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STEADY-SQUEEZING STATE FOR THE SIMPLEST LINEAR SYSTEM

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1. Goals of the work and basic relations

Linear systems[1)] are often an acid test for beginning a study of a phenomenon which has not been the focus of interest among previous workers. In spite of their simplicity, the linear systems permit us to establish immediately many important physical and mathematical peculiarities of the object under investigation [1]. The case in point is what we call "steady-squeezing states". It may be noted that well-known coherent states [2] and (dynamic) squeezed ones [3-5] were first introduced in the linear framework. As a logical addition, it is therefore natural to begin the study of steady-squeezing states in the linear approximation.

On the other hand, it should be emphasized that squeezed states are of general interest, since the bosonic systems for which the (dynamic) squeezed states have been already observed [6,7] are not only present in quantum optics but also, for instance, in the theory of solids [8], in the theory of superfluidity [9,10], in nuclear physics [11,12], in high-energy physics [13,14], etc. So, the squeezed states are very attractive objects especially as a phenomenon concerning foundations of quantum mechanics.

Constant squeezing in the quadrature components is obviously a property of a system during the evolution in time. Consequently, for explicit analysis it is necessary to concretize the Hamiltonian of the system, determining the single-parametric dynamic group of the system [1]. In this Letter, on the basis of results obtained in a previous paper [15], the set of the wave functions {|sss>} realizing the steady-squeezing state will be found for the linear model system, characterized by the Hamiltonian

$$H = \omega a^{+}a + f^{*}a^{2} + f a^{+2} . \qquad (1)$$

Here $\omega > 0$ is the frequency of the Bose field quanta, f is the time-independent coupling constant. Besides the annihilation a and creation a^{+} operators one can define the canonical coordinate X_1 and momentum X_2. All these operators are introduced in the usual way

$$[\, a, a^{+}] = 1, \qquad [\, X_1, X_2] = \frac{i}{2} , \qquad (2)$$

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1) i.e. the systems for which the operator equations of motion are the linear ones; the Hamiltonian of such a system is a general bilinear form over the canonical coordinates and momenta.

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$$a = X_1 + i X_2, \qquad a^+ = X_1 - i X_2 . \tag{3}$$

Let <...> mean the averaging over the sought-for steady-squeezing wave function |sss>. Therefore, for an operator $\hat{O}$ which we consider well-defined on the wave function |sss> one can write down

$$<\hat{O}> = <sss|\hat{O}|sss>. \tag{4}$$

The variance operators for the state |sss> may be consequently written as

$$\Delta a = a - <a>, \qquad \Delta a^+ = a^+ - <a^+>, \tag{5}$$

$$\Delta X_i = X_i - <X_i>, \quad i=1,2. \tag{6}$$

Explicit expressions determining the time evolution of the operators $\Delta X_1^2(t)$, $\Delta X_2^2(t)$ and $(\Delta X_1(t)\Delta X_2(t) + \Delta X_2(t)\Delta X_1(t))/2$ for system (1) show that if the averages for these operators obey the relations[2)]

$$\frac{1}{2}<\Delta X_1(t_0)\Delta X_2(t_0) + \Delta X_2(t_0)\Delta X_1(t_0)> = - \frac{2D\sin(\theta)}{1-2D\cos(\theta)+D^2}<\Delta X_1^2(t_0)>, \tag{7}$$

$$<\Delta X_2^2(t_0)> = R <\Delta X_1^2(t_0)>, \qquad R= \frac{1+2D\cos(\theta)+D^2}{1-2D\cos(\theta)+D^2}. \tag{8}$$

at the initial time t_0, then the variances $<\Delta X_1^2(t)>$ and $<\Delta X_2^2(t)>$ are steady, i.e. are equal to their own initial values $<\Delta X_1^2(t_0)>$ and $<\Delta X_2^2(t_0)>$.

The R parameter in formulae (7)-(8) characterizes the degree of the asymmetry between the values of the variances and, therefore, the degree of the squeezing if it occurs. Any one of the three independent averages in eqs. (7)-(8) may be given at will and one can always put it equal to such a value that there will be squeezing in the system. The quantities D and θ are defined by means of the Hamiltonian constants:

$$D = \frac{1}{2|f|} (\omega - \sqrt{\omega^2-4|f|^2}), \qquad \theta = \arg f. \tag{9}$$

It is very important to make here the following remark. One should not think that the problem of finding the steady-squeezing state is only the trivial problem of finding the eigenstates for Hamiltonian (1). For a system for which the initial state is one of

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2) There are the misprints in basic relations (52) and (54) of paper [15]: the factor 1/2 before the left-hand sides has been dropped. Also, in the formula (58) of [15] the "+" sign should be there instead of the "-" sign (see Errata [16]).

its eigenstates the averages are steady in time for all the operators which are time-independent in the Schrödinger picture. But the author's task is, and that is just the difference, to find such state |sss> that only the averages $<\Delta X_1^2(t)>$, $<\Delta X_2^2(t)>$ and $<\Delta X_1(t)\Delta X_2(t) + \Delta X_2(t)\Delta X_1(t)>/2$ were constant in time although the averages of any other operators may have arbitrary evolution. Further we will show that the function |sss> may be chosen so that the average $<a(t_0)>$ may have any a priori given value. Therefore, for example, the evolution of the average $<a(t)>$ is not stationary at all (see the explicit expressions for the time evolution of the operators in [15]). An additional condition of the problem is the squeezing for one of the quadrature variances, i.e. the fulfilment of the inequality $<\Delta X_i^2(t)> < 1/4$ for one of the numbers i=1,2.

Let the amplitude $<a(t)>$ of the Bose field be a previously given parameter at the initial time moment t_0:

$$\bar{a} = <a(t_0)> = <sss|a(t_0)|sss>. \tag{10}$$

Since it is the only parameter (besides the Hamiltonian constants) in relations (7)-(8), it is just the one which parametrizes the steady-squeezing wave function |sss>. Using identities

$$\Delta X_1^2 \equiv \frac{1}{4} (1 + 2\Delta a^+\Delta a + \Delta a^2 + \Delta a^{+2}), \tag{11}$$

$$\Delta X_2^2 \equiv \frac{1}{4} (1 + 2\Delta a^+\Delta a - \Delta a^2 - \Delta a^{+2}), \tag{12}$$

$$\frac{1}{2} (\Delta X_1\Delta X_2 + \Delta X_2\Delta X_1) \equiv \frac{1}{4i} (\Delta a^2 - \Delta a^{+2}), \tag{13}$$

we obtain the following relations instead of (7)-(8)

$$<\Delta a^2(t_0)> = M (1 + 2<\Delta a^+(t_0)\Delta a(t_0)>), \tag{14}$$

$$<\Delta a^{+2}(t_0)> = M^*(1 + 2<\Delta a^+(t_0)\Delta a(t_0)>), \tag{15}$$

$$M = - \frac{D \exp(i\theta)}{1+ D^2} = - \frac{\exp(i\theta)}{2} \tanh(2\,\mathrm{artanh}(D)). \tag{16}$$

Equations (14)-(15) are not quite convenient for determining the |sss> function. They have an infinite set of solutions and as it will be noted in the forthcoming subsection this set cannot be written down in a compact or, at least, in a visible form. The reason is that eqs.(14)-(15) connect the averages, i.e. the bilinear forms over the coefficients of wave function |sss> in any complete basis. One can determine the |sss> function in a unique way if one has an equation linear with respect to the |sss> function with the solution obeying relations (7)-(8). This is the only equation for given values of $\bar{a}$ and M; let us find it.

2. Fluctuation representation (F-representation) and the equation for steady-squeezing wave function

For some wave function $|\Psi\rangle$ such that the average $\langle\Psi|a(t_0)|\Psi\rangle$ makes sense one can define the operators:

$$F_\Psi = a - \langle\Psi|a|\Psi\rangle, \quad F_\Psi^+ = a^+ - \langle\Psi|a^+|\Psi\rangle \tag{17}$$

naturally obeying the commutation relation

$$[F_\Psi, F_\Psi^+] = 1. \tag{18}$$

Let us now make sure that operators F_Ψ and F_Ψ^+ are the creation and annihilation ones for the fluctuation quanta over the classical value of the field energy $E_{cl} = \omega|\langle\Psi|a|\Psi\rangle|^2$ at the given field amplitude $\langle\Psi|a|\Psi\rangle$. At first we note that the vacuum for F_Ψ and F_Ψ^+ operators which we denote as $|0\rangle_\Psi$ is simply the coherent state for the operator a corresponding to the eigenvalue $\langle\Psi|a|\Psi\rangle$:

$$F_\Psi|0\rangle_\Psi=0; \quad a|0\rangle_\Psi=\langle\Psi|a|\Psi\rangle|0\rangle_\Psi; \quad |0\rangle_\Psi=\exp\{-\tfrac{1}{2}|\langle\Psi|a|\Psi\rangle|^2+\langle\Psi|a|\Psi\rangle a^+\}|0\rangle; \tag{19}$$

where $|0\rangle$ is the vacuum for the initial field: $a|0\rangle=0$. Therefore the states with definite number of fluctuation quanta over the classical value of the field energy $E_{cl}= \omega|\langle\Psi|a|\Psi\rangle|^2$ at the given field amplitude $\langle\Psi|a|\Psi\rangle$ are

$$|n\rangle_\Psi= \frac{(F_\Psi^+)^n}{\sqrt{n!}}|0\rangle_\Psi=\exp\{-\tfrac{1}{2}|\langle\Psi|a|\Psi\rangle|^2+\langle\Psi|a|\Psi\rangle a^+\}\sum_{k=1}^{n}\frac{(-\langle\Psi|a^+|\Psi\rangle)^{n-k}}{(n-k)!}\sqrt{\frac{n!}{k!}}|k\rangle, \tag{20}$$

where $|k\rangle$ are the Fock states for the initial field. Calculating the mean of the quantum-number operator of the initial field over the state $|n\rangle_\Psi$ with the definite number of the fluctuation quanta one obtains:

$${}_\Psi\langle n|a^+a|n\rangle_\Psi= {}_\Psi\langle n|(F_\Psi^+ +\langle\Psi|a^+|\Psi\rangle)(F_\Psi+\langle\Psi|a|\Psi\rangle)|n\rangle_\Psi = n_\Psi+|\langle\Psi|a|\Psi\rangle|^2, \tag{21}$$

while calculating the field amplitude over those very states one obtains:

$${}_\Psi\langle n|a|n\rangle_\Psi = {}_\Psi\langle n|(F_\Psi+\langle\Psi|a|\Psi\rangle)|n\rangle_\Psi = \langle\Psi|a|\Psi\rangle. \tag{22}$$

Consequently, *annihilation and creation operators of the fluctuation quanta change the value of the energy of the system, but keep the given field amplitude $\langle\Psi|a|\Psi\rangle$ constant.*

To find the steady-squeezing wave function it is convenient to pass into the F-representation with $|\Psi\rangle=|sss\rangle$. Since in the case $\langle\Psi|a|\Psi\rangle=\langle sss|a|sss\rangle=\bar{a}$, then according to eqs. (5), (6) and (10)

$$F_{sss}(t_0)= \Delta a(t_0); \qquad F_{sss}^+(t_0)= \Delta a^+(t_0). \tag{23}$$

In virtue of this relationship eqs.(14)-(15) takes the form

$$\langle F^{+2}_{sss}(t_0)\rangle = M^*(1+2\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle), \tag{24}$$

$$\langle F^2_{sss}(t_0)\rangle = M\,(1+2\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle). \tag{25}$$

Let now $|n\rangle_{sss}$ be the states (20) in which wave function $|\Psi\rangle$ has been replaced by $|sss\rangle$. We will find this $|sss\rangle$ function in the form of the expansion over the $|n\rangle_{sss}$ states:

$$|sss\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle_{sss}\,. \tag{26}$$

Determining the coefficients C_n one can easily pass into the initial Fock-representation for the operators a and a^+ using (19)-(20) for wave function $|\Psi\rangle=|sss\rangle$, i.e. replacing mean $\langle\Psi|a|\Psi\rangle$ by the value of $\bar{a}$ parameter in these formulae.

Obviously we cannot find any uniformly described class of wave functions $|sss\rangle$ if we will use eqs.(24)-(25) for the mean values. There are only two equations while the coefficients C_n form an infinite set. However, we can require that instead of (24)-(25) one of the relations for the operators

$$F^{+2}_{sss}(t_0)|sss\rangle = M^*(1+2\,F^+_{sss}(t_0)F_{sss}(t_0))|sss\rangle, \tag{27}$$

$$F^2_{sss}(t_0)|sss\rangle = M\,(1+2\,F^+_{sss}(t_0)F_{sss}(t_0))|sss\rangle \tag{28}$$

should be fulfilled. If at least one of the equalities (27)-(28) is valid, then relations (24)-(25) are satisfied automatically. Yet, we show that equation (27) is not fulfilled for any $|sss\rangle$ of the form (26) with finite nonzero norm. Substituting expansion (26) into (27) we obtain

$$\sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}\;C_n|n+2\rangle_{sss} = M^*\sum_{n=0}^{\infty}(1+2n)\;C_n|n\rangle_{sss}\,. \tag{29}$$

At the left-hand side of (29) the expansion begins from basis function $|2\rangle_{sss}$. It means that

$$c_0 = c_1 = 0. \tag{30}$$

But if it is so, then the expansion at left-hand side of (29) begins from $|4>_{sss}$ as a matter of fact, which in its turn means that two more coefficients are equal to zero:

$$c_2 = c_3 = 0, \tag{31}$$

whence it follows that the expansion at left-hand side of (29) has already began from $|6>_{sss}$. Continuing such reasoning we finally obtain:

$$c_n = 0,\ n=0,1,2,3,\ldots\infty . \tag{32}$$

Thus equation (27) has no solutions of the form (26) besides zero solution. In other words the kernel of the operator $F^{+2}_{sss}(t_0) - M^*(1+2F^+_{sss}(t_0)F_{sss}(t_0))$ is empty (except zero):

$$\ker(F^{+2}_{sss}(t_0) - M^*(1+2\,F^+_{sss}(t_0)F_{sss}(t_0))) = 0. \tag{33}$$

Nothing else is left for us, but to find the set of states $\{|sss>\}$ as a kernel of operator $F^2_{sss}(t_0) - M(1+2\,F^+_{sss}(t_0)F_{sss}(t_0))$:

$$\{|sss>\} = \ker(F^2_{sss}(t_0) - M\,(1+2\,F^+_{sss}(t_0)F_{sss}(t_0))), \tag{34}$$

i.e. to require the fulfilment of the equation (28). So, *we take (34) as a definition of the class of steady-squeezing states with the given complex amplitude* $\bar{a}$ *for system (1) while equation (28) will be the basic one for the calculation of coefficients in expansion (26).*

3. Steady-squeezing state and some properties of it

Substituting expansion (26) into equation (28) we find that

$$\sum_{n=0}^{\infty} \sqrt{(n-1)n}\ c_n|n-2>_{sss} = M \sum_{n=0}^{\infty} (1+2n)\ c_n|n>_{sss}. \tag{35}$$

Multiplying equality (35) by bra-state $_{sss}<k|$ we obtain a recurrent relation:

$$c_{k+2} = M\,\frac{1+2k}{\sqrt{(k+1)(k+2)}}\,c_k\,,\quad k=0,1,2,\ldots \tag{36}$$

which connects the coefficients c_n with numbers n of the same parity. For this reason we will further distinguish two wave functions: the $|sss>_{ev}$ wave function , the expansion of which over the number states $|n>_{sss}$ contains only even n, and $|sss>_{od}$ wave function, which is a

superposition of the number states $|n>_{sss}$ with odd n. These wave functions are of the form:

$$|sss>_{ev} = c_0 \sum_{k=0}^{\infty} \frac{M^k\circ 1\circ(1+4\circ 0)(1+4\circ 1)\ldots(1+4(k-1))}{\sqrt{(2k)!}}\ |2k>_{sss}, \tag{37}$$

$$|sss>_{od} = c_1 \sum_{l=0}^{\infty} \frac{M^l\circ 1\circ(3+4\circ 0)(3+4\circ 1)\ldots(3+4(l-1))}{\sqrt{(2l+1)!}}\ |2l+1>_{sss}. \tag{38}$$

Here one should make sure that the norms $_{ev}<sss|sss>_{ev}$ and $_{od}<sss|sss>_{od}$ are finite in a region of the values of the parameter M. Firstly, we investigate the convergence of the series determining the norm of the $|sss>_{ev}$ wave function:

$$_{ev}<sss|sss>_{ev} = \sum_{k=0}^{\infty} |c_{2k}|^2, \tag{39}$$

$$|c_{2k}|^2 = \frac{|M|^{2k}\{1\circ(1+4\circ 0)(1+4\circ 1)\ldots(1+4(k-1))\}^2}{(2k)!}\ |c_0|^2. \tag{40}$$

Calculating the limit

$$\lim_{k\to\infty} \frac{|c_{2k+2}|^2}{|c_{2k}|^2} = |M|^2 \lim_{k\to\infty} \frac{(1+4k)^2}{(2k+1)(2k+2)} = 4|M|^2, \tag{41}$$

we find that series (39) converges in the region $|M| < 1/2$. However, this inequality is fulfilled at all finite values of the quantities D and θ by virtue of (16). Thus, the steady-squeezing wave function $|sss>_{ev}$ exists at all the values of the constants ω and f, allowing the diagonalization of the Hamiltonian (1)[3], and it is determined by the series (37).

The above statements remain valid for the wave function $|sss>_{od}$ as well, which is obvious from the following formulae:

$$_{od}<sss|sss>_{od} = \sum_{l=0}^{\infty} |c_{2l+1}|^2, \tag{42}$$

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[3] i.e. at $\omega > 2|f|$.

$$|C_{2l+1}|^2 = \frac{|M|^{2l}\{1\circ(3+4\circ0)(3+4\circ1)\dots(3+4(l-1))\}^2}{(2l+1)!}|C_1|^2. \tag{43}$$

$$\lim_{l\to\infty}\frac{|C_{2l+3}|^2}{|C_{2l+1}|^2} = |M|^2\lim_{l\to\infty}\frac{(3+4l)^2}{(2l+2)(2l+3)} = 4|M|^2. \tag{44}$$

Consequently, the wave function $|sss>_{od}$ exists and is determined by the series (38) at the same assumptions as the $|sss>_{ev}$ wave function. We normalize both these functions to one:

$$_{ev}<sss|sss>_{ev} = {}_{od}<sss|sss>_{od} = 1 \tag{45}$$

and obtain the normalizing constants C_0 and C_1 (upto a constant phase factor):

$$|C_0| = \left[\sum_{k=0}^{\infty}\frac{|M|^{2k}\{1\circ(1+4\circ0)(1+4\circ1)\dots(1+4(k-1))\}^2}{(2k)!}\right]^{-\frac{1}{2}} \tag{46}$$

$$|C_1| = \left[\sum_{l=0}^{\infty}\frac{|M|^{2l}\{1\circ(3+4\circ0)(3+4\circ1)\dots(3+4(l-1))\}^2}{(2l+1)!}\right]^{-\frac{1}{2}}. \tag{47}$$

The general form of the normalized $|sss>$ wave function is

$$|sss> = A\ |sss>_{ev} + B\ |sss>_{od}\ ; \qquad |A| = \sqrt{1-|B|^2},\ |B| \le 1. \tag{48}$$

It is evident that $|sss>_{ev}$ and $|sss>_{od}$ are orthogonal:

$$_{ev}<sss|sss>_{od} = 0, \tag{49}$$

and the field amplitude is equal to $\bar{a}$ both for $|sss>_{ev}$ and for $|sss>_{od}$:

$$_{ev}<sss|a|sss>_{ev} = {}_{od}<sss|a|sss>_{od} = <sss|a|sss> = \bar{a}. \tag{50}$$

Let us show that the averages for any finite powers of operators $F_{sss}(t_0)$, $F^+_{sss}(t_0)$ and $F^+_{sss}(t_0)F_{sss}(t_0)$ are finite and consequently the averages of any finite powers of $a(t_0)$, $a^+(t_0)$ and $a^+(t_0)a(t_0)$ are finite too. As an example, we demonstrate this for the operator $[F^+_{sss}(t_0)F_{sss}(t_0)]^N$ and function $|sss>_{ev}$:

$$_{ev}<sss|[F^+_{sss}(t_0)F_{sss}(t_0)]^N|sss>_{ev} = \sum_{k=0}^{\infty}(2k)^N|C_{2k}|^2 =$$

$$= |C_0|^2\sum_{k=0}^{\infty}(2k)^N\frac{|M|^{2k}\{1\circ(1+4\circ0)(1+4\circ1)\dots(1+4(k-1))\}^2}{(2k)!}. \tag{51}$$

Calculating the limit

$$\lim_{k\to\infty}\frac{(2k+2)^N|C_{2k+2}|^2}{(2k)^N\ |C_{2k}|^2} = |M|^2\lim_{k\to\infty}\frac{(2k+2)^N}{(2k)^N}\lim_{k\to\infty}\frac{(1+4k)^2}{(2k+1)(2k+2)} = 4|M|^2, \tag{52}$$

we find that the series (51) determining the value of $_{ev}<sss|[F^+_{sss}(t_0)F_{sss}(t_0)]^N|sss>_{ev}$ converges in the whole region of the physical values of the parameter M: $0 \le |M| < 1/2$. One can easily carry out analogous calculations for all above-mentioned averages, but besides relations (41) and (44) one should take into account the following ones:

$$_{ev}<sss|F^+(t_0)^{m_1}F(t_0)^{m_2}|sss>_{od} = 0, \tag{53}$$

if $|m_1 - m_2|$ is an even number,

$$_{ev}<sss|F^+(t_0)^{m_1}F(t_0)^{m_2}|sss>_{ev} = {}_{od}<sss|F^+(t_0)^{m_1}F(t_0)^{m_2}|sss>_{od} = 0. \tag{54}$$

if $|m_1 - m_2|$ is an odd number.

So there exist uniquely defined averages for any finite powers of $a(t_0)$, $a^+(t_0)$ and $a^+(t_0)a(t_0)$ operators over the wave functions $|sss>_{ev}$ and $|sss>_{od}$.

4. Variances and uncertainty relation in steady-squeezing states

Using relations (11)-(12), equations (14)-(16) and designation (23) we find that the forms of the quadrature component variances and of the uncertainty relation in steady-squeezing state (48) are

$$<\Delta X_1^2(t)> = <\Delta X_1^2(t_0)> = \frac{1}{4}(1 + 2<\Delta a^+(t_0)\Delta a(t_0)>)(1+2\ \mathrm{Re}\ M) = \tag{55}$$
$$= \frac{1}{4}(1 + 2<F^+_{sss}(t_0)F_{sss}(t_0)>)(1 - 2|M|\cos(\theta)).$$

$$<\Delta X_2^2(t)> = <\Delta X_2^2(t_0)> = \frac{1}{4}(1 + 2<\Delta a^+(t_0)\Delta a(t_0)>)(1-2\ \mathrm{Re}\ M) = \tag{56}$$
$$= \frac{1}{4}(1 + 2<F^+_{sss}(t_0)F_{sss}(t_0)>)(1 + 2|M|\cos(\theta)).$$

$$\langle\Delta X_1^2(t)\rangle\langle\Delta X_2^2(t)\rangle = \langle\Delta X_1^2(t_0)\rangle\langle\Delta X_2^2(t_0)\rangle =$$

$$= \frac{1}{16}\,(1 + 2\,\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle)^2(1 - 4|M|^2\cos^2(\theta)). \tag{57}$$

One can see that variances $\langle\Delta X_1^2(t)\rangle$ and $\langle\Delta X_2^2(t)\rangle$ turn into one another if the sign of $\cos(\theta)$ changes. Therefore we will consider only the case with $\cos(\theta) > 0$ meaning the squeezing in quadrature X_1. Further we note that the following derivative is positive:

$$\frac{\partial|M|}{\partial|f|} = \frac{\partial|M|}{\partial D}\,\frac{\partial D}{\partial|f|} = \frac{1}{\sqrt{\omega^2-4|f|^2}} > 0. \tag{58}$$

It means that $|M|$ is a monotonously growing function of the modulus of the self-coupling constant $|f|$ at fixed ω. At the same time $|M|$ is bounded at $\omega > 2|f|$ by virtue of (16). *Thus all values of variances (55)-(56) and uncertainty relation (57) depend on two parameters given on narrow fixed segments:* $0 \le \cos(\theta) \le 1$ *and* $0 \le |M| < 1/2$.

It should be mentioned here that if $|M| = 0$ then $|f| = 0$, so Hamiltonian (1) of the system becomes the Hamiltonian of the free oscillator. In this case the wave function $|sss\rangle_{ev}$ degenerates into the coherent state for the operator $a(t_0)$ corresponding to the eigenvalue $\bar{a}$. Naturally, the mean $\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle$ becomes equal to zero, the variances become minimal and equal to each other, the uncertainty relation becomes minimized. The wave function $|sss\rangle_{od}$, in its turn, changes over to the eigenstate of the $F^+_{sss}(t_0)F_{sss}(t_0)$ operator with the eigenvalue equal to one, while both the variances and the uncertainty relation for this state are not minimized.

As the value of the parameter $|M|$ becomes nonzero then the value of the parameter $|f|$ begins increasing by virtue of (58). Therefore to know the qualitative behaviour of the variances when the parameter $|f|$ increases from zero to $\omega/2$ it is sufficient to investigate the dependence of the variances in the $|M|$ parameter region from zero to one half. It is trivial to make this investigation numerically using the series of type (51) for calculation of the main quantity $\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle$.

Calculations of the variances were carried out with an accuracy of 10^{-10} on the "Microway 80386/387" computer for the following values of $\cos(\theta)$: 0.25; 0.50; 0.60; 0.70; 0.75; 0.80; 0.90; 0.95; 1.0 separately for the $|sss\rangle_{ev}$ and $|sss\rangle_{od}$ wave functions. There are not any reasons to calculate the above quantities for the superpositions of

$|sss\rangle_{ev}$ and $|sss\rangle_{od}$ because variances (55)-(56) have linear dependence with respect to $\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle$ which turns out to be simply a sum of corresponding contributions of the $|sss\rangle_{ev}$ and $|sss\rangle_{od}$ wave functions by virtue of their orthogonality (49) when we average $F^+_{sss}(t_0)F_{sss}(t_0)$ over (48):

$$\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle = (1-|B|^2)\ {}_{ev}\langle sss|F^+_{sss}(t_0)F_{sss}(t_0)|sss\rangle_{ev} + |B|^2\ {}_{od}\langle sss|F^+_{sss}(t_0)F_{sss}(t_0)|sss\rangle_{od}, \tag{59}$$

i.e. variances (55)-(56) are linear with respect to the contributions from each of the wave functions $|sss\rangle_{ev}$ and $|sss\rangle_{od}$.

As it should be expected the number of fluctuation quanta $\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle$ turned out to be a rapidly increasing function of $|M|$ both for the $|sss\rangle_{ev}$ and for the $|sss\rangle_{od}$ wave functions independently of the $\cos(\theta)$ value (Fig.1). In spite of this behaviour of $\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle$ the degree of the squeezing is unbounded at $\cos(\theta)=1$ both for $|sss\rangle_{ev}$ and for $|sss\rangle_{od}$ (Fig.2); in this case it means that in expression (55) the decrease of the factor $(1-2|M|)$ is faster than the increase of the factor $(1+2\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle)$ everywhere on the segment $0\le|M|<1/2$. However, if $\cos(\theta)\ne1$ then variance (55) has unexpected minimum, i.e. beginning from some $|M|$ value the factor $(1+2\langle F^+_{sss}(t_0)F_{sss}(t_0)\rangle)$ increases faster than the factor $(1-2|M|\cos(\theta))$ decreases. It is seen from Fig.2 that at any positive $\cos(\theta)$ value the variance $\langle\Delta X_1^2(t)\rangle$ over the $|sss\rangle_{ev}$ state has the interval corresponding to the squeezing. For the $|sss\rangle_{od}$ wave function the squeezing exists only for the $\cos(\theta)$ values close to one. The uncertainty relation is not minimized for both states and turns out to be an unbounded function of $|M|$ at all values of $\cos(\theta)$ (Fig.3).

It is also clear from the above consideration that the $|sss\rangle_{ev}$ and $|sss\rangle_{od}$ wave functions are not the Gaussian pure states (GPS)[18]. Actually, one can obtain all single-mode GPS by "rotating" the set of minimum uncertainty states [18]. It means that single-mode GPS have bounded uncertainty relation, in particular, for single-mode GPS one can always minimize the uncertainty relation choosing the phase of the field amplitude. On the contrary, in the case considered in this Letter the uncertainty relation does not depend on the phase of $\bar{a}$ at all. Thus, the $|sss\rangle_{ev}$ and $|sss\rangle_{od}$ states we found do not belong to the set of single-mode GPS.

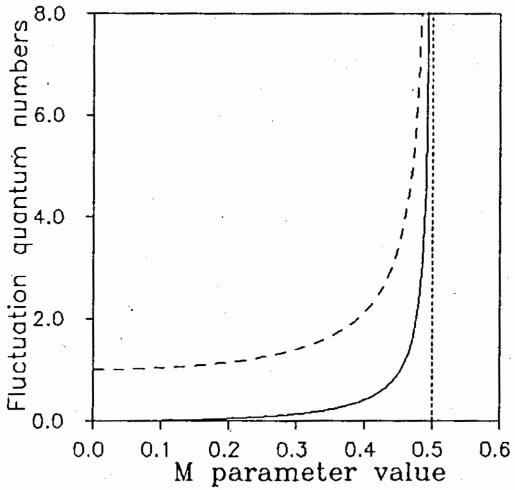

Fig.1. Number of the fluctuation quanta plotted against the value of the parameter $|M|$; solid line: for the wave function $|sss\rangle_{ev}$; dashed one: for the wave function $|sss\rangle_{od}$. Both lines go to infinity as $|M|$ approach the value 0.5.

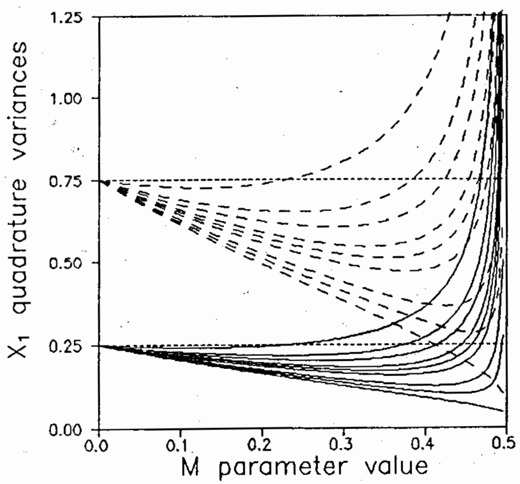

Fig.2. The value of $\langle\Delta X_1^2(t)\rangle$ plotted against the value of the parameter $|M|$ at $\cos(\theta)$ values: 0.25; 0.50; 0.60; 0.70; 0.75; 0.80; 0.90; 0.95; 1.0 both for the wave function $|sss\rangle_{ev}$ (solid lines) and for the wave function $|sss\rangle_{od}$ (dashed lines). The higher the value of $\cos(\theta)$ the lower is the location of the corresponding curve.

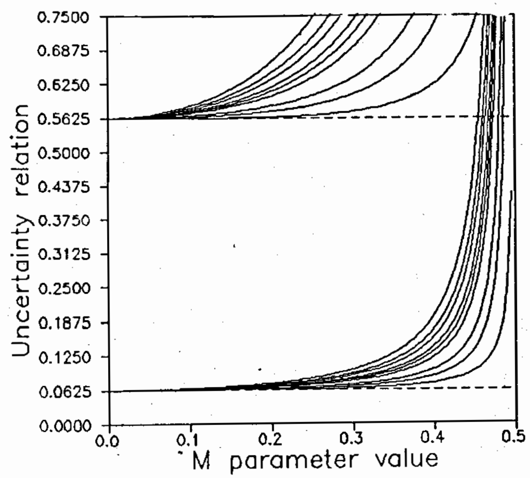

Fig.3. The uncertainty product $\langle\Delta X_1^2(t)\rangle\langle\Delta X_2^2(t)\rangle$ plotted against the value of the parameter $|M|$ at $\cos(\theta)$ values: 0.25; 0.50; 0.60; 0.70; 0.75; 0.80; 0.90; 0.95; 1.0 both for the wave function $|sss\rangle_{ev}$ (solid lines) and for the wave function $|sss\rangle_{od}$ (dashed lines). The higher the value of $\cos(\theta)$ the lower is the location of the corresponding curve.

5. Final remarks

The steady-squeezing states with an a priori given complex amplitude of the oscillator have been explicitly built up for the simplest linear system. The properties of these states have been described. On the other hand, the interpretation of the results obtained does not give a clear picture of the behaviour of the wave functions in time because the Heisenberg representation was used. So, we need the Schrödinger picture to follow the evolution of the states found. We will use the P-representation [19] for this purpose in a forthcoming article.

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Стационарно-сжатое состояние
для простейшей линейной системы

Определены волновые функции, реализующие постоянное сжатие одной из квадратурных компонент для осциллятора с самодействием при заданной комплексной амплитуде колебаний. Множество таких волновых функций образует ядро специального оператора и распадается на два взаимно-ортогональных подмножества: подмножество волновых функций, являющихся суперпозицией состояний с четным числом квантов флуктуаций над классическим значением энергии осциллятора, и подмножество волновых функций, построенных на состояниях с нечетным числом квантов флуктуаций. Величина стационарного сжатия как функция константы самодействия не монотонна, а имеет неожиданный минимум. Стационарно-сжатые состояния не минимизируют соотношение неопределенностей и не являются гауссовыми.

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Steady-Squeezing State for the Simplest
Linear System

Wave functions have been determined which realize constant squeezing for a quadrature component of the oscillator with self-action at an a priori given complex amplitude of the oscillator. The set of these wave functions forms a kernel of the special operator and is divided into two mutually orthogonal subsets: the subset of superpositions of the states with the even number of quanta of fluctuations over the classical value of the oscillator energy and the subset of superpositions of the states with the odd number of such quanta. The value of the squeezing is not a monotonous function of the self-action constant and it has an unexpected minimum. Steady-squeezing states do not minimize the uncertainty relation and are not Gaussian.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

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