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STABILITY OF THE MOVING BUBBLES IN THE BOSE CONDENSATE

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1. INTRODUCTION

This work is devoted to the so-called $\psi^3 - \psi^5$ nonlinear Schroedinger (NLS) equation:

$$i\psi_t + \Delta\psi - \alpha_1\psi + \alpha_3\psi \mid \psi \mid^2 - \alpha_5\psi \mid \psi \mid^4 = 0, \tag{1.1}$$

where $\Delta \equiv \partial^2/\partial x_1^2 + ... + \partial^2/\partial x_D^2$, and $\alpha_1, \alpha_3, \alpha_5$ are positive constants.[1] Equation (1.1) arises in a great number of physical contexts. Chronologically, the first one seems to be a gas of bosons interacting via the two-body attractive and three-body repulsive δ-function potential which is described quasiclassically just by (1.1) [1,2]. Other condensed matter applications include e.g. the ψ - theory of superfluidity [3] and the problem of defectons [4]; more recently similar equations were also discussed in connection with the high- T_c superconductivity [5]. In the theory of magnetism the problem of spin complexes and magnetic solitons can be reduced, under the weak excitation condition, to the solution of the very same eq.(1.1). For example, if we have the one-axis ferromagnet in the longitudinal magnetic field $H_z = H$, with the anisotropic energy $E_{an} = -\delta M_z^2/2 + \gamma M_z^4/4$, then the coefficients of eq.(1.1) have the following meaning: $\alpha_1 = \delta - \gamma + H$, $\alpha_3 = (\delta - 3\gamma)/2$, $\alpha_5 = -(\delta + 3\gamma)/2$ [6].

The applications of the $\psi^3 - \psi^5$ NLS equation are not limited to the condensed matter problems, however. It is met in various branches of biophysics [7], nonlinear optics [8], plasmas [9], and even in the nuclear hydrodynamics [10]. Depending on the context, the corresponding space dimension, D varies from 1 to 3.

In this paper we shall be concerned with the range of parameters such that $\alpha_1\alpha_5/\alpha_3^2 < 1/4$. Here, introducing $\rho_0 > 0$ and A,

$$\frac{A}{\rho_0} = -2 + \frac{3}{4}\,\frac{\alpha_3^2}{\alpha_1\alpha_5}\left(1 + \sqrt{1 - 4\frac{\alpha_1\alpha_5}{\alpha_3^2}}\right)$$

($0 < A/\rho_0 < 1$), and making the transformation $\psi(x,t) = \gamma\phi(\beta x, \beta^2 t)$ with $\beta^2 = \frac{1}{2}\alpha_5\gamma^4$ and $\gamma^2 = \frac{3}{2}(\alpha_3/\alpha_5)(A+2\rho_0)^{-1}$, eq.(1.1) can be rewritten as

$$i\phi_t + \Delta\phi + (\mid \phi \mid^2 - \rho_0)(2A + \rho_0 - 3 \mid \phi \mid^2)\phi = 0. \tag{1.2}$$

This form is more convenient for finding solutions.

[1] There is no loss of generality in requiring that α_1 be positive, for the trivial substitution $\psi(x,t) = e^{i\omega t}\tilde{\psi}(x,t)$ changes α_1 to any desired value: $\tilde{\alpha}_1 = \alpha_1 + \omega$.

In applications both the vanishing boundary conditions are considered, $\phi(\mathbf{x},t) \rightarrow 0$ and the non-vanishing ones:

$$| \phi(\mathbf{x},t) |^2 \rightarrow \rho_0 \quad as \ \mathbf{x}^2 \rightarrow \infty. \tag{1.3}$$

It is the latter case that will be dealt with below. Since in the boson gas problem the homogeneous time-independent solution $\phi \equiv \sqrt{\rho_0}$ of eq.(1.2) is usually interpreted as the Bose condensate, the conditions (1.3) are called "the condensate boundary conditions". [In this paper we always apply to the boson gas interpretation of eq. (1.1) for it seems to be the most explicit.]

