect to the equivalence relation of the classes of conjugated subgroups would be inadmissible in some physical applications. For example it may lead to inequality of the colours in the colour space or to definition of nonequivalent colour groups associated with equivalent permutation representations $D_G^{H'}$ [7] (the last might correspond to equivalent physical properties, such as domain with the same energies, etc.)

The choice between the rest criteria depends on the presumable application of the colour space groups. For the crystallographic applications the definition GD3 (direct generalization of SD3) is appropriate. The definition SD8 is recommended in solid state physics, where the physical properties are related with the irreducible representations of the space groups. In this case colour groups equivalent by SD3 might be associated with unequivalent representations $D_G^{H'}$ of the space groups. By the similar reasons the criteria SD8 has been adopted in Ref[6-12] and has been called "a physical equivalence of the colour groups"[11], When a physical phenomenon is associated with so-called "quasi-equivalent representations"/20/ the transition from SD8 to SD3 would be easier than backwards.

APPENDIX. Proof of the Assertion 1.

Let first to assume that two groups $G_1^{(P_1)}$ and $G_2^{(P_2)}$ are IGT by an isomorphism $\delta: G_1 \longrightarrow G_2$ and β in (16). Let

$$\beta(f_1^{(1)}) = f_j^{(2)}. \tag{1.1}$$

It is easy to see that

$$\delta(H_1') = H_2^j , \tag{1.2}$$

where

$$\pi_1(H_1')f_1^{(1)} = f_1^{(1)} \quad , \quad \pi_2(H_2^j)f_j^{(2)} = f_j^{(2)}. \tag{1.3}$$

From the structure of the colour groups it follows that δ maps the class of conjugated subgroups $\{g H_1' g^{-1} \mid g \in G_1\}$ onto the class $\{g H_2^j g^{-1} \mid g \in G_2\}$, i. e. there exists $g_o \in G_2$, such that

$$H_2' = g_o H_2^j g_o^{-1} = g_o \delta(H_1') g_o^{-1} .$$

Conversely, let there exists an isomorphism δ and $g_o \in G_2$ which satisfy (18). Then G_i, $i = 1,2$, are IGT of $F^{(i)}$ and of the sets of the left cosets $\{g_j^{(i)} H_i'\}$ by the pairs $(a_o^{(i)}, \beta^{(i)})$, where $a_o^{(i)}$ are the trivial automorphisms of G_i and $\beta^{(i)}$ are the one-one correspondences (8). Due to the isomorphism δ G_1 and G_2 are IGT of the sets $\{g_j^{(1)} H_1'\}$ and $\{\delta(g_j^{(1)} H_1')\}$ by the pair (δ, β_1), where

$$\beta_1 : g_j^{(1)} H_1' \longrightarrow \delta(g_j^{(1)} H_1') \quad , i = 1,\ldots,n \tag{1.4}$$

$$H_1' \longrightarrow \delta(H_1') .$$

As the subgroup $\delta(H_1')$ and H_2' are conjugated in G_2 by $g_o \in G_2$

there is one-one correspondence β_2 from the set $\{\delta(g_j^{(1)} H_1')\} = \{g_1^{(2)} \delta(H_1')\}$ and the set $\{g_j^{(2)} H_2'\}$, such that G_2 is IGT of them by the pair $(a_o^{(2)}, \beta_2)$. Therefore we have that $G_1^{(P_1)}$ and $G_2^{(P_2)}$ are IGT of $F^{(1)}$ and $F^{(1)}$ according with (16) by the pair (δ, β), where

$$\delta' = a_o^{(2)-1} \quad a_o^{(2)} \quad a_o^{(1)} = \delta$$

$$\beta = \beta^{(2)-1} \beta_2 \beta_1 \beta^{(1)} .$$

Table 1. Equivalence of colour groups belonging to different families

GD	IGT	$H' \subset G$	$G^{(P)} \subset S_n \times A$
GD1	δ- an isomorphism β- one-one correspondence	$\delta: G_1 \longrightarrow G_2$ $H_2' = \delta(H_1')$ δ - isomorphism	$\delta: G_1 \longrightarrow G_2$ $\pi_2(\delta(g^1)) = s\ \pi_1(g^1)\ s^{-1}$ δ - isomorphism, $s \in S_n$
GD2	$\delta \leftrightarrow$ conjugation by $a \in A$ β- one-one correspondence	$G_2 = a\ G_1 a^{-1}$ $H_2' = a\ H_1' a^{-1}$ $a \in A$	$G_2^{(P_2)} = \langle s;a \rangle\ G_1^{(P_1)}\ \langle s;a \rangle^{-1}$ $\langle s;a \rangle \in S_n \times A$
GD3	$\delta \leftrightarrow$ conjugation by $a \in A^+$ β- one-one correspondence	$G_2 = a\ G_1 a^{-1}$ $H_2' = g\ a\ H_1' a^{-1} g^{-1}$ $a \in A^+$, $g \in G_2$	$G_2^{(P_2)} = \langle s;a \rangle\ G_1^{(P_1)}\ \langle s;a \rangle^{-1}$ $\langle s;a \rangle \in S_n \times A^+$
GD4	$\delta \leftrightarrow$ conjugation by $a \in A^+$ β- one-one correspondence, $\beta(f_1^{(1)}) = f_1^{(2)}$	$G_2 = a\ G_1 a^{-1}$ $H_2' = a\ H_1' a^{-1}$ $a \in A^+$	$G_2^{(P_2)} = \langle s;a \rangle\ G_1^{(P_1)}\ \langle s;a \rangle^{-1}$ $\langle s;a \rangle \in Stab_{S_n}(1) \times A^+$

Table 2. Equivalence of colour groups isostructural to the same G

SD	IGT	$H' \subset G$	$G^{(P)} \subset S_n \times A$
SD1	δ- automorphism β- one-one correspondence	$H'_2 = \delta(H'_1)$ δ- automorphism of G	$\pi_2(\delta(g)) = s\, \pi_1(g)\, s^{-1}$ δ-automorphism of G, $s \in S_n$
SD2	δ ↔ conjugation by a $N_A(G)$ β- one-one correspondence	$H'_2 = a\, H'_1 a^{-1}$ $a \in N_A(G)$	$G_2^{(P_2)} = <s;a>\, G_1^{(P_1)}\, <s;a>^{-1}$ $<s;a> \in S_n \times N_A(G)$
SD3	δ ↔ conjugation by a $\in N_A{+}(G)$ β- one-one correspondence	$H'_2 = g\, a\, H'_1 a^{-1} g^{-1}$ $a \in N_A{+}(G)$, $g \in G$	$G_2^{(P_2)} = <s;a>\, G_1^{(P_1)}\, <s;a>^{-1}$ $<s;a> \in S_n \times N_A{+}(G)$
SD4	δ ↔ conjugation by a $\in N_A{+}(G)$ β- one-one correspondence, $\beta(f_1^{(1)}) = f_1^{(2)}$	$H'_2 = a\, H'_1 a^{-1}$ $a \in N_A{+}(G)$	$G_2^{(P_2)} = <s;a>\, G_1^{(P_1)}\, <s;a>^{-1}$ $<s;a> \in Stab_{S_n}(1) \times N_A{+}(G)$
SD5	δ ↔ conjugation by a $\in N_E(G)$ β- one-one correspondence	$H'_2 = a\, H'_1 a^{-1}$ $a \in N_E(G)$	$G_2^{(P_2)} = <s;a>\, G_1^{(P_1)}\, <s;a>^{-1}$ $<s;a> \in S_n \times N_E(G)$
SD6	δ ↔ conjugation by a $\in N_E{+}(G)$ β- one-one correspondence	$H'_2 = g\, a\, H'_1 a^{-1} g^{-1}$ $a \in N_E{+}(G)$, $g \in G$	$G_2^{(P_2)} = <s;a>\, G_1^{(P_1)}\, <s;a>^{-1}$ $<s;a> \in S_n \times N_E{+}(G)$
SD7	δ ↔ conjugation by a $\in N_E{+}(G)$ β- one-one correspondence, $\beta(f_1^{(1)}) = f_1^{(2)}$	$H'_2 = a\, H'_1 a^{-1}$ $a \in N_E{+}(G)$	$G_2^{(P_2)} = <s;a>\, G_1^{(P_1)}\, <s;a>^{-1}$ $<s;a> \in Stab_{S_n}(1) \times N_E{+}(G)$
SD8	δ Int(G) β- one-one correspondence	$H'_2 = g\, H'_1 g^{-1}$ $g \in G$	$G_2^{(P_2)} = <s;g>\, G_1^{(P_1)}\, <s;g>^{-1}$ $<s;g> \in S_n \times G$
SD9	δ $Int^{+}(G)$ β- one-one correspondence	$H'_2 = g\, H'_1 g^{-1}$ $g \in G$	$G_2^{(P_2)} = <s;g>\, G_1^{(P_1)}\, <s;g>^{-1}$ $<s;g> \in S_n \times G^{+}$
SD 10	δ $Int^{+}(G)$ β- one-one correspondence, $\beta(f_1^{(1)}) = f_1^{(2)}$	$H'_2 = g\, H'_1 g^{-1}$ $g \in G^{+}$	$G_2^{(P_2)} = <s;g>\, G_1^{(P_1)}\, <s;g>^{-1}$ $<s;g> \in Stab_{S_n}(1) \times G^{+}$

Table 3. Classification of colour-point groups

GD		SD	
GD	Number of $G^{(P)}$	SD	Number of $G^{(P)}$
GD1	130	SD1	
GD2 GD3 GD4	244	SD2, SD3 SD4, SD5 SD6, SD7	244
		SD8, SD9	279
		SD10	280

Table 4. Classification of colour groups isomorphic to D_2

	G/H' [G:H'] = 4	G/H' [G:H'] = 2	G/H' [G:H'] = 1
GD1 (3)	D_2/C_1	D_2/C_2^z	D_2/D_2
SD1 (9)	C_{2v}/C_1 C_{2h}/C_1	C_{2v}/C_2 C_{2h}/C_2	C_{2v}/C_{2v} C_{2h}/C_{2h}
GD2 SD2 (12)		C_{2v}/C_s^x C_{2h}/C_s C_{2h}/C_i	
SD9 (15)		C_{2v}/C_s^y D_2/C_2^x D_2/C_2^y	

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О проблеме эквивалентности цветных групп Р-типа

Анализируются возможные определения эквивалентности цветных кристаллографических групп Р-типа на базе концепции изоморфизма групп как преобразования множеств. Показана взаимосвязь и применимость различных критериев в математической кристаллографии и физике твердого тела.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Перевод авторов

Alexandrova D.A., Kotzev J.N. E17-90-397

On the Problem of the Equivalence of P-Type Permutational Colour Groups

An approach aimed at equivalence classification of P-type colour groups is proposed. It is based on the concept of isomorphisms of groups as groups of transformations of colour sets. The relation between the proposed approach and those considering the colour groups as group-subgroup pairs or subdirect products of S_nxG is clarified. The choice of an appropriate equivalence criterion for the purposes of the mathematical crystallography and solid state physics is discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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