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EXACT SOLUTION OF THE NEW ONE-DIMENSIONAL FERMION SYSTEM ON A LATTICE

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1. INTRODUCTION

In the very rich branch of theoretical physics concerned with the investigation of various exactly solvable models the problems of interacting many-fermion systems are both as of the most interest and of the most source of difficulties [1-8]. Historically these problems were first posed by Thirring [9] and Schwinger [10] in relativistic field theory of massless fermions in one space dimension. The nontrivial interaction was taken to be bilinear in the densities having essentially bosonic properties. After the proper modification of the vacuum and transition to the particle-hole picture these models were reduced to a free-boson problem, but the description of global properties of the system in all charge sectors was not trivial because of the difficulties inherent in local quantum field theory. By the correct procedure of bosonization the more general Luttinger model [11] was also diagonalized. In each charge sector the excitations were represented as collective fluctuations of the bosonic nature [12]. The origin of solvability of the massive generalization of all these models also lies in the possibility of bosonization [13] and reduction to the well-known integrable sine-Gordon model [14].

The progress in the investigation of solvable nonrelativistic fermion systems was also almost complete based on the results obtained earlier in the boson models. For example, the eigenvalue problem for a finite number of fermions interacting via a delta-function potential was solved by McGuire [15], Gaudin [16], Yang [3] and Takahashi [17] only after the complete investigation of the analogous bosonic system by the Bethe ansatz [18]. The description of the latter in the second quantization scheme can be reduced to the problem of solution of the quantum nonlinear Schrödinger equation directly connected with the general quantum inverse scattering method [19]. For the corresponding fermion problem such a connection seems to be much less transparent [20].

For the purposes of solid-state physics the most important problem of that kind is the search for nontrivial solvable models of interacting electrons in the periodic field of an ionic lattice. In the first quantization picture one must consider the usual nonrelativistic many-particle Hamiltonian

$$H^{(1)} = \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} + W(x_j) \right] + \sum_{j>k}^{N} V(x_j - x_k), \quad W(x_j + a) = W(x_j). \qquad (1)$$

The fermionic nature of interacting objects lies in the symmetry properties of $H^{(1)}$ eigenvectors. It would be natural to conclude that the search for solvable models must be based on the investigation of the internal symmetries of (1) for various potentials W and V. However, in this way, one finds only very strange, highly singular potentials having no clear physical interpretation. If W and V are not reduced to singularities at some isolated points, the single known example of integrable models of the type (1) is the Sutherland [21] system ($V(x) = \lambda \sin^{-2}\omega x$) in the three-parametric periodic external field $W(x) = A_1 \cos(2\omega x + \delta) + A_2 \cos 4\omega x$ [22]. The transition to the second quantization by the standard procedure is questionable in that case because of the strong V singularity.

On the other hand, the use of the second-quantized Hamiltonians historically was more fruitful. Stimulated by the famous Lieb - Wu solution [23] of the one-band Hubbard model [24] based on the slight modification of the Bethe ansatz, a number of fundamental studies of one-dimensional fermionic models have been performed [6,7,25-30]. The Lieb - Wu results were also extended to the more preferable cases of arbitrary electton density [29] and two degenerated bands [31].

All extensions of that type of the Hubbard model contain a typical shortage. The hopping term describing the process of electron transition from one localized state to another is supposed, except for the trivial case of constant infinite-range hopping [32,33], to be restricted to nearest-neighbour states. The approximations corresponding to this regime become much more clear after the inverse transition from the original Hubbard Hamiltonian to the first-quantized model of type (1). One finds that the nearest-neighbour hopping may be obtained only if W(x) in (1) is chosen as an attractive Cronig - Penney potential $W = -A \sum_n \delta(x - an)$ in the limit of the infinite strength A of delta functions. The interaction in the Hubbard model corresponds to the McGuire - Yang pair delta potential with a strength inversely proportional to A (cf. [34]). So the first quantized description of the Hubbard Hamiltonian is highly singular and seems to be far from the real physical situation.

The aim of the present paper is the investigation of the possibilities of constructing the second-quantized solvable models with a more general type of nontrivial hopping than the nearest-neighbour one ("overjumping" model). The full lattice fermionic Hamiltonian can be obtained from (1) by introducing the second-quantized operators on the base of orthonormal Wannier functions. In the one-band approximation that reads

$$H^{(2)} = \sum_{i,j,\sigma} t_{ij} a^+_{i\sigma} a_{j\sigma} + \sum_{\substack{i,j,k,\ell \\ i,j,k,\ell \in \mathbf{Z}}} \sum_{\sigma,\sigma'} \Phi^{\sigma\sigma'}_{ijk\ell} a^+_{i\sigma} a^+_{j\sigma'} a_{k\sigma'} a_{\ell\sigma}, \qquad (2)$$

here $a_{j\sigma}$ are the annihilation operators for the electrons of spin projection σ in the Wannier state at site j, $\{a_{j\sigma}, a^+_{k\sigma'}\} = \delta_{jk}\delta_{\sigma\sigma'}$, t_{ij} is the real symmetric hopping tensor. The structure of $\Phi^{\sigma\sigma'}_{ijk\ell}$ is specified as follows:

$$\Phi^{\sigma\sigma'}_{ijk\ell} = \frac{U}{2}\delta_{\sigma,-\sigma'}\delta_{jk}\delta_{i\ell}\delta_{ij} + \frac{(1-\delta_{ij})}{2}(V_{ij}\delta_{jk}\delta_{i\ell} + I_{ij}\delta_{ik}\delta_{j\ell}). \qquad (3)$$

The first term in (3) coincides with the usual Hubbard one and the two others correspond to direct and exchange intersite interactions. The main question posed here consists in the following: is it possible to find any combination of three second-rank symmetric tensors t, V, I such that the model becomes solvable?

In this paper we present only the partial solution of that problem. There are at least two motivations which forced us to present this stage of investigation. First, we believe that it will be stimulating for searching the general solution since nobody (to our knowledge) has been able to propose any systematic method of comstructing solvable fermionic models with nontrivial non-nearest-neighbour hopping. Even for the standard one-dimensional Hubbard model the relatively complicated extra integrals of motion and connection to Lax and Yang - Baxter regular procedures have been established only very recently [35-37]. Second, mathematical problems of constructing the exact solutions in the nonlocal case seem to require a much more sophisticated technique for fermionic systems. Even for bosons (for example magnons in quasi-Heisenberg spin chains) this problem is still open. These reasons inspire us to describe here the nontrivial exact eigenvectors of the extended version (2-3) of the Hubbard Hamiltonian. Various cases of the extended Hubbard model have been of great interest in recent years in describing quasi-one-dimensional conductors [37-39]. In view of all the difficulties and shortcomings of various approximate solutions of the Hubbard model and its extended versions any rigorous information is of particular importance. The complementary study of the exact and approximate solutions[40] of those models certainly can provide an insight into the real nature of the relevant general solution.

The paper is organized as follows. In Sec. II we derive the algebraic equations for states of two interacting electrons. In Secs. III and IV we find the solutions to these equations for some tensors t, V, I with the use of the Weierstrass theory of elliptic functions. The last Section is devoted to the discussion of some hypotheses and ways of further development.

II. THE EQUATIONS FOR TWO-ELECTRON STATES ON AN INFINITE LATTICE

Let us write the basic Hamiltonian (2-3) in a more convenient form,

$$H^{(2)} = h_1 + h_2 + h_3 ,$$

$$h_1 = \sum_{i,j} t_{ij} (a^+_{i\uparrow} a_{j\uparrow} + a^+_{i\downarrow} a_{j\downarrow}) ,$$

$$h_2 = U \sum_i N_{i\uparrow} N_i + \frac{1}{2} \sum_{i \neq j} V_{ij} (N_{i\uparrow}+N_{i\downarrow})(N_{j\uparrow}+N_{j\downarrow}) , \qquad (4)$$

$$h_3 = -\frac{1}{2} \sum_{i \neq j} I_{ij} [N_{i\uparrow} N_{j\uparrow} + N_{i\downarrow} N_{j\downarrow} + a^+_{i\uparrow} a_{i\downarrow} a^+_{j\downarrow} a_{j\uparrow} + a^+_{i\downarrow} a_{i\uparrow} a^+_{j\uparrow} a_{j\downarrow}],$$

where $\uparrow$ and $\downarrow$ denote up and down electron spins, $N_{j\uparrow} = a^+_{j\uparrow} a_{j\uparrow}$. We shall consider in detail the case an infinite lattice, i.e. $-\infty << i, j < \infty$. The modifications to periodic boundary conditions will be discussed in the last section. We shall also suppose that all three tensors t, V, I depend on their indices only through the distance between the sites $|i-j|$.

The simplest eigenvector of $H^{(2)}$ is the pure vacuum $|0\rangle$ having zero eigenvalue (such a vacuum state without any particles is taken for convenience):

$$a_{j\uparrow} |0\rangle = a_{j\downarrow} |0\rangle = 0 \quad \text{for all} \quad j \in \mathbf{Z} . \qquad (5)$$

It is evident that in this case one-electron states are described by quasi-free or plane-wave-like states with quasimomenta $p \in (0, 2\pi)$:

$$\psi_{p\uparrow} = \sum_k \exp(ikp) a^+_{k\uparrow} |0\rangle, \quad \psi_{p\downarrow} = \sum_k \exp(ikp) a^+_{k\downarrow} |0\rangle .$$

The dependence of the eigenvalues on p is completely determined by the Fourier transform of the hopping tensor

$$\epsilon_p = \sum_k t_{j,j+k} \exp(ikp). \qquad (6)$$

The eigenvalue problem for two-electron states is much less trivial. Let us introduce the relevant "elementary" states

$$\psi^{\uparrow\uparrow}_{\ell m} = a^+_{\ell\uparrow} a^+_{m\uparrow} |0\rangle, \quad \psi^{\downarrow\downarrow}_{\ell m} = a^+_{\ell\downarrow} a^+_{m\downarrow} |0\rangle, \quad \psi^{\uparrow\downarrow}_{\ell m} = a^+_{\ell\uparrow} a^+_{m\downarrow} |0\rangle .$$