In Ref.[11] the equation (1.2) was found to possess, under the conditions (1.3), a new type of soliton solutions describing rarefaction bubbles in the condensate.[2] These nontopological solitons exist at $0 < A < \rho_0$ and have the remarkable property to survive passing to arbitrary higher dimensions. (In the case of lumps[3] this property is not surprising but it becomes nontrivial for solitons with non-vanishing boundary conditions. To compare, note that the kinks and vortices of the repulsive ψ^3 NLS *do not* have stationary localized analogues in $D = 3$ [12]). However, regardless of the dimension, the *quescent* "bubbles" proved unstable [11,13].

In the present study we address ourselves to the question of whether the *travelling* bubble can be stable. Here our treatment will be restricted to the one-dimensional situation for in this case the bubble solution to eq. (1.2) is known explicitly [11]:

$$\phi_s(x,t) = \phi_s(\tilde{x}) = \frac{\sqrt{2\rho_0}\cosh(\xi/2 - i\mu)}{\sqrt{(2\rho_0 - A)(A^2+v^2)^{-1/2} + \cosh\xi}} \quad , \tag{1.4}$$

where $\xi = (c^2 - v^2)^{1/2}\tilde{x}$,

$$\tilde{x} = x - vt, \tag{1.5}$$

v denotes the velocity of the soliton ($| v | < c$), and $c = [4\rho_0(\rho_0 - A)]^{1/2}$ is the velocity of sound. The "twisting angle" μ is also defined by v:

$$\sin 2\mu = \frac{v}{2}\left(\frac{c^2 - v^2}{A^2 + v^2}\right)^{1/2} \quad , \tag{1.6}$$

[2] Actually we believe that the solitonic bubbles can have a definite interpretation in each field where the $\psi^3 - \psi^5$ equation arises, and there should be concrete physical effects that the bubbles are responsible for. The first results in this direction are presumably due to L.Masperi who has pointed out in his recent work [14] that this type of excitations is capable of explaining the phase diagram of the $He^3 - He^4$ mixture.

[3] As usual, by lumps the multidimensional solitons are meant that are localized along *any* direction.

$-\pi/4 < \mu < \pi/4$. Asymptotically we have

$$\phi_s(\tilde{x}) \rightarrow \sqrt{\rho_0}e^{\mp i\mu} \qquad as \qquad \tilde{x} \rightarrow \pm\infty. \tag{1.7}$$

We expect the $D = 1$ case to have much in common with the two- and three- dimensional situations that we plan to analyse in future publications.

It was conjectured earlier on the base of some approximate analysis that one-dimensional bubbles can stabilize when moving sufficiently fast [11]. Later, numerical simulations were performed [15] revealing that the bubble becomes anomalously long-lived when v exceeds certain velocity v_c. However, the authors of Ref.[15] were unable to determine whether the bubble becomes really stable or the instability growth rate diminishes extremely but remains non-zero at $v > v_c$.

Our main result here[4] is that the critical velocity v_c such that the "bubble" is stable in the supercritical domain, indeed exists. This conclusion is obtained in the numerical solution of eq.(1.2) linearized in the vicinity of the soliton (1.4). We also provide a simple qualitative explanation of the critical velocity appearance as well as of the slow bubble's instability itself, in terms of the Bose gas problem.

2. LINEARIZED STABILITY

In order to examine the stability of the "bubble" (1.4) against small perturbations we take a solution to eq.(1.2) of the form $\phi(x,t) = \phi_s(\tilde{x}) + \delta\phi(x,t)$ and linearize with respect to $\delta\phi$. Assuming for $\delta\phi$ the ansatz $\delta\phi(x,t) = [f(\tilde{x}) + ig(\tilde{x})]e^{\lambda t}$ with $\tilde{x}$ as in (1.5) and f, g, λ real, we arrive at the eigenvalue problem

$$H_v y = \lambda J y \tag{2.1}$$

$$y(\pm\infty) = 0, \tag{2.2}$$

where

$$H_v = -\frac{d^2}{dx^2}\, I \ + v\frac{d}{dx}\, J + U_v(x), \tag{2.3}$$

$$y(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

I is the identity matrix, and

$$U_v(x) = -\begin{pmatrix} F + 2F_\rho\phi_R^2 & 2F_\rho\phi_R\phi_I \\ 2F_\rho\phi_R\phi_I & F + 2F_\rho\phi_I^2 \end{pmatrix}. \tag{2.4}$$

[4]This result has been announced briefly in [16].

