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E17-90-336

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# PHASE PROPERTIES OF THE ANHARMONIC OSCILLATOR STATES

Submitted to "Optics Communications"

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1990

## 1. Introduction

A problem of producing squeezed states of light has attracted a considerable amount of interest of physicists in recent years [1]. Squeezed states have phase sensitive noise and it is interesting to reveal their phase properties. Phase properties of the ideal squeezed states have been examined by Sanders et al. [2], Yao [3] and Fan et al. [4] with the use of the Susskind-Glogower [5] phase formalism with the non-unitary phase operators. Another squeezed states, which are different from the ideal squeezed states, or the two-photon coherent states of Yuen [6], are the states that can be produced by the anharmonic oscillator. Squeezing in such a model has been shown by Tanaś and Kielich [7] for the two-mode version and by Tanaś [8] for the one-mode version of the anharmonic oscillator. Phase properties of such states have been considered by Gerry [9] from the point of view of the Susskind-Glogower formalism, and by Lynch [10] from the point of view of the so called measured phase operators introduced by Barnett and Pegg [11]. Recently, Pegg and Barnett [12-14] have introduced a new formalism with a Hermitian phase operator which allows to smooth away difficulties inherent in the Susskind-Glogower formalism. This new formalism has recently been used by Vaccaro and Pegg [15] to re-examine the problem of the phase properties of the ideal squeezed states. They have shown that there are significant differences between the results based on the two formalisms, especially for fields of low excitation.

In this paper, we examine the phase properties of the anharmonic oscillator using the Pegg-Barnett formalism. The phase probability density is obtained and the expectation values and variance of the Hermitian phase operator are calculated. The expectation values and variances for the cosine and sine functions of such an operator are also calculated. A comparison is made with earlier results based on the Susskind-Glogower formalism and the measured phase concept.

## 2. The Hermitian phase operator

The new formalism recently introduced by Pegg and Barnett [12-14] to describe the phase properties of a single mode field has successfully overcome the difficulties associated with the existence of the Hermitian phase operator. As the Hermitian phase



БИБЛИОТЕКА

operator is now at our disposal, we can pose anew some old questions of phase properties of the field and formulate some new questions. For example, phase mean values, phase variances as well as phase probability densities can be calculated for various states of the field. Vaccaro and Pegg [15] have recently studied the phase properties of the ideal squeezed states of light.

The idea of Pegg and Barnett [12-14] is based on introducing a finite (s+1)-dimensional subspace $\Psi$ spanned by the number states $|0>, |1>, \ldots, |s>$. The Hermitian phase operator operates on this subspace. After the expectation values have been calculated in $\Psi$, the value of s is allowed to tend to infinity. A complete orthonormal basis of (s+1) states is defined on $\Psi$ as [12-14]

$$|\theta_m> \equiv (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n>, \tag{1}$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s+1) \;, \quad (m=0,1,\ldots,s) \;. \tag{2}$$

The value of $\theta_0$ is arbitrary and defines a particular basis set of (s+1) mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m><\theta_m| \;, \tag{3}$$

and the states (1) are eigenstates of the phase operator (3) with the eigenvalues $\theta_m$, which are restricted to lie within a phase window between $\theta_0$ and $\theta_0+2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ when operating on $|\theta_m>$ gives the eigenvalue $\exp(i\theta_m)$. This operator can be written as [12-14]

$$\exp(i\hat{\phi}_\theta) \equiv |0><1| + |1><2| + \ldots + |s-1><s| + \exp[i(s+1)\theta_0]\,|s><0| \;, \tag{4}$$

and its Hermitian conjugate is

$$[\exp(i\hat{\phi}_\theta)]^{+} = \exp(-i\hat{\phi}_\theta) \;, \tag{5}$$

with the same set of eigenstates $|\theta_m>$ but with eigenvalues $\exp(-i\theta_m)$.

To relate this new operator to the Susskind and Glogower phase operator one can use the relation [15]



$$<\exp(im\hat{\phi}_\theta)> = <[\exp(i\hat{\phi}_\theta)]^m> =$$

$$= \lim_{s\to\infty} < \left\{ \sum_{n=0}^{s-m} |n><n+m| + \exp[i(s+1)\theta_0] \sum_{n=0}^{m-1} |s-n><m-1-n| \right\} > =$$

$$= < \hat{\exp}(im\phi_{SG}) > +$$

$$+ \lim_{s\to\infty} < \left\{ \exp[i(s+1)\theta_0] \sum_{n=0}^{m-1} |s-n><m-1-n| \right\} >, \tag{6}$$

where the Susskind-Glogower phase operator is given by

$$\hat{\exp}(im\phi_{SG}) \equiv \sum_{n=0}^{\infty} |n><n+m| \,. \tag{7}$$

It should be emphasized that in the case of Pegg-Barnett definition we have the exponential of the Hermitian phase operator while in the Susskind-Glogower case the exponential operator is defined as a whole.

For "physical states" [12-14] one obtains the following relations [15]

$$<\exp(im\hat{\phi}_\theta)>_p = <\hat{\exp}(im\phi_{SG})>_p \;, \tag{8}$$

where the subscript p refers to a physical state expectation value.

For the cosine and sine phase operators the relations are [15]

$$<\cos\hat{\phi}_\theta>_p = \frac{1}{2} < \exp(i\hat{\phi}_\theta) + \exp(-i\hat{\phi}_\theta) >_p =$$

$$= <\hat{\cos}\phi_{SG}>_p \;, \tag{9}$$

$$<\sin\hat{\phi}_\theta>_p = \frac{1}{2i} < \exp(i\hat{\phi}_\theta) - \exp(-i\hat{\phi}_\theta) >_p =$$

$$= <\hat{\sin}\phi_{SG}>_p \;, \tag{10}$$

$$\langle\cos^2\hat{\phi}_\theta\rangle_p = \frac{1}{4}\langle \exp(i2\hat{\phi}_\theta) + \exp(-i2\hat{\phi}_\theta) + 2\rangle_p$$

$$= \langle\hat{\cos}^2\phi_{SG}\rangle_p + \frac{1}{4}\langle(|0\rangle\langle 0|)\rangle_p \ , \qquad (11)$$

$$\langle\sin^2\hat{\phi}_\theta\rangle_p = -\frac{1}{4}\langle \exp(i2\hat{\phi}_\theta) + \exp(-i2\hat{\phi}_\theta) - 2\rangle_p$$

$$= \langle\hat{\sin}^2\phi_{SG}\rangle_p + \frac{1}{4}\langle(|0\rangle\langle 0|)\rangle_p \ . \qquad (12)$$

The above relations will be used by us to describe phase properties of the anharmonic oscillator.

## 3. The anharmonic oscillator evolution

The anharmonic oscillator model, phase properties of which we discuss in the paper, is described by the Hamiltonian

$$H = \hbar\omega a^+a + \frac{1}{2}\hbar\kappa a^{+2}a^2 \ , \qquad (13)$$

where $a$ and $a^+$ are the annihilation and creation operators of the field mode, and $\kappa$ is the coupling constant which is real and can be related to the nonlinear susceptibility $\chi^{(3)}$ of the medium if the anharmonic oscillator is used to describe propagation of laser light in a nonlinear Kerr medium.

Since the number of photons $a^+a$ is a constant of motion the state evolution of the system is described, in the interaction picture, by the Schrödinger equation

$$i\hbar\frac{d}{dt}U(t) = H_I U(t) \ , \qquad (14)$$

where $U(t)$ is the time evolution operator and $H_I$ is the nonlinear part of the Hamiltonian (13). In the propagation problem of light propagating in a Kerr medium one can make the replacement $t=-z/v$ to describe the spatial evolution of the field instead of the time evolution.

The solution of equation (14) is given by [16]

$$U(\tau) = \exp\left[i\frac{\tau}{2}\hat{n}(\hat{n}-1)\right] \ , \qquad (15)$$

where

$$\tau=\kappa z/v \qquad (16)$$

is the dimensionless length of the nonlinear medium (or time in the time domain), and $\hat{n}=a^+a$ is the number operator. If the state of the incoming beam is a coherent state $|\alpha_0\rangle$, the resulting state of the outgoing beam is given by

$$|\psi(\tau)\rangle = U(\tau)|\alpha_0\rangle$$

$$= \exp(-|\alpha_0|^2/2)\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\exp\left[i\frac{\tau}{2}n(n-1)\right]|n\rangle \ . \qquad (17)$$

If we introduce the notation $\alpha_0= N^{1/2}\exp(i\varphi_0)$, the state (17) can be written as

$$|\psi(\tau)\rangle = \sum_{n=0}^{\infty} b_n \exp\left\{i\left[n\varphi_0+ \frac{\tau}{2}n(n-1)\right]\right\}|n\rangle \ , \qquad (18)$$

where

$$b_n = \exp(-N/2)N^{n/2}/\sqrt{n!} \ . \qquad (19)$$

Since the number of photons is a constant of motion, and is equal to N, the state (18) is a "physical state" in the sense of Pegg and Barnett [13,14] for any finite N, and obviously $\lim_{n\to\infty} b_n=0$. This means that the formulas for the physical states given in the previous Section can be directly applied to this state, and all necessary phase properties of the outgoing beam can be calculated.

## 4. Phase properties of the field

In this paper we consider the phase properties of the field the state of which is given by the superposition (18). Initially this state is a coherent state with the phase $\varphi_0$ and, thus, it is a partial phase state. We choose $\theta_0$ in the way that is convenient for partial phase states [14]

$$\theta_0 = \varphi_0- \frac{\pi s}{s+1} \ , \qquad (20)$$

and introduce a new phase index

$$\mu = m - \frac{s}{2} \ , \qquad (21)$$

which goes in integer steps from $-\frac{s}{2}$ to $\frac{s}{2}$. This makes the distribution symmetric in $\mu$. With such assumption, according to equations (1),(2) and (18), we obtain for the phase probability distribution the following expression

$$|<\theta_\mu|\psi(\tau)>|^2 = \frac{1}{s+1} + \frac{2}{s+1}\sum_{n>k} b_n b_k \times \cos\left\{(n-k)\mu 2\pi/(s+1) - \frac{\tau}{2}\Big[n(n-1)-k(k-1)\Big]\right\}, \tag{22}$$

where $b_n$ are given by equation (19). When $\tau=0$ this expression goes over into the corresponding expression given by Pegg and Barnett [14]. In the limit as s tends to infinity, we can replace $\mu 2\pi/(s+1)$ by $\theta$ and $2\pi/(s+1)$ by $d\theta$. This gives us a continuous phase probability distribution given by

$$P(\theta) = \frac{1}{2\pi}\left\{1+2\sum_{n>k} b_n b_k \cos\Big[(n-k)\theta - \frac{\tau}{2}[n(n-1)-k(k-1)]\Big]\right\}, \tag{23}$$

with the normalization

$$\int_{-\pi}^{\pi} P(\theta)d\theta=1 \quad . \tag{24}$$

For $\tau=0$, expression (23) describes the phase probability distribution for a coherent state. When the nonlinear evolution is on ($\tau\neq0$) the distribution $P(\theta)$ acquires new and very interesting features. Some of these features are illustrated in Fig.1 where the plots of $P(\theta)$ in the polar coordinate system are given for various values of $\tau$ and two different values of the mean number of photons N. For N=0.25 and $\tau=0$, i.e., for the coherent state with the mean number of photons equal to 0.25, the distribution $P(\theta)$ has more or less elliptic shape which, however, has nothing to do with squeezing of the field. Quite opposite, when squeezing in the system increases the shape of $P(\theta)$ becomes less elliptic, for the maximum of squeezing which appears for $\tau=\pi$ [17] the shape of $P(\theta)$, although symmetric, is not at all elliptic. This means that the shape of $P(\theta)$ is not related to squeezing in the simple way that the elliptic shape of $P(\theta)$ means squeezing. On the other hand, the shape of $P(\theta)$ can be related to the shape of the quasi-probability distribution $Q(\alpha,\alpha^*)$ which is clearly visible from Fig.1b, where, for $\tau=\pi$, the phase distribution $P(\theta)$ splits into two separate parts. This corresponds to the superposition of two coherent states to which the anharmonic oscillator evolves [18] in this case. When $\tau$ is taken as $2\pi/n$ (n=2,3,4,...) the shape of $P(\theta)$ shows n-fold symmetry, confirming generation of discrete



superpositions of coherent states with 2,3,4,... components [19].