By this definition we immediately find

$$h_1 \psi^{\uparrow\uparrow}_{\ell m} = \sum_j (t_{j\ell} \psi^{\uparrow\uparrow}_{jm} - t_{jm} \psi^{\uparrow\uparrow}_{j\ell}), \quad h_1 \psi^{\uparrow\downarrow}_{\ell m} = \sum_j (t_{j\ell} \psi^{\uparrow\downarrow}_{jm} + t_{jm} \psi^{\uparrow\downarrow}_{\ell j}),$$

$$h_2 \psi^{\uparrow\uparrow}_{\ell m} = V_{\ell m} \psi^{\uparrow\uparrow}_{\ell m}, \quad h_2 \psi^{\uparrow\downarrow}_{\ell m} = (U\delta_{\ell m} + V_{\ell m}) \psi^{\uparrow\downarrow}_{\ell m}, \qquad (7)$$

$$h_3 \psi^{\uparrow\uparrow}_{\ell m} = -I_{\ell m} \psi^{\uparrow\uparrow}_{\ell m}, \quad h_3 \psi^{\uparrow\downarrow}_{\ell m} = I_{\ell m} \psi^{\uparrow\downarrow}_{m\ell} .$$

Let us now construct two-electron eigenvectors as follows

$$\psi_{\uparrow\uparrow} = \sum_{\ell, m} Z^{(a)}_{\ell m} \psi^{\uparrow\uparrow}_{\ell m}, \quad \psi_{\downarrow\downarrow} = \sum_{\ell, m} Z^{(a)}_{\ell m} \psi^{\downarrow\downarrow}_{\ell m}, \quad \psi_{\uparrow\downarrow} = \sum_{\ell, m} \tilde{Z}_{\ell m} \psi^{\uparrow\downarrow}_{\ell m}, \qquad (8)$$

where the tensors $Z^{(a)}$ and $\tilde{Z}$ must be determined from the eigenvalue equation $H^{(2)} \psi = \epsilon \psi$. Note that according to the Pauli principle $Z^{(a)}$ is antisymmetric while $\tilde{Z}$ has no definite symmetry properties.

By using (7) and (8) one can obtain the equations for $Z^{(a)}$ and $\tilde{Z}$ in the explicit form:

$$\sum_{j=-\infty}^{\infty} (t_{j\ell} Z^{(a)}_{jm} - t_{jm} Z^{(a)}_{j\ell}) + (V_{\ell m} - I_{\ell m} - \epsilon) Z^{(a)}_{\ell m} = 0 , \qquad (9)$$

$$\sum_{j=-\infty}^{\infty} (t_{j\ell} \tilde{Z}_{jm} + t_{jm} \tilde{Z}_{\ell j}) + (V_{\ell m} + U\delta_{\ell m} - \epsilon) \tilde{Z}_{\ell m} + I_{\ell m} \tilde{Z}_{m\ell} = 0. \qquad (10)$$

The tensor $\tilde{Z}_{\ell m}$ in (10) can always be represented as a sum of its symmetric and antisymmetric components,

$$\tilde Z_{\ell m} = \tilde Z^{(a)}_{\ell m} + \tilde Z^{(s)}_{\ell m}, \quad \tilde Z^{(s,a)}_{\ell m} = \frac{1}{2}(\tilde Z_{\ell m} \pm \tilde Z_{m\ell})$$

each of them must obey (10). It is easy to see that the resulting equation for $\tilde Z^{(a)}_{\ell m}$ coincides with (9) and for the symmetric $\tilde Z^{(s)}_{\ell m}$ (10) reduced to

$$\sum_{j=-\infty}^{\infty} (t_{j\ell}\, \tilde Z^{(s)}_{jm} + t_{jm}\, \tilde Z^{(s)}_{j\ell}) + (U\delta_{\ell m} + V_{\ell m} + I_{\ell m} - \epsilon)\, \tilde Z^{(s)}_{\ell m} = 0. \qquad (11)$$

If the U term is here omitted, (11) is just the equation for two-magnon wave functions in a quasi-Heisenberg XXZ ferromagnet with the Hamiltonian

$$\tilde H = \sum_{j,k=-\infty}^{\infty} [\, t_{jk} (\sigma^x_j \sigma^x_k + \sigma^y_j \sigma^y_k) + \frac{1}{2}(V_{jk} + I_{jk})\, \sigma^z_j \sigma^z_k \,]. \qquad (12)$$

So even in the general case of non-nearest-neighbour hopping we obtain a certain correspondence between fermionic and bosonic models despite the absence of the Jordan - Wigner type diffeomorphism. This corresponden may serve as the key to the choice of the hopping tensor: it must be taken from some integrable spin model of the type (12). Recently possible candidates for this role have been proposed [41-43]. In the remaining part of this paper we shall consider the most general case [43]

$$t_{jk} = \begin{cases} 0, & j = k \\ t\, \dfrac{\pi^2}{\kappa^2} [\sinh \dfrac{\pi}{\kappa}(j-k)]^{-2}, & j \neq k \end{cases} \qquad \begin{matrix} \mathrm{Im}\,\kappa = 0, \\ \mathrm{Re}\,\kappa > 0 \end{matrix} \qquad (13)$$

and investigate the solutions of "bosonic" eq. (10) and eq.(9) of the "pure fermionic" nature.

III. SYMMETRIC SOLUTIONS

Let us start with the calculation of the energy of one-electron state (6) with the hopping (13) briefly discussed also in [43].

The closed form of the trigonometric sum

$$\epsilon(p) = t_0 \sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2} \exp(ikp)\, [\, \sinh \frac{\pi}{\kappa} k]^{-2}, \quad |\mathrm{Im}\, p| < \frac{2\pi}{\kappa} \qquad (14)$$

may be obtained by the methods of Weierstrass elliptic function theory. Let us construct the function of the complex variable x,

$$F(x) = t_0 \sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2} \exp(ikp)\, [\, \sinh \frac{\pi}{\kappa}(k+x)]^{-2}, \qquad (15)$$

which has the following quasiperiodicity properties:

$$F(x+1) = \exp(-ip)\, F(x), \quad F(x+\omega) = F(x), \quad \omega = i\kappa. \qquad (16)$$

This function has a pole singularity at the point $x = 0$. The expansion of F in the vicinity of this point contains the term with $\epsilon(p)$,

$$F(x) \simeq t_0 \left(\frac{1}{x^2} - \frac{\pi^2}{3\kappa^2}\right) + \epsilon(p) + O(x). \qquad (17)$$

Note that it is the only singularity of $F(x)$ on the torus $\mathbf{C}/\Gamma$ obtained by factorization of the complex plane $\mathbf{C}$ by the lattice of quasiperiods $\Gamma = m_1 + m_2\,\omega$, $m_1, m_2 \in \mathbf{Z}$. Now we construct the function $G(x)$ with the same quasiperiodicity (16) and singular term as in (17) by using Weierstrass functions $\wp(x)$, $\zeta(x)$, $\sigma(x)$ defined on the same torus $\mathbf{C}/\Gamma$:

$$G(x) = -A \frac{\sigma(x+r)}{\sigma(x-r)} \exp(\delta x)\, [\wp(x) - \wp(r) + \Delta(\zeta(x+r) - \zeta(x) - \zeta(2r) + \zeta(r))].$$

The constants A, r, δ, Δ are determined from the relations

$$\delta + \zeta(\tfrac{1}{2})\, r = -ip, \quad \delta\omega + \zeta(\tfrac{\omega}{2})\, r = 0, \quad A = t_0, \quad A\Delta = (2\zeta(r) + \delta)\cdot t_0,$$

which can be simply obtained by using the standard properties of Weierstrass functions [44]. The result is

$$A = t_0, \quad r = -\frac{p\omega}{\pi}, \quad \delta = \frac{p}{\pi}\zeta(\frac{\omega}{2}), \quad \Delta = \pi^{-1}[\, p\,\zeta(\frac{\omega}{2}) - 2\zeta(\frac{p\omega}{4\pi})]. \qquad (18)$$

The difference $R(x) = F(x) - G(x)$ is, by construction, an analytic function of x without any singularities in the full complex plane $\mathbf{C}$. According to the Liouville theorem [44] $R(x)$ would be equal to a constant and due to the relation $R(x+1) = \exp(-ip) R(x)$ this constant must be exactly zero. By comparing the terms independent of x in the decompositions of $F(x)$ and $G(x)$ near $x = 0$, we finally have

$$\epsilon(p) = t_0 \left[\frac{\pi^2}{3\kappa^2} - \wp(r) - (2\zeta(r) + \delta)(\zeta(2r) + \delta) - \frac{(2\zeta(r) + \delta)^2}{2} \right] \tag{19}$$

with r and δ given by (18).

The next step is the calculation of a slightly more complicated sum

$$S(p, \ell) = \sum_{\substack{k=-\infty \\ k \neq 0, -\ell}}^{\infty} \frac{\pi^2}{\kappa^2} \exp(ikp) \left[\sinh \frac{\pi}{\kappa} k \right]^{-2} \coth \frac{\pi}{\kappa}(k + \ell), \quad \ell \in \mathbf{Z} \tag{20}$$

The appropriate function which may be constructed by Weierstrass theory is

$$F_1(x) = \sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2} \exp(ikp) \left[\sinh \frac{\pi}{\kappa}(k + x) \right]^{-2} \coth \frac{\pi}{\kappa}(k + \ell + x)$$

with the quasiperiodicity relations (16) and pole singularity like (17). Arguments of the type used above lead to the following closed form of (20):

$$S(p, \ell) = \frac{\pi}{\kappa} \coth \frac{\pi\ell}{\kappa} \left[\epsilon(p) - \frac{\pi^2}{\kappa^2 \sinh^2 \frac{\pi\ell}{\kappa}} (1 + 2\exp(-ip\ell)) \right] + \frac{\pi^2}{\kappa^2 \sinh^2 \frac{\pi\ell}{\kappa}} (1 - \exp(-ip\ell)) f(p), \tag{21}$$

where

$$f(p) = \frac{p}{\pi} \zeta\left(\frac{\omega}{2}\right) - \zeta\left(\frac{p\omega}{2\pi}\right). \tag{22}$$

Now we are ready to construct the solution of eq.(11) as follows

$$Z^{(s)}_{\ell m} = (1 - \delta_{\ell m}) \{ \coth\gamma [\exp(i(p_1\ell + p_2 m)) + \exp(i(p_2\ell + p_1 m))] + [\exp(i(p_1\ell + p_2 m)) - \exp(i(p_2\ell + p_1 m))] \coth \frac{\pi}{\kappa}(\ell - m) \}. \tag{23}$$

