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A THREE STATE POTTS MODEL WITH COLOUR GROUP SYMMETRY

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I. Introduction

The present article is based on an earlier one by Schick and Griffiths[1]. The reason for returning to the Potts model discussed there is twofold. First, we identify the symmetry of this model[1] as colour group symmetry. Thus colour space groups[2-6] are needed to find the degeneracies of the respective Hamiltonian, being therefore groups of dynamical invariance. We would like to stress that this is a new field of application of the idea of colour space groups introduced earlier on purely geometrical grounds. Second, by applying a new method, discussed elsewhere[7], we find that the phase diagram corresponding to the present model deviates in part from the one constructed in Ref.1. We also explicitly determine the invariance of the low symmetry phases which are once again given by nontrivial colour groups. Finally it is quite enlightening to compare our results to those that would be obtained by standard Landau argumentation.

The material is organized as follows:in part II we focus our attention on the precise symmetries of the referred Potts Hamiltonian[1], while in part III we present the results concerning the thermodinamical behaviour of the system under investigation.

II. The Symmetry of the Hamiltonian[1]

The usual q-state Potts model[8] is a generalization of the Ising model. It represents an attractive or repulsive two-particle interaction between nearest neighbours. The particles (infinite in number) are situated on a certain lattice and each of them occupies one of q possible states. It is well known that the thermodynamics of such a system is equivalent to the one of a quantum mechanical model with Hamiltonian whose eigen states are into a one-to-one correspondence with the mentioned discrete "Potts" states. Therefore in what follows we restrict ourselves to the quantum - mechanical terminology. In this terms the Hamiltonian given by Schick and Griffiths is [1],

$$\hat{H} = -\frac{1}{2}K\sum_{\langle ij\rangle}\sum_{l=1}^{3}\hat{P}_i^l\hat{P}_j^l - L\sum_{\langle ijk\rangle}\sum_{l=1}^{3}\hat{P}_i^l\hat{P}_j^l\hat{P}_k^l \quad . \qquad (1)$$

Here K is the usual two-particle interaction constant and $\langle ij \rangle$ means sum over the nearest couple neighbours. The second term represents a three – particle interaction with constant L, the sum ranging over the nearest triple neighbours. The particles are situated in the high symmetry Wyckoff position (point group- C_{6v}=6mm) of a two dimensional hexagonal lattice with symmetry group $G'=p6mm$. As q=3 the Hilbert space of the present three state model is an infinite tensor product of 3-dimensional linear spaces. The operator $\hat{P}_i^l$ is a projection operator corresponding to the l-th state of the i-th particle,

$$\hat{P}_i^1 = E(3)\otimes E(3)\otimes\ldots\otimes\underbrace{\begin{bmatrix}1 & & \\ & 0 & \\ & & 0\end{bmatrix}}_{i\text{-th place}}\otimes\ldots \quad ,l=1. \qquad (2)$$

To simplify our considerations and to focus on the essential symmetries of $\hat{H}$ we impose the usual Born-Karman conditions. Thus the system reduces to a finite N-particle system with symmetry group $G \cong G'/T_N$. As always T_N is an infinite subgroup of T, the invariant translation subgroup of G'; $T_N=\{(E|N_1\vec{a}_1+N_2\vec{a}_2)\,|N_1N_2=N\}$ is generated by two sufficiently large (non collinear) translations belonging to T and $[T:T_N]=N$. The group G is not isomorphic to $p6mm$ although it is essentially "the same". To be short in what follows we shall denote G by $p6mm$, having in mind that $p6mm \xrightarrow{\phi} G \subset S_N$, $\ker\phi = T_N$, $|G|=[G':T_N]=12N$. Even now when the Hilbert space of the system is finite dimensional the symmetry group of the Hamiltonian (i.e. the set of operators that commute with it [9]) is infinite. This group $\tilde{G}_H$, contains an infinite invariant subgroup U of unitary operators, diagonal in the basis of eigen vectors of $\hat{H}$. It is only the group $\bar{G}_H=\tilde{G}_H/U$ that is physically interesting and for the 3-state Potts model $\bar{G}_H$ is straight forwardly identified to be a subgroup of S_{3N}. The latter is the group of permutations of 3N objects, which are the 3N eigen vectors of $\hat{H}$. Note that $\hat{P}_i^1$ are the projectors adapted to these vectors.

Consider the group,

$$W \circledS S_N = (\underbrace{S_3\times S_3\times\ldots\times S_3}_{N}) \circledS S_N = S_3 \textcircled{w} S_N \subset S_{3N} \ . \qquad (3)$$

In Eq.(3) we used the notation $\textcircled{w}$, which stands for a wreath product [10] of the occurring groups. For the sake of simplicity hereafter we shall use the symbol $A \subset B$ for both the cases when A is a subgroup of B and when A is isomorphic to a subgroup of B.

The group $W \circledS S_N$ permutes independently the states into the different one-particle spaces and also performs all possible permutations of these spaces. Therefore the action of the group element on $\hat{P}_i^1$ is given by,

$$(p_1,\ldots,p_N|p)\hat{P}_i^1 = \hat{P}_{p(i)}^{p_i(1)} = U_{(p_1,\ldots,p_N|p)}\,\hat{P}_i^1\,U^{-1}_{(p_1,\ldots,p_N|p)} \ ,$$

$p_i \in S_3$, $i=1,\ldots,N$; $p \in S_N$; $U_{(p_1,\ldots,p_N|p)}$ -the unitary operator corresponding to $(p_1,\ldots,p_N|p)$. (4)

As a consequence of Eq.(4) we obtain that,

$$(p_1,\ldots,p_N|p)(p'_1,\ldots,p'_N|p') = (p_{p'(i)}p'_1,\ldots,p_{p'(k)}p'_N|pp')$$

where $p'(i)=1$,..., $p'(k)=N$. (5)

Equation(5) is exactly the multiplication law of a wreath product [10] and clarifies the meaning of Eq.(3).

In what follows we shall be interested in G_H – a subgroup of $W \circledS S_N$ that leaves $\hat{H}$ invariant. The main reason is that $W \circledS S_N$ acts transitively on the eigen vectors of $\hat{H}$. Thus even though $G_H \subset \bar{G}_H$, the decomposition of the Hilbert space into $\bar{G}_H$-invariant subspaces should coincide with its decomposition into G_H-invariant subspaces. These arguments and the definition of $\hat{H}$ (cf.Eq.(1)) immediately imply that,

$$G_H \subset W \circledS G \subset W \circledS S_N \ . \qquad (6)$$

Thus G_H, the group of essential invariance of $\hat{H}$, being a subgroup of $W \circledS G$ is therefore a P- or W-type colour group in the terminology of Refs 2-3.

For arbitrary values of the interaction constants (K and L) it is also easy to conclude from Eq.(1) that ,

$$G_H = (\underbrace{[S_3\times S_3\times\ldots\times S_3]}_{N}) \circledS G \subset W \circledS G = S_3 \textcircled{w} G,$$

where $[\underbrace{S_3\times S_3\times\ldots\times S_3}_{N}] = \mathrm{diag}\,(\underbrace{S_3\times S_3\times\ldots\times S_3}_{N}) = \{\underbrace{(p,p,\ldots,p)}_{N} \mid p \in S_3\}$. (7)

Therefore,

$$G_H = S_3 \times G . \tag{8}$$

To proceed further, following Ref.1, we decompose the lattice into 3 sublattices A, B and C respectively. On figure 1 A is denoted by ⊙, B – by ●, and C – by ×. Clearly the symmetry group of sublattice A – G_A, has a point group identical to that of $G = p6mm$ and a translation group with enlarged minimal translation vectors. They might be chosen to be,

$$\vec{t}_1' = \vec{t}_1 - \vec{t}_2 \quad ; \quad \vec{t}_2' = \vec{t}_1 + 2\vec{t}_2 \quad . \tag{9}$$

We find the structure of G_A and fix the coset representatives of G with respect to G_A,

$$G_A = p'6mm \subset p6mm = G ,$$

$$G = G_A \cup (E|t_1)G_A \cup (E|2t_1)G_A = G_A \cup g_2G_A \cup g_3G_A . \tag{10}$$

Bypassing we note that,

$$G_B = g_2G_Ag_2^{-1} , \quad G_C = g_3G_Ag_3^{-1} ,$$

and

$$G_A \cap G_B \cap G_C = G_{ABC} . \tag{11}$$

The groups G_B and G_C are the invariance groups of the sublattices B and C respectively. (On figure 1 the elementary cells of these lattices are shown.) The group $G_{A,B,C}$ is an invariant subgroup of G which is a stabilizer of the three sublattices.

$$G_A = G_{ABC} \cup g_0G_{ABC} , \quad G_{ABC} = p'31m \text{ (cf. Ref.11)} . \tag{12}$$

We chose,

$$g_0 = (m'|\vec{0}), \quad m' = (x,2x) \text{ (see figure 1)}. \tag{13}$$

The order of the multiples S_3 in W could be adapted to the decomposition of G with respect to G_A (cf. Eq.(10)). Indeed each of the mentioned factors S_3 corresponds to one lattice site. The sites might be labeled by the different translantions in G. Finally we separate the factors in three groups corresponding to the sublattices

A, B and C. Thus we construct the group W',

$$W' \cong (\underbrace{[S_3\times S_3\times\ldots\times S_3]}_{N/3}\times\underbrace{[S_3\times S_3\times\ldots\times S_3]}_{N/3}\times\underbrace{[S_3\times S_3\times\ldots\times S_3]}_{N/3}) \subset W ,$$

$$W' = S_3\times S_3\times S_3 . \tag{14}$$

The fact that the Hamiltonian $\hat{H}$ demonstrates interactions between particles belonging to different sublattices is a hint that $G_H \subset W' \circledS G \subset W \circledS G$. This conjecture agrees with Eq.(8), that gives G_H for the "general position" in the (K, L) space. Let's define M as,

$$M = \frac{3}{2} K + L . \tag{15}$$

In Ref.1 it was noticed that when M=0 the symmetry group of $\hat{H}$ spontaneously rises to $G_H^{M=0}$. To see what is this group we proceed as follows. We decompose the lattice into triangles as shown on figure 2. Each site belongs to only one triangle and in each triangle there is only one site from the sublattices A, B or C. Therefore the sites might be labeled by iA(iB, iC), where i is the label of i-th the triangle. Clearly this decomposition is invariant under the group G_{ABC}. As $[G:G_{ABC}] = 6$, there are six such decompositions. The remaining five ones could be obtained from the present decomposition applying the coset representatives of G with respect to G_{ABC} (cf. Eqs (10-13)). If we introduce an additional label s = 1,..,6 , then we obtain an unambiguous notation for every site in each of the six decompositions, namely $i_sA(i_sB, i_sC)$. Now the Hamiltonian might be rewritten in the form,

$$\hat{H} = -\frac{1}{2} K \sum_{s=1}^{6} \sum_{i_s} \sum_{l=1}^{3} \left[\hat{P}^l_{i_sA}\hat{P}^l_{i_sB} + \hat{P}^l_{i_sB}\hat{P}^l_{i_sC} + \hat{P}^l_{i_sC}\hat{P}^l_{i_sA} \right] -$$

$$- L \sum_{s=1}^{6} \sum_{i_s} \sum_{l=1}^{3} \hat{P}^l_{i_sA}\hat{P}^l_{i_sB}\hat{P}^l_{i_sC} , \tag{16}$$

or

$$\hat{H} = \sum_{s=1}^{6} \sum_{i_s} \hat{H}_s , \tag{17}$$

where the meaning of $\hat{H}_s$ is obvious from Eqs (16,17). It is already straight forward to find by direct calculation the group of invariance of $\hat{H}_s$. For M=0 this is,

$$\begin{aligned} W^{M=0}=\{&(p_1,p_1,p_1),(p_1,p_2,p_3),(p_4,p_5,p_6)\\ &(p_2,p_2,p_2),(p_1,p_3,p_2),(p_4,p_6,p_5)\\ &(p_3,p_3,p_3),(p_2,p_1,p_3),(p_5,p_4,p_6)\\ &(p_4,p_4,p_4),(p_3,p_1,p_2),(p_5,p_6,p_4)\\ &(p_5,p_5,p_5),(p_2,p_3,p_1),(p_6,p_4,p_5)\\ &(p_6,p_6,p_6),(p_3,p_2,p_1),(p_6,p_5,p_4)\} , \end{aligned}$$

$$W^{M=0} \cong C_3 \; s \; S_3 \; . \qquad (18)$$

The isomorphism is given by,

$$S_3 \cong [S_3 x S_3 x S_3] \; ,$$

$$C_3 = \{e,a,a^2\} \cong \{(p_1,p_1,p_1),(p_1,p_2,p_3),(p_1,p_3,p_2)\} \; , \qquad (19)$$

while the automorphisms of C_3 required to define the semidirect product are,

$$(a)^{p_1}=(a)^{p_2}=(a)^{p_3}=(a^2)^{p_4}=(a^2)^{p_5}=(a^2)^{p_6}=a \; ,$$

$$(a^2)^{p_1}=(a^2)^{p_2}=(a^2)^{p_3}=(a)^{p_4}=(a)^{p_5}=(a)^{p_6}=a^2 \; . \qquad (20)$$

In Eqs (18-20) we used the shorthand notation,

$$p_1=(1)(2)(3), p_2=(123), p_3=(132), p_4=(12), p_5=(23), p_6=(13). \qquad (21)$$

Having in mind the invariance of $\hat{H}_s$ we see that

$$G_H^{M=0} = W^{M=0} \circledS G \subset W' \circledS G \; . \qquad (22)$$

In Ref.1 $G_H^{M=0}$ was given through its generators. Now we found its structure explicitly. For the reader's convenience we represent the introduced group-subgroup relations on diagram 1.

$$\begin{aligned} &S_{3N}\\ &\cup\\ &W \circledS S_N = \underbrace{(S_3 x S_3 x \ldots x S_3)}_{N} \circledS S_N = S_3 \; ⓦ \; S_N\\ &\cup\\ &W \circledS G \;\; = \underbrace{(S_3 x S_3 x \ldots x S_3)}_{N} \circledS G \;\; = S_3 \; ⓦ \; G\\ &\cup\\ &W' \circledS G = (S_3 x S_3 x S_3) \circledS G\\ &\cup\\ &G_H^{M=0} = W^{M=0} \circledS G \; , \quad W^{M=0} \cong C_3 \circledS S_3\\ &\cup\\ &G_H = S_3 \times G \end{aligned}$$

diagram 1.

III. Thermodynamics of the Three State Potts Model

The thermodynamical behaviour (phase transitions, critical phenomena) of a system with Hamiltonian $\hat{H}$ (cf. Eq.(1)) is described by the corresponding free energy F=F(T;K,L). The direct approach requires the calculation of

$$F = -k \; T \; \ln \; Sp \; [\exp \; (-\hat{H}(K,L)/kT)] . \qquad (23)$$

Unfortunately even for the usual Potts model (L=0) this is only possible at the transition temperature[/12/]. Therefore certain approximation technique should be applied. One of the appropriate possibilities in the present case is the position space renormalization group[/13/]. It was realized in Ref.1. A second possibility is to find the generalized free energy (Landau-Ginzburg-Wilson Hamiltonian) employing a method we shall name hereafter the "inhomogeneous mean field"[/7,14/]. The LGW - Hamiltonian might be studied following one of the standard procedures. One could extract the main (leading) degree of freedom performing thus the Landau approximation[/15/] or to take into account the critical fluctuations by means of the usual renorm group approach[/16/].

An entirely phenomenological approach consists in applying the Landau theory of phase transitions[/15/] starting with symmetrical arguments only. Naturally thus one can not find any quantitative approximation for $F=F(T;K,L)$, but nevertheless the possible qualitative behaviours of the system could be predicted. A key question in this case would be, what is the "symmetry of the physical problem". The standard procedure requires the use of the crystal group and the transformation properties of some physical (tensor) field defined on localized site functions /17/. A better choice is to use the full symmetry of the Hamiltonian $\hat{H}$ /7/. The latter approach was partly realized in Ref.1.

We shall present the results found using the inhomogeneous mean field technique. The LGW Hamiltonian obtained this way is studied further on by the Landau approximation. A comparison with the results of the standard Landau theory is also given.

In order to construct the LGW – Hamiltonian it is convenient to pass to new operator variables in $\hat{H}$,

$$\hat{M}_i^1 = \frac{\sqrt{2}}{2}(\hat{P}_i^1 - \hat{P}_i^2) = E(3)\otimes\ldots\otimes\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & & \\ & -\frac{1}{\sqrt{2}} & \\ & & 0 \end{bmatrix}}_{\text{i-th place}}\otimes\ldots$$

$$\hat{M}_i^2 = -\frac{1}{\sqrt{6}}(\hat{P}_i^1 + \hat{P}_i^2) + \frac{2}{\sqrt{6}}\hat{P}_i^3 = E(3)\otimes\ldots\otimes\underbrace{\begin{bmatrix} -\frac{1}{\sqrt{6}} & & \\ & -\frac{1}{\sqrt{6}} & \\ & & \frac{2}{\sqrt{6}} \end{bmatrix}}_{\text{i-th place}}\otimes\ldots$$

$$\hat{M}_i^3 = \frac{1}{\sqrt{3}}(\hat{P}_i^1 + \hat{P}_i^2 + \hat{P}_i^3) = \frac{1}{\sqrt{3}}E(3)\otimes\ldots\otimes E(3)\otimes\ldots = \frac{1}{\sqrt{3}}\hat{E}\,. \qquad (24)$$

From Eq.1 we obtain,

$$\hat{H} = -\frac{1}{2}\left(K+\frac{2}{3}L\right)\sum_{\langle ij\rangle}\sum_{l=1}^{2}\hat{M}_i^1\hat{M}_j^1 - \frac{1}{6\sqrt{6}}L\sum_{\langle ijk\rangle}(\hat{M}_i^2\hat{M}_j^2\hat{M}_k^2 - \hat{M}_i^2\hat{M}_j^1\hat{M}_k^1 - \hat{M}_i^1\hat{M}_j^2\hat{M}_k^1 - \hat{M}_i^1\hat{M}_j^1\hat{M}_k^2) + c\hat{E}\,. \qquad (25)$$

Here c is a constant and the whole term $c\hat{E}$ is physically insignificant. Using the methods developed in Ref.7 we obtain the following LGW-Hamiltonian,

$$\Phi(\vec{M}) = -\frac{1}{2}\left(K+\frac{2}{3}L\right)\sum_{\langle ij\rangle}\sum_{l=1}^{2}M_{i,1}M_{j,1} - \frac{1}{6\sqrt{6}}L\sum_{\langle ijk\rangle}(M_{i,2}M_{j,2}M_{k,2} - M_{i,2}M_{j,1}M_{k,1} - M_{i,1}M_{j,2}M_{k,1} - M_{i,1}M_{j,1}M_{k,2}) + kT/3\sum_{i=1}^{N}[(\sqrt{6}M_{i,2}+1)\times$$

$$\times\ln(\sqrt{6}M_{i,2}+1)+\left(-\frac{3\sqrt{2}}{2}M_{i,1}-\frac{\sqrt{6}}{2}M_{i,2}+1\right)\ln\left(-\frac{3\sqrt{2}}{2}M_{i,1}-\frac{\sqrt{6}}{2}M_{i,2}+1\right)+$$

$$+\left(\frac{3\sqrt{2}}{2}M_{i,1}-\frac{\sqrt{6}}{2}M_{i,2}+1\right)\ln\left(\frac{3\sqrt{2}}{2}M_{i,1}-\frac{\sqrt{6}}{2}M_{i,2}+1\right) - NkT\ln 3\,,$$

$$\vec{M} = (M_{1,1}, M_{1,2},\ldots,M_{i,1},M_{i,2},\ldots,M_{N,2})\,. \qquad (26)$$

In this equation $M_{i,1}$ and $M_{i,2}$ are classical variables that give the thermodynamical mean value of the operators $\hat{M}_i^1$, $\hat{M}_i^2$. They could be viewed as Cartesian coordinates of the states of the i-th "classical particle", which is in fact by construction, a whole subsystem of particles /7/. The function $\Phi(\vec{M})$ is defined into a subset of the space of all variables $\vec{M}$. Its domain resticted to the subspace $(M_{i,1}, M_{i,2})$ forms an equilateral triangle and is given on figure 3 (compare with Eq.(24)). Equation (26) implies that $\Phi(\vec{M})=\Phi(T;K,L,\vec{M})$ is invariant under transformations in the space $\vec{M}$, performed by the group $S_3 \times G$, a result that is a consequence of the invariance of $\hat{H}$ /7/. When M=0 (cf. Eq(15)) the symmetry of $\Phi(\vec{M})$ rises to $G_H^{M=0}=W^{M=0}Ⓢ G$.

Now the main degree of freedom which depends on both K and L is to be found. If we follow the standard procedure of Landau's approximation[15] and take interest only in the second degree terms in $\Phi(\vec{M})$ then we shall conclude that there are only two subspaces in M which are connected with phase transitions. The first of these is $(M_{\Gamma,1}, M_{\Gamma,2})$,

$$M_{\Gamma,l} = \frac{1}{N} \sum_{i=1}^{N} M_{i,l} \;, l=1,2 \;. \tag{27}$$

This subspace is "the main one" when $K + \frac{2}{3}L > 0$. The variables $M_{\Gamma,l}$ transform according the two-dimensional irreducible representation (irrep) of $S_3 \times G$ − $E \times \Gamma 1$, where E is the two dimensional irrep of S_3. The corresponding low symmetry phase is found by minimization and is a "ferromagnetic" one – all "classical particles" simultaneously tend to one of the Potts states, say 1. The invariance group of this phase is ,

$$G_1^F = (p_1, G) \cup (p_5, G) \cong C_2 \times G \;,\; G_1^F \subset G_H \;. \tag{28}$$

The decomposition of G_H with respect to G_1^F is ,

$$G_H = G_1^F \cup (p_2|g_1)\, G_1^F \cup (p_2|g_1)\, G_1^F \;,\; g_1 = (E|\vec{0}) \;, \tag{29}$$

and therefore there are three domains.