The phase distribution $P(\theta)$ can be used to calculate the mean and the variance of the phase operator defined by equation (3). The results are

$$<\psi(\tau)|\hat{\phi}_\theta|\psi(\tau)> = \varphi_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sum_{n>k} b_n b_k \times \cos\left\{(n-k)\theta - \frac{\tau}{2}\Big[n(n-1) - k(k-1)\Big]\right\}\theta d\theta = \varphi_0 - 2\sum_{n>k} b_n b_k \frac{(-1)^{n-k}}{n-k}\sin\left\{\frac{\tau}{2}\Big[n(n-1)-k(k-1)\Big]\right\}, \tag{25}$$

$$<\psi(\tau)|(\Delta\hat{\phi}_\theta)^2|\psi(\tau)> = \frac{\pi^2}{3} + 4\sum_{n>k} b_n b_k \frac{(-1)^{n-k}}{(n-k)^2}\cos\left\{\frac{\tau}{2}\Big[n(n-1)-k(k-1)\Big]\right\} - \left\{2\sum_{n>k} b_n b_k \frac{(-1)^{n-k}}{n-k}\sin\left\{\frac{\tau}{2}\Big[n(n-1)-k(k-1)\Big]\right\}\right\}^2 . \tag{26}$$

For $\tau=0$, we get the results for a coherent state with the phase $\varphi_0$ [14]. The nonlinear evolution of the system leads to a shift of the mean phase and essentially changes the variance. An example is illustrated in Fig.2, where the evolution of the phase and its variance is plotted against $\tau$, for N=4. The line $\pi^2/3$ marks the variance for the state with random distribution of phase. We have assumed $\varphi_0=0$, and the window of the phase values is taken between $-\pi$ and $\pi$. At the initial stage of the evolution the mean value of the phase as well as the variance go up. Later on the mean phase oscillates around zero while the variance becomes close to $\pi^2/3$ until $\tau$ approaches $2\pi$, where the initial values are restored. So, most of the period of the evolution the field spends in a state which is very close to the state with randomly distributed phase. This effect is even more pronounced when the number of photons increases.

Phase characteristics of the field such as the phase distribution $P(\theta)$, the expectation value of the phase operator and its variance can be obtained within the Pegg-Barnett formalism only, and cannot be compared to any other approach so far.

Other phase characteristics are the cosine and the sine functions of the phase and their variances. Using equations (7)-(12) and the state (18), we obtain the following results

$$\langle\psi(\tau)|\cos\hat{\phi}_\theta|\psi(\tau)\rangle = N^{1/2}e^{-N}\sum_{n=0}^{\infty}\frac{N^n\cos(n\tau+\varphi_0)}{n!(n+1)^{1/2}} , \tag{27}$$

$$\langle\psi(\tau)|\sin\hat{\phi}_\theta|\psi(\tau)\rangle = N^{1/2}e^{-N}\sum_{n=0}^{\infty}\frac{N^n\sin(n\tau+\varphi_0)}{n!(n+1)^{1/2}} , \tag{28}$$

$$\langle\psi(\tau)|\cos^2\hat{\phi}_\theta|\psi(\tau)\rangle = \frac{1}{2} + \frac{1}{2}\,Ne^{-N}\sum_{n=0}^{\infty}\frac{N^n\cos[(2n+1)\tau+2\varphi_0]}{n![(n+1)(n+2)]^{1/2}} , \tag{29}$$

$$\langle\psi(\tau)|\sin^2\hat{\phi}_\theta|\psi(\tau)\rangle = \frac{1}{2} - \frac{1}{2}\,Ne^{-N}\sum_{n=0}^{\infty}\frac{N^n\cos[(2n+1)\tau+2\varphi_0]}{n![(n+1)(n+2)]^{1/2}} . \tag{30}$$

It should be emphasized that according to the Pegg-Barnett formalism we deal with the sine and cosine functions of the Hermitian phase operator $\hat{\phi}_\theta$, while in the Susskind-Glogower formalism the sine and cosine operators are defined as separate entities. In the case of the sine and cosine, according to (9) and (10), there is no difference between the expectation values obtained in both approaches because the state of the field produced in the anharmonic oscillator model is a physical state. However, the two approaches give different results for the squares of the sine and cosine of the phase. Our formulas (29) and (30) show that $\cos^2\hat{\phi}_\theta+\sin^2\hat{\phi}_\theta=1$, which is not the case in the Susskind-Glogower formalism. For $\tau=0$, again the results correspond to a coherent state with the phase $\varphi_0$. When $\tau\neq0$, the results (27)-(30) are rather complicated. However if N is not too large they can be easily evaluated numerically. This allows us to compare results for the variance of the cosine of the phase obtained by different formalisms. This is shown in Fig.3. Since differences between various approaches are essential only for small N and vanish as N becomes large, we have chosen N=0.25 to illustrate them clearly. In Fig.3 the evolution of the variance of the cosine of the phase is shown for the Pegg-Barnett, Susskind-Glogower and so called measured phase of Barnett



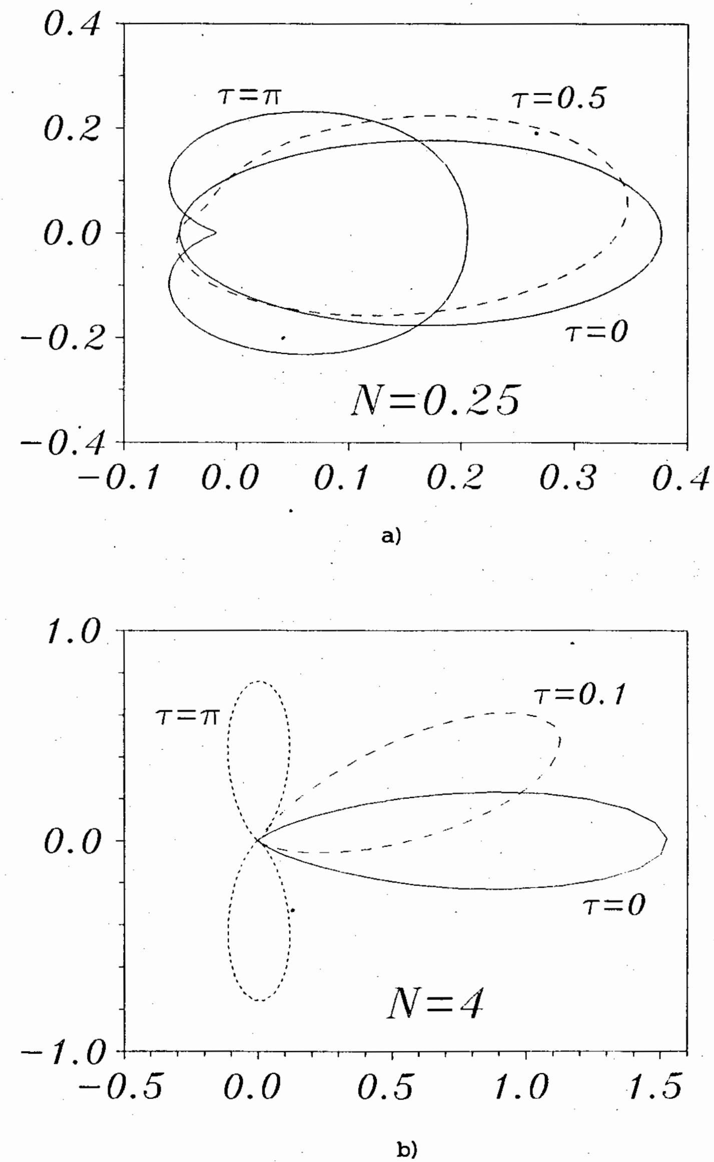


Fig.1 Plot of the phase distribution $P(\theta)$ against $\theta$ in the polar coordinate system: a) for $\tau=0$, 0.5 and $\pi$; N=0.25; b) for $\tau=0$, 0.1 and $\pi$; N=4

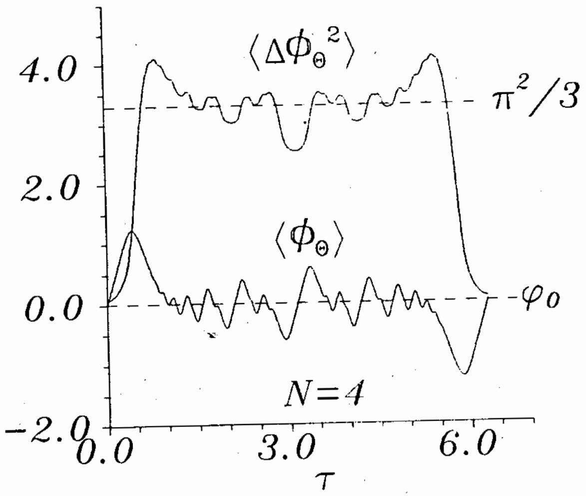


**Fig.2** Plot of the mean value of the phase $\langle\hat{\phi}_\theta\rangle$ and its variance $\langle(\Delta\hat{\phi}_\theta)^2\rangle$ against $\tau$, for N=4

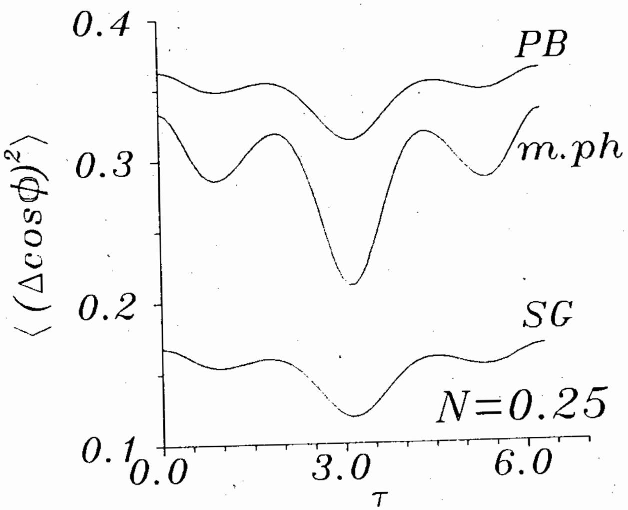


**Fig.3** Plot of the variance of the phase cosine against $\tau$, for N=0.25; *PB*-Pegg-Barnett result, *SG*-Susskind-Glogower result, *m.ph*-measured phase result



and Pegg [11] formalisms. The Susskind-Glogower variance is shifted with respect to the Pegg-Barnett result by $\frac{1}{4}e^{-N}$, according to equation (11). The measured phase result is slightly different in shape, although it reproduces main features of the Pegg-Barnett result, and locates itself between the other two results. The phase properties of the anharmonic oscillator, in a bit different form, have been discussed by Gerry [9] who used the Susskind-Glogower formalism, and by Lynch [10] from the point of view of the measured phase formalism. The measured phase formalism is directly related to squeezing and the shape of the variance of the cosine of the phase is in fact the appropriately normalized variance of the field [8,17].

## 5. Conclusions

We have discussed the phase properties of the field generated in course of the evolution of the anharmonic oscillator from the point of view of the new phase formalism introduced recently by Pegg and Barnett [12-14]. This new formalism constructs the Hermitian phase operator and allows to pose, for the first time, the question of the properties of the phase itself. We have obtained analytical formulas for the phase distribution, the expectation value of the phase operator and its variance. The phase distribution $P(\theta)$ for the anharmonic oscillator model has interesting features, some of which have been illustrated graphically. In particular, this distribution confirms earlier results [18,19] that discrete superpositions of coherent states can be generated during the evolution of the anharmonic oscillator. We have shown that the shape of the phase distribution in the polar coordinate system cannot be directly related to squeezing of the field. The evolution of the mean value of the phase and its variance show some oscillations, but for N>1, the phase variance becomes close to $\pi^2/3$-the value for the randomly distributed phase, except for the short time limit. The variances of the sine and cosine functions of the phase operator have been calculated. A comparison of our results with earlier results is made showing essential differences when the number of photons is small. For large number of photons all the results become indistinguishable.

The phase distribution of the field can be considered as an alternative description with respect to the quasiprobability distribution $Q(\alpha,\alpha^*)$, which has often been used to describe the anharmonic oscillator evolution [16,19-22]. It is probably the most spectacular to compare the two descriptions for $\tau=2\pi/n$ when discrete superpositions of coherent states appear [18,19], but this will be done elsewhere [23].

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Received by Publishing Department
on May 17, 1990.




Ганцог Ц., Танась Р. E17-90-336
Фазовые свойства состояний ангармонического осциллятора


Изучены фазовые свойства ангармонического осциллятора с применением нового эрмитового фазового формализма Пегга и Барнетта. Фазовое распределение, среднее значение и дисперсии фазового оператора вычислены и проиллюстрированы графически. Результаты для дисперсии косинуса фазы получены и сравнены с ранее полученными результатами на основе других подходов.



Работа выполнена в Лаборатории теоретической физики ОИЯИ.

**Препринт Объединенного института ядерных исследований. Дубна 1990**


Gantsog Ts., Tanaś R. E17-90-336
Phase Properties of the Anharmonic Oscillator States


Phase properties of the anharmonic oscillator are studied using the new Hermitian phase formalism of Pegg and Barnett. The phase distribution, the expectation value of the phase operator and its variance are calculated and illustrated graphically. The results for the variance of the cosine of the phase are obtained and compared to earlier results based on other approaches.



The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

**Preprint of the Joint Institute for Nuclear Research. Dubna 1990**