In eq.(2.4) ϕ_R and ϕ_I stand for the real and imaginary part of eq.(1.4): $\phi_s(\tilde{x}) = \phi_R(\tilde{x}) + i\phi_I(\tilde{x})$, and $\rho = |\phi_s|^2 = \phi_R^2 + \phi_I^2$. Finally, $F = F(\rho) = (\rho - \rho_0)(2A + \rho_0 - 3\rho)$, $F_\rho = dF/d\rho = 2(A + 2\rho_0 - 3\rho)$, and the tilde over x has been omitted in (2.3)-(2.4).

It is not difficult to realize that any solution of eq.(1.2) depends, up to a scaling transformation, only on a single parameter A/ρ_0. Accordingly, we set, without loss of generality, $\rho_0 = 1$. The eigenvalue problem (2.1)-(2.2) was analysed then numerically at 20 equally spaced values of A from the range (0,1). For each A a single positive eigenvalue $\lambda = \lambda_1$ was found, depending continuously on v.

The function $\lambda_1(v)$ which looks similar for each A, is depicted in Fig.1. It is seen that $\lambda_1(v)$ tends to zero as v approaches some critical velocity v_c, indicating that the soliton is stable for $v \geq v_c$. To determine the stability domain it remains only to find the precise value of v_c; actually, we have to be able to compute v_c to any prescribed accuracy. However, at this point we meet with certain computational difficulties whose nature can be understood by analysing the structure of the spectrum of the operator $J^{-1}H_v$.

3. THE SPECTRUM STRUCTURE

First, let us determine the location of the continuous spectrum. At spatial infinities we have $\phi_R(\pm\infty) = \sqrt{\rho_0}\cos\mu$, $\phi_I(\pm\infty) = \mp\sqrt{\rho_0}\sin\mu$,

$$U_v(\pm\infty) = c^2 \begin{pmatrix} \cos^2\mu & \mp\sin\mu\cos\mu \\ \mp\sin\mu\cos\mu & \sin^2\mu \end{pmatrix},$$

and the characteristic equation for the system (2.1) is readily found:

$$k^4 + c^2k^2 + (\lambda - ivk)^2 = 0 \tag{3.1}$$

(we assume $y(x) \propto e^{ikx}$). Now suppose $\lambda = i\nu$ with ν real, $\nu \neq 0$. Eq.(3.1) yields then

$$\nu = vk \pm |k| \sqrt{k^2 + c^2}. \tag{3.2}$$

From (3.2) it follows that for any $\lambda = i\nu \neq 0$ eq.(3.1) has two real and two complex-conjugate roots. It is not difficult to show that in such a circumstance eq.(2.1) has two independent bounded solutions, so that the imaginary axis of λ is occupied by the double-degenerate continuous spectrum.

Next, let $\lambda = 0$. Then eq.(3.1) has two coinciding zero roots which we shall denote $k_\pm$, $k_+ = k_- = 0$, and a pair of complex conjugate roots, $k_{3,4} = \pm i\sqrt{c^2 - v^2}$. Consequently, we can have not more than two independent bounded solutions here, and in fact there are exactly two. The first one corresponds to the translational symmetry of eq.(1.2),

$$y_0(x) = \frac{d}{dx}\begin{pmatrix} \phi_R(x) \\ \phi_I(x) \end{pmatrix}, \tag{3.3}$$

where $\phi_R(\tilde{x})$ and $\phi_I(\tilde{x})$ are the real and imaginary part of the "bubble" $\phi_s(\tilde{x})$. The second zero mode pertains to the U(1) symmetry:

$$\hat{y}_0(x) = \begin{pmatrix} -\phi_I(x) \\ \phi_R(x) \end{pmatrix}. \tag{3.4}$$

[That y_0 and $\hat{y}_0$ obey eq.(2.1) with $\lambda = 0$ follows from the nonlinear equation (1.2) just by substituting $\phi(x,t) = \phi_s(\tilde{x})$]. As $|x| \to \infty$, y_0 corresponds to $k_{3,4}$: $y_0(x) \to exp(-\sqrt{c^2 - v^2} \mid x \mid)$ and $\hat{y}_0$ corresponds to $k_\pm : \hat{y}_0(x) \to const.$