The second term describes the distortion of the states of two free-propagating electrons due to interaction between them. By substituting (23) into (11) and using (19) and (21) it is easy to find that (11) is satisfied for all quasimomenta p_a, $|\mathrm{Im}\, p_a| < \frac{2\pi}{\kappa}$, $a = 1,2$, if

$$U = 0, \quad V_{\ell m} + I_{\ell m} = t \frac{2\pi^2}{\kappa^2} \left[\sinh \frac{\pi}{\kappa}(\ell - m) \right]^{-2}, \tag{24}$$

$$\coth\gamma = \frac{\kappa}{2\pi} [f(p_1) - f(p_2)]. \tag{25}$$

The eigenvalue which corresponds to the solution (23) is

$$\epsilon(p_1, p_2) = \epsilon(p_1) + \epsilon(p_2). \tag{26}$$

The tape of potentials (24) which is obtained above leads to the constraint that any site of the lattice according to (23) cannot be doubly occupied. The case of real p_1, p_2 corresponds to the scattering states. Let us show that the bound states are also possible for the complex quasimomenta (cf. [45] for Hubbard case). If $|\mathrm{Im}\, p_a| > 0$, the only situations in which $Z^{(s)}_{\ell m}$ vanishes as $|\ell - m| \to \infty$ are $|\coth\gamma| = 1$. Taking for definiteness $p_{1,2} = \frac{P}{2} \pm iQ$, $\mathrm{Im}\, P = 0$, $\mathrm{Re}\, Q > 0$ we have from (25) the equation

$$\frac{\kappa}{2\pi} \left[f\left(\frac{P}{2} + iQ\right) - f\left(\frac{P}{2} - iQ\right) \right] = 1.$$

Let $\frac{P\omega}{4\pi} = i\lambda$, $-\frac{iQ\omega}{2\pi} = \tau$ where λ is real and $0 < \mathrm{Re}\,\tau < 1$. The above equation reduces to

$$\phi_\lambda(\tau) = \frac{4\tau}{\omega} \zeta\left(\frac{\omega}{2}\right) + \frac{2\pi}{\kappa} + \zeta(i\lambda - \tau) - \zeta(i\lambda + \tau) = 0. \tag{27}$$

At fixed $\lambda \neq 0$ the function $\phi_\lambda(r)$ has the following simple properties:

$$\phi_\lambda(0) = \frac{2\pi}{\kappa}, \quad \phi_\lambda(1) = -\frac{2\pi}{\kappa}, \quad \phi_\lambda(\tfrac{1}{2}) = 0, \quad \frac{d\phi_\lambda}{dr}(0) < 0.$$

The zero at $r = 1/2$ is trivial because after its substitution into (23) $Z^{(s)}_{\ell m}$ vanishes. The nontrivial zero exists if and only if the derivative of $\phi_\lambda(r)$ on the interval (0,1/2) changes the sigh, i.e.

$$\frac{d\phi_\lambda}{dr}\Big|_{r=1/2} = \frac{4}{\omega}\zeta(\frac{\omega}{2}) + \wp(i\lambda - \frac{1}{2}) + \wp(i\lambda + \frac{1}{2}) = \tag{28}$$

$$= \frac{\pi^2}{\kappa^2}\sum_{n=-\infty}[\sinh^{-2}\frac{\pi}{\kappa}(i\lambda + n - \frac{1}{2}) + \sinh^{-2}\frac{\pi}{\kappa}(i\lambda + n + \frac{1}{2})] > 0,$$

The analysis of the behaviour of this sum as a function of λ shows that at least for $\cot\frac{\pi\lambda}{\kappa} > \coth\frac{\pi}{2\kappa}$ the inequality (28) holds. So from (27) we conclude that for some values of real Q there are real total quasimomenta P defined by the formula

$$\wp(\frac{P\omega}{4\pi}) = \wp(\frac{iQ\omega}{2\pi}) + \wp'(\frac{-iQ\omega}{2\pi})[2\zeta(\frac{-iQ\omega}{2\pi}) - \frac{2\pi}{\kappa} + \frac{2iQ}{\pi}\zeta(\frac{\omega}{2})]^{-1}. \tag{29}$$

The possible case of complex r needs a more thorough consideration. It is likely that the bound states (29) have the smallest possible energy at the fixed total two-electron quasimomentum but the proof of this statement is now not complete.

IV. ANTISYMMETRIC SOLUTIONS

In the two previous Sections we have fixed the hopping tensor and the combination $V_{\ell m} + I_{\ell m}$ by (13) and (24) in the Hamiltonian (2-3). The remaining term $V_{\ell m} - I_{\ell m}$ can be specified by the requirement of the presence of exact solutions to the equation (9) for the antisymmetric tensor $Z^{(a)}_{\ell m}$. The simplest situation is evidently realized when $V_{\ell m} - I_{\ell m} \equiv 0$, i.e. the electrons with the same spin projection do not interact (note that just the same property is inherent in the Hubbard model). The solution of (9) in this case has the form of the antisymmetrized combination of one-particle solutions

$$Z^{(a)\,(I)}_{\ell m} = \exp[i(p_1\ell + p_2 m)] - \exp[i(p_2\ell + p_1 m)] \tag{30}$$

with the eigenvalue $\epsilon(p_1,p_2) = \epsilon(p_1) + \epsilon(p_2)$.

One can try to introduce a distortion due to nonzero interaction $V_{\ell m} - I_{\ell m}$ by adding to (30) the term analogous to (23),

$$Z^{(a)\,(II)}_{\ell m} \sim \{\exp[i(p_1\ell + p_2 m)] + \exp[i(p_2\ell + p_1 m)]\}\coth\frac{\pi}{\kappa}(\ell - m).$$

However, the use of the relations (19), (21) shows that (9) cannot be satisfied for any ($V_{\ell m} - I_{\ell m}$) and combinations of quasimomenta. One needs to construct the principally new ansatz to distort (30) by some interaction between electrons.

For that purpose let us consider the following trigonometric sum similar to (21),

$$\tilde{S}(p,\ell) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{\pi^2}{\kappa^2}\exp(ipk)[\sinh\frac{\pi}{\kappa}k]^{-2}\tanh\frac{\pi}{\kappa}(k+\ell) \tag{31}$$

and find for that an analytic expression through Weierstrass functions. As in the previous section, one can construct the function

$$F_\ell(x) = \sum_{k=-\infty}^{\infty}\frac{\pi^2}{\kappa^2}\exp(ipk)[\sinh\frac{\pi}{\kappa}(k+x)]^{-2}\tanh\frac{\pi}{\kappa}(k+\ell+x) \tag{32}$$

that is quasiperiodic as F(x) (16) but has on the torus $\mathbf{C}/\Gamma$ two pole singularities at the points $x = 0$ and $x = \omega/2$ generated by the terms with $k = 0$ and $k = -\ell$ in (32):

$$F_\ell(x) \simeq \tanh\frac{\pi\ell}{\kappa}(\frac{1}{x^2} + \frac{\pi}{x\kappa\cosh\frac{\pi\ell}{\kappa}\sinh\frac{\pi\ell}{\kappa}} - \frac{\pi^2}{\kappa}(\frac{1}{3} + \cosh^{-2}\frac{\pi\ell}{\kappa})) +$$

$$+ \tilde{S}(p,\ell) + O(x) \quad \text{near } x = 0, \tag{33}$$

$$F_\ell(x) \simeq -\frac{\pi}{\kappa}\exp(-ip\ell)[\cosh\frac{\pi\ell}{\kappa}]^{-2}(x - \frac{\omega}{\kappa})^{-1} \quad \text{near } x = \frac{\omega}{2}.$$

The proper combination $G_\ell(x)$ of Weierstrass functions having the same structure of singularities and quasiperiodicity contains five constants,

$$G_\ell(x) = -\frac{A\sigma(x+r)}{\sigma(x-r)}\exp(x\delta)[\wp(x) - \wp(r) + \Delta_1(\zeta(x+r) - \zeta(x) - \zeta(2r) +$$

$$+\zeta(r)) + \Delta_2(\zeta(x+r) - \zeta(x-\frac{\omega}{2}) - \zeta(2r) + \zeta(r-\frac{\omega}{2}))]$$

which must be determined from (16) and pole terms in (33),

$$A = \tanh\frac{\pi\ell}{\kappa}, \quad \Delta_1 = 2\zeta(r) + \delta - \frac{\pi}{\kappa}A^{-1}\cosh^{-2}(\frac{\pi\ell}{\kappa}),$$

$$\Delta_2 = -\frac{\pi}{\kappa}A^{-1}\cosh^{-2}(\frac{\pi\ell}{\kappa})\exp(-ip\ell)$$

and r and δ as in (18).

Comparing the first nonsingular terms in (33) and the decomposition of $G_\ell(x)$ near $x=0$ we have

$$\tilde{S}(p,\ell) = \tanh\frac{\pi\ell}{\kappa}[\frac{\pi^2}{\kappa^2\cosh^2(\frac{\pi\ell}{\kappa})} + \epsilon(p)] + \qquad (34)$$

$$+\frac{\pi}{\kappa}\cosh^{-2}\frac{\pi\ell}{\kappa}[f(p) - \exp(-ip\ell)g(p)],$$

where

$$f(p) = \frac{p}{\pi}\zeta(\frac{\omega}{2}) - \zeta(\frac{p\omega}{2\pi}),$$

$$g(p) = \zeta(r) + \zeta(\frac{\omega}{2}) - \zeta(2r) + \zeta(r-\frac{\omega}{2}).$$

Note that $f(p)$ and $g(p)$ are odd functions of their argument. Let us search for the solutions of eq. (9) in the form

$$\tilde{Z}^{(a)}_{\ell m} = \coth\gamma[\exp(i(p_1\ell + p_2 m)) - \exp(i(p_2\ell + p_1 m))] + \qquad (35)$$

$$+[\exp(i(p_1\ell + p_2 m)) + \exp(i(p_2\ell + p_1 m))]\tanh\frac{\pi}{\kappa}(\ell - m).$$

Calculating by (19) and (34) the infinite sums in the left-hand side we obtain

$$\sum_{j=-\infty}^{\infty}(t_{j\ell}\tilde{Z}^{(a)}_{jm} - t_{jm}\tilde{Z}^{(a)}_{j\ell}) + (V_{\ell m} - I_{\ell m} - \epsilon)\tilde{Z}^{(a)}_{\ell m} =$$

$$= (\epsilon(p_1) + \epsilon(p_2) - \epsilon + \frac{2\pi^2}{\kappa^2}\frac{\}t_0}{\cosh^2\frac{\pi}{\kappa}(\ell - m)} + V_{\ell m} - I_{\ell m})\tilde{Z}^{(a)}_{\ell m} +$$

$$+ t_0[\frac{\kappa}{2\pi}(f(p_1) - f(p_2)) - \coth\gamma]\frac{2\pi^2}{\kappa^2\cosh^2\frac{\pi}{\kappa}(\ell - m)}[\exp(i(p_1\ell + p_2 m)) -$$

$$- \exp(i(p_2\ell + p_1 m))] - \frac{\pi t_0}{\kappa\cosh^2\frac{\pi}{\kappa}(\ell - m)} \times \qquad (36)$$

$$\times[g(p_1) + g(p_2)][\exp(i(p_1 + p_2)m) - \exp(i(p_1 + p_2)\ell) = 0.$$

The first term in the right-hand side of (36) reduces to zero if

$$I_{\ell m} - V_{\ell m} = \frac{2\pi^2 t_0}{\kappa^2\cosh^2\frac{\pi}{\kappa}(\ell - m)}, \qquad (37)$$

$$\epsilon = \epsilon(p_1) + \epsilon(p_2).$$

The absence of the second term is, as for symmetric solutions, guaranteed by the proper choice of the parameter $\coth\gamma$,

$$\coth\gamma = \frac{\kappa}{2\pi}[f(p_1) - f(p_2)].$$

Most troubles are concerned with the last term having no analog in the symmetric case. It vanishes if and only if the electron quasimomenta are not arbitrary but restricted by the transcendental relation

$$g(p_1) + g(p_2) = 0. \tag{38}$$

The simplest possibility is $p_1 = -p_2$ which corresponds to zero total quasimomentum. So the ansatz (35) with (25) and (38) gives not a general but only a partial one-parametric solution to (9). At this stage we could not find any modification of (35) free from the restriction (38). However we hope that it will be possible in the future.

Let us discuss also the problem of antisymmetric bound states with zeroth total quasimomentum. The decrease of $\tilde{Z}^{(a)}_{\ell m}$, when $|\ell - m| \to \infty$, can be reached only if

$$\frac{\kappa}{\pi} f(p) = 1, \quad 0 < \mathrm{Im}\, p < \frac{2\pi}{\kappa}, \quad p_1 = -p_2 = p\,. \tag{39}$$

In the case of a pure imaginary quasimomentum, $p = \frac{2\pi i \tau}{\kappa}$, $\mathrm{Im}\,\tau = 0$, the solution of (39) is equivalent to the search for the zeroes of the function $\phi(\tau) = \zeta(\tau) + \frac{2i\tau}{\kappa}\zeta(\frac{\omega}{2}) - \frac{\pi}{\kappa}$ on an interval $0 < \tau < 1$ with the exception of the point $\tau = 1/2$ at which $\tilde{Z}^{(a)}_{\ell m}$ identically vanishes. On the left and right ends of that interval $\phi(\tau)$ tends to $\pm\infty$. Its derivative $\frac{d\phi}{d\tau}$ can be written as an infinite sum,

$$\frac{d\phi}{d\tau} = -\wp(\tau) + \frac{2i}{\kappa}\zeta(\frac{\omega}{2}) = -\frac{\pi^2}{\kappa^2}\sum_{n=-\infty}^{\infty}[\sinh\frac{\pi}{\kappa}(\tau - n)]^{-2},$$

which is negative at all real τ. So $\phi(\tau)$ has the only zero at $0 < \tau < 1$. By using the well-known properties of ζ function,

$$\zeta(\tau+1) = \zeta(\tau) + 2\zeta(\frac{1}{2}), \quad \zeta(\frac{1}{2})\omega - \zeta(\frac{\omega}{2}) = \pi i,$$

it is easy to see that this zero is located just at the exceptional point $\tau = 1/2$, so the bound states in this model do not exist at zeroth total quasimomentum.

V. DISCUSSION

Returning to the original Hamiltonian (4) we see that the simplest two-electron problem on an infinite lattice has the exact solutions if

$$U = 0, \quad t_{\ell m} = t_0 \frac{\pi^2}{\kappa^2 \sinh^2 \frac{\pi}{\kappa}(\ell - m)},$$

$$V_{\ell m} = t_0 \frac{\pi^2}{\kappa^2}[\sinh^{-2}\frac{\pi}{\kappa}(\ell - m) - \delta\cosh^{-2}\frac{\pi}{\kappa}(\ell - m)], \tag{40}$$

$$I_{\ell m} = t_0 \frac{\pi^2}{\kappa^2}[\sinh^{-2}\frac{\pi}{\kappa}(\ell - m) + \delta\cosh^{-2}\frac{\pi}{\kappa}(\ell - m)], \quad \delta = 0 \text{ or } 1$$

in both symmetric and antisymmetric cases. From our point of view this fact indicated that such models seem to be solvable. More convincing arguments may be obtained by analyzing the states with three or more electrons. In the complete symmetric sector corresponding to the Heisenberg-chain type model one can show that the wave functions have a Bethe structure, i.e. depend only on one phase, which is expressed through quasimomenta as (25). The states of mixed symmetry with more than two electrons have not been investigated but our previous experience shows that exact solutions can also be obtained.

Note that the short-range hopping (40) is reduced to the nearest neighbour Hubbard one in the limit $\kappa \to 0$ after a trivial "renormalization" $t_0 \to t_0 \frac{\kappa^2}{\pi^2}\exp(\frac{2\pi}{\kappa})$. However, the total Hamiltonian does not tend to the Hubbard one because of $U = 0$ and surviving of the exchange nearest neighbour intersite interaction term in (40). To our knowledge models of that type have not been considered as candidates for being solvable. It would be interesting to check for them the possibility of constructing the standard Lax representation as it was done in 36 for the Hubbard case.

By analogy with spin chains we can also assume that models of the type (40) would be solvable if $V_{\ell m}$ and $I_{\ell m}$ are modified as follows

$$V_{\ell m} = \frac{t_0\pi^2}{2\kappa^2}[j(j+1)\sinh^{-2}\frac{\pi}{\kappa}(\ell - m) - k(k+1)\cosh^{-2}\frac{\pi}{\kappa}(\ell - m)],$$

$$I_{\ell m} = \frac{t_0 \pi^2}{2\kappa^2}[j(j+1)\sinh^{-2}\frac{\pi}{\kappa}(\ell-m) + k(k+1)\cosh^{-2}\frac{\pi}{\kappa}(\ell-m)]. \qquad (41)$$

That "discrete" analogs of the well-known Poschl - Teller potentials correspond to the possibility of expressing through Weierstrass functions the trigonometric sums which are similar to (21) and (31),

$$S^{(n)}(p,\ell) = \sum_{\substack{k \neq 0, -\ell \\ -\infty}}^{\infty} \frac{\pi^2}{\kappa^2} \frac{\exp(ipk)}{\sinh^2(\frac{\pi}{\kappa}k)} P_n(\tanh\frac{\pi}{\kappa}(k+\ell), \coth\frac{\pi}{\kappa}(k+\ell),$$

where P_n is an arbitrary polynomial of n-th power. However, no rigorous proof of the existence of exact solutions of type (23), (35) in that case is now yet found.

Another type of interesting problems is the investigation of excitations in more realistic and complicated than (5) "vacuum" basic states. The picture of particle-hole and hole-hole interactions in a half-filled (ferromagnetically ordered ground state) band seems to be very similar to the case considered above but the propagation of excitations on the vacuum of antiferromagnetic type needs a more thorough analysis.

The proper consideration of periodic boundary conditions consists in the change of trigonometric hopping and interaction terms (40) into elliptic ones,

$$t_{\ell m} = t_0 \tilde{\mathcal{P}}(\ell - m), \quad V_{\ell m} = t_0[\tilde{\mathcal{P}}(\ell - m) + \delta\tilde{\mathcal{P}}(\ell - m + \frac{\omega}{2})],$$

$$I_{\ell m} = t_0[\tilde{\mathcal{P}}(\ell - m) - \delta\tilde{\mathcal{P}}(\ell - m + \frac{\omega}{2})],$$

where the real period of double periodic Weierstrass $\tilde{\mathcal{P}}$ functions must be equal to the number of sited N and the imaginary period coincides with that of (40). As N tends to infinity,

$$\tilde{\mathcal{P}}(x) \to \frac{\pi^2}{\kappa^2}(\frac{1}{3} + \sinh^{-2}\frac{\pi x}{\kappa}), \quad \tilde{\mathcal{P}}(x + \frac{\omega}{2}) \to \frac{\pi^2}{\kappa^2}(\frac{1}{3} - \cosh^{-2}\frac{\pi x}{\kappa})$$

and the previous structure (40) appears. The two-particle exact solutions in the symmetric second were also investigated earlier but had a much more complicated form [43].

To summarize, at this stage we tried to construct a solvable nonlocal generalization of the original Hubbard model without the U term. The case including that term requires further efforts. It is quite possible that a model like that, if exists, needs a more sophisticated than (3) form of interaction in the second-quantized representation. Nevertheless we hope that the solutions discussed above give an insight into the more general problem of introducing solvable lattice models. From the algebraic point of view the most interesting questions here arise in constructing analytic integrals of motion commuting with the Hamiltonian (2). The finding of a regular procedure for that gives, according to the common wisdom, the most preferable proof of the solvability. In a much more simple Hubbard situation this procedure has been proposed only recently [35-37] and even the simplest integrals are relatively complicated [37]. In our case of non-nearest-neighbour hopping the standard routine based on the search for a local transition matrix seems to be nonapplicable. For similar quasi-Heisenberg nonlocal spin chains only few integrals of motion were constructed but any regular scheme was not found [43]. This line of research seems to be most intriguing and promising for future studies.

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Точное решение новой одномерной фермионной системы на решетке

Мы рассматриваем обобщенную модель Хаббарда на бесконечной одномерной решетке, включающую прямое и обменное межузельное взаимодействие нерелятивистских фермионов спина 1/2. Модель допускает возможность перескока электронов между любыми узлами решетки, однако сохраняет короткодействие, характерное для стандартной модели Хаббарда. В одно- и двухэлектронном секторах найдены явные точные решения путем использования теории эллиптических функций Вейерштрасса. Обсуждаются вопросы построения общего решения и дальнейшего развития теории интегрируемых решеточных ферми-систем.

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Exact Solution of the New One-Dimensional Fermion System on a Lattice

We investigate the problem of constructing the solvable models of nonrelativistic spin 1/2 interacting fermions on an infinite lattice. The extended Hubbard model which includes the direct and exchange intersite interaction is considered. The hopping term in the Hamiltonian is not restricted to nearest neighbour sites (the "overjumping" model) but the short-range character of the standard Hubbard model is retained. The solutions in one- and two-particle sectors are obtained in an explicit form by using the theory of Weierstrass elliptic functions. The possibilities of further development of models of that type are also considered.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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