The second subspace is $(M_{\vec{K},1}, M_{\vec{K},2}, M_{\vec{K}',1}, M_{\vec{K}',2})$,

$$M_{\vec{K},l} = \frac{1}{N} \sum_{i=1}^{N} \exp(-2\pi i\vec{K})\, M_{i,l} \;,\; l=1,2 \;,$$

$$M_{\vec{K}',l} = \frac{1}{N} \sum_{i=1}^{N} \exp(-2\pi i\vec{K}')\, M_{i,l} \;,\; l=1,2 \;,$$

$$\vec{K} = (\tfrac{1}{3}, \tfrac{1}{3}, 0) \quad \text{(cf. Ref.18)}$$

$$\vec{K}' = 6^{+}(\tfrac{1}{3}, \tfrac{1}{3}, 0) = (-\tfrac{1}{3}, \tfrac{2}{3}, 0) \;. \tag{30}$$

These variables are the principal ones when $K + \frac{2}{3}L < 0$, and transform according the four dimensional irrep $E \times K1$ of $S_3 \times G$. Now the low symmetry phase turns to be an "antiferromagnetic" one which means that each three nearest "classical particles" tend to three different Potts states[1]. This would mean that most of the particles on sublattice A are for example in state 1, the ones on B – in state 2, and those on C – in state 3. The invariance group of this phase is,

$$G_{123}^{AF} = G_{ABC} \cup (p_4|g_0)G_{ABC} \cup (p_2|g_2)G_{ABC} \cup (p_5|g_2g_0)G_{ABC}$$

$$\cup\, (p_3|g_3)G_{ABC} \cup (p_6|g_3g_0)G_{ABC} \;,\; G_{123}^{AF} \subset G_H \;. \tag{31}$$

The decomposition of G_H with respect to G_{123}^{AF} is,

$$G_H = \bigcup_{i=1}^{6} (p_i|g_1)G_{123}^{AF} \;, \tag{32}$$

which means that there are six antiferromagnetic states.

When M=0 $(E \times \Gamma 1) \oplus (E \times K1)$ is a six dimensional irrep of $G_H^{M=0}$ [1], while it's a reducible representation of $S_3 \times G$. This situation corresponds to Landau's case of phase transition with order parameter belonging to a reducible representation in a "point" of the phase diagram[15]. Now the ferromagnetic and the antiferromagnetic phases are with the same energy and occur as domains of the same phase. Indeed the subgroup of $G_H^{M=0}$, that preserves the antiferromagnetic phase discussed above is,

$$\begin{aligned} G_{123}^{M=0} = \{ & (p_1,p_1,p_1|G_{ABC}) \;, && (p_5,p_6,p_4|G_{ABC}), \\ & (p_2,p_2,p_2|g_2G_{ABC}) \;, && (p_4,p_5,p_6|g_2G_{ABC}), \\ & (p_3,p_3,p_3|g_3G_{ABC}) \;, && (p_6,p_4,p_5|g_3G_{ABC}), \\ & (p_4,p_4,p_4|g_0G_{ABC}) \;, && (p_1,p_2,p_3|g_0G_{ABC}), \\ & (p_5,p_5,p_5|g_2g_0G_{ABC}), && (p_2,p_3,p_1|g_2g_0G_{ABC}), \\ & (p_6,p_6,p_6|g_3g_0G_{ABC}), && (p_3,p_1,p_2|g_3g_0G_{ABC}) \} \;. \end{aligned} \tag{33}$$

The first column is isomorphic to G and therefore,

$$G_{123}^{M=0} \cong G \cup bG = G \,s\, C_2 \;,\; b \longleftrightarrow (p_5,p_6,p_4|g_1) \;. \tag{34}$$

The decomposition of $G_H^{M=0}$ with respect to $G_{123}^{M=0}$ is,

$$G_H^{M=0} = \bigcup_{i=1}^{9} \bar{g}_i G_{123}^{M=0} \;,\; \bar{g}_i = (p_i,p_i,p_i|g_1) \;,\; i=1..6$$

$$\bar{g}_7 = (p_1,p_3,p_2|g_1),\; \bar{g}_8 = (p_2,p_1,p_3|g_1),\; \bar{g}_9 = (p_3,p_2,p_1|g_1) \;. \tag{35}$$

Obviously the coset representatives $\bar{g}_i$, i=1,..,6, generate the six antiferromagnetic domains, while the remaining ones generate the three ferromagnetic domains.

All these considerations were made, as we pointed out, when it was taken for granted that the second degree terms are the leading ones. However when $|K + \frac{2}{3}L| \ll |L|$ this is not the case and one

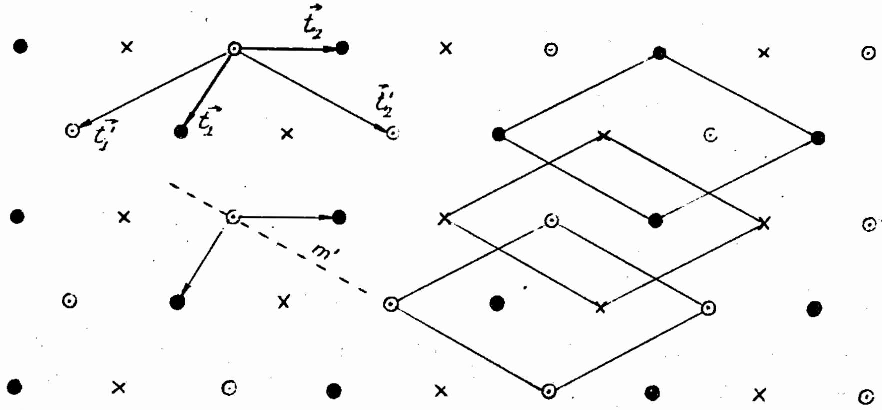

figure 1

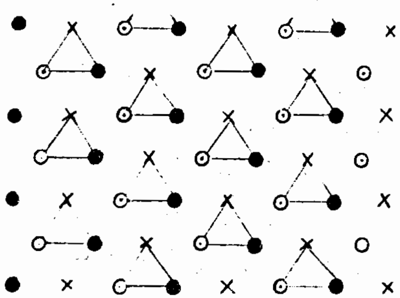

figure 2

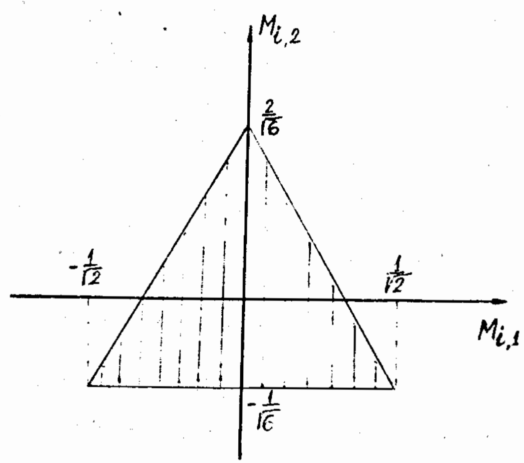

figure 3

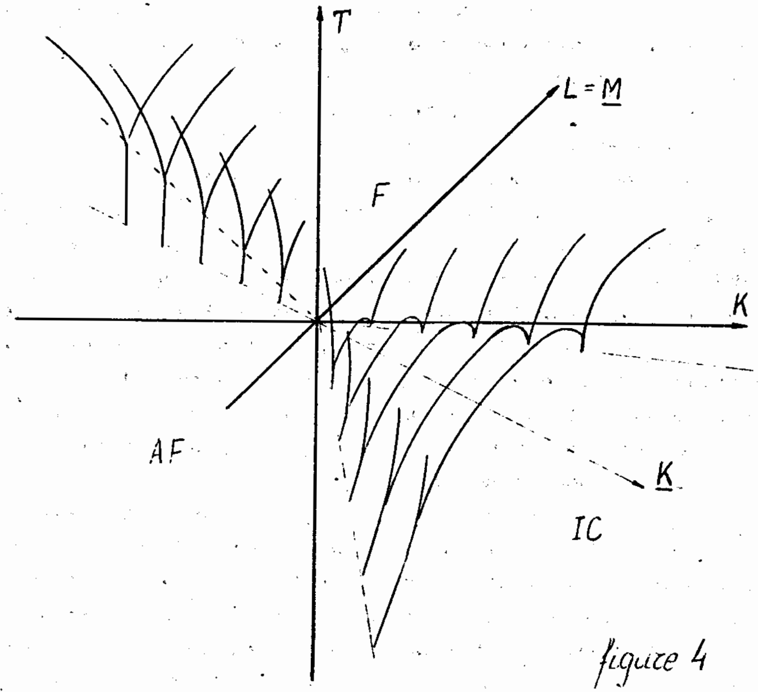

figure 4

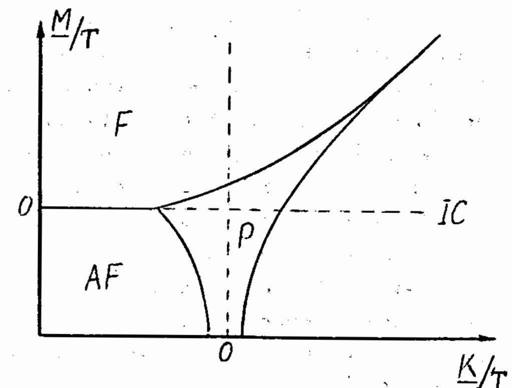

figure 5

F - ferromagnetic phase
AF - antiferromagnetic phase
IC - incommensurate phase
P - paramagnetic phase

should take into account the third degree terms. A careful analysis shows that the results that we reached are valid when $K + \frac{2}{3}L = 0$ and $L > 0$. In the opposite direction in the (K,L) space: $K + \frac{2}{3}L = 0$, $L < 0$ and in the region around it, however, neither the ferromagnetic nor the antiferromagnetic phases are stable. In other words the absolute minimum is no longer in the subspaces $(M_{\Gamma,1})$, or $(M_{\vec{K},1}, M_{\vec{K}',1}), l=1,2$. It flows away to different degrees of freedom, passing from one to another. Definitely in this situation a series of incommensurate phase transitions occurs, which is a new feature we notice on the phase digram. The latter is given on figure 4. All the phase transitions seem to be of first order, at least on this level of investigation. This is due to the presence of third order terms in $\Phi(\vec{M})$. It is quite possible however, that if one takes into account the critical fluctuations, then the transitions could turn to be second order ones as found in Ref.1.

To compare the results of Ref.1 and the present paper we reproduce figure 4 in a new form on figure 5. Already in figure 4 we used a second set of variables $\underline{K} = K$ and $\underline{M} = M$ instead of K and L (cf. Eq.(15)). On figure 5 a projective space, obtained from the space $(T,\underline{K},\underline{M})$ is plotted. In other words one point on the latter figure $(\underline{K}/T,\underline{M}/T)$ corresponds to a whole line $(\lambda T,\lambda\underline{K},\lambda\underline{M})$ on figure 4. The region IC of incommensurate phase transitions is not given in Ref.1.

Finally we briefly review the results obtained using the standard phenomenological Landau theory. As we already mentioned they crucially depend on the symmetry of the free energy and the transformation properties of the order parameter. Using the natural transformation properties of $\vec{M}$ we conclude that the only commensurate phases in this problem could be related to the representations $E\times\Gamma1$, $E\times K1$ and $E\times M1$ of $S_3\times G$. (The definition of the representations $\Gamma1$, $K1$ and $M1$ of G are given in Ref.18.) A lattice of subgroups - possible low symmetry phases, could also be obtained and naturally G_1^F and G_{123}^{AF} are two of these. The transitions should be first order ones as the symmetry does not exclude third order invariants.

In the standard modification of Landau's theory we have to find what are the tensorial transformation properties of the order parameter with respect to G. Virtually the same results as those just quotated are obtained if we chose the representation $\Gamma5$ of G for this purpose. The commensurate phases would then be related to the representations $\Gamma5 \times \Gamma1$, $\Gamma5 \times K1$, $\Gamma5 \times M1$ of G. To find these "natural" transformation properties of the order parameter, however and the determination of $\Gamma5$ in particular is to greater extent a guess than a result of a well defined procedure.

Thus we see that our results agree with the phenomenological theory but are less general and instead of a variety of possibilities consist in particular facts. This concretization is not strange having in mind that we used much more initial information (cf. Eq.(26)), than only the covariance properties of $\vec{M}$.

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"Трехцветная" модель Поттса с цветной группой симметрии

Показано, что симметрия гамильтониана трехцветной модели Поттса описывается нетривиальными цветными группами W-типа и P-типа /подгруппы сплетения симметрической группы S_3 с кристаллографической группой симметрии C^1_{6v}/. В приближении неоднородного среднего поля построены соответствующие фазовые диаграммы. Найдена новая серия несоизмеримых фазовых переходов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1990

Kotzev J.N., Peev M.K. E17-90-386

Three State Potts Model with Colour Group Symmetry

Nontrivial colour groups of W-type and P-type (subgroups of the writh product of S_3 and C^1_{6v}) are found to requisite for describing the invariance of a three state Potts model. The corresponding phase diagram is constructed using the inhomogeneous mean field approximation. A series of incommensurate phase transitions is newly found.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1990