Finally we consider the case of real λ. In terms of $\kappa = ik$ eq.(3.1) is rewritten as

$$\kappa^4 - c^2\kappa^2 + (\lambda - v\kappa)^2 = 0. \tag{3.5}$$

The function $\lambda(\kappa)$ defined in this way is plotted in Fig.2. It is seen that when $|\lambda| < \tilde{\lambda}$ (where $\tilde{\lambda}$ is the smaller maximum of $\lambda(\kappa)$), eq.(3.5) has 4 real roots, $\kappa_3 < \kappa_- < 0 < \kappa_+ < \kappa_4$. Consequently, the eigenfunction $y_1(x)$ pertaining to the positive eigenvalue λ_1 whose finding was reported in the previous section, satisfies, for $\lambda_1 < \tilde{\lambda}$

$$\begin{array}{lll} y_1 \to C_- exp\{\kappa_+ x\} & as & x \to -\infty \\ y_1 \to C_+ exp\{\kappa_- x\} & as & x \to +\infty \end{array}, \tag{3.6}$$

where $C_\pm$ are constant columns. As $\lambda_1 \to 0$ we have $\kappa_\pm \to 0, \kappa_3 \to -\sqrt{c^2 - v^2}$, $\kappa_4 \to \sqrt{c^2 - v^2}$.

So, what happens to $y_1(x)$ when $v \to v_c - 0$ and $\lambda_1(v) \to 0$? Since at $\lambda = 0$ the eigenfunctions (3.3) and (3.4) exhaust the set of all bounded solutions, $y_1(x)$ should pass to the linear combination $c_0 y_0(x) + \hat{c}_0 \hat{y}_0(x)$, $c_0, \hat{c}_0 = const$. No additional zero modes emerge at $v = v_c$, and this point is in no way isolated therefore.

Thus one cannot determine v_c simply as such v for which $\lambda_1(v)$ vanishes. In order to find the point of intersection of the curve $\lambda_1(v)$ with the abscissa axis one has to apply some sort of extrapolation procedure instead. Our extrapolation was based on a Taylor series expansion[5]

$$\lambda_1(v) = c_1(v - v_c) + c_2(v - v_c)^2 + ... \tag{3.7}$$

in the vicinity of v_c (for $v \leq v_c$).

4. NUMERICAL COMPUTATION

Suppose v approaches $v_c - 0$ and $\kappa_\pm \to 0$. Then if we wanted the boundary conditions (2.2) to be satisfied to any prescribed accuracy, we would have to extend the integration interval to infinity. This computational difficulty can be circumvented by invoking the asymptotics (3.6) and passing to the conditions of the form $y_x - \kappa_+ y = 0$ at $x = -\infty$ and $y_x - \kappa_- y = 0$ at $x = +\infty$. Next, for $\lambda \to 0$ eq.(3.5) yields $\kappa_\pm = \lambda/(v \pm c) + O(\lambda^2)$ so that in the vicinity of the critical velocity the latter conditions simplify to

$$\left(\frac{dy}{dx} - \frac{\lambda}{v \mp c} y\right)\Bigg|_{x=\pm\infty} = 0. \tag{4.1}$$

Our computational policy was to solve the *inverse* spectral problem, i.e., for several sufficiently small values of λ we solved the problem (2.1),(4.1) to determine the corresponding v. In doing this we utilized the modified continuous analogue of Newton's method; details of the numerical algorithm are relegated to a separate paper [17]. We have found that $c_1 \neq 0$ in the expansion (3.7). The critical velocity was obtained then by the linear extrapolation of the resulting curve $\lambda_1(v)$ to $\lambda_1 = 0$. The values of v_c pertaining to different A are collected in Table 1; the dependence $v_c(A)$ is illustrated by Fig.3 .

It is seen that when $A/\rho_0 \to 1, v_c \to c/2$ holds which is in complete agreement with earlier results. Namely, in Ref.[11] it was shown that for $A \sim \rho_0$, eq.(1.2) reduces approximately to the Boussinesq equation whereas the Boussinesq soliton is known to be stable just for $v \geq c/2$.

5. INTERPRETATION

1. Although the analitical proof of the static "bubble's" instability is already available [11,13], no qualitative interpretation of this phenomenon

[5]That the function $\lambda_1(v)$ is analitical at $v = v_c$ is suggested by the results of numerical computation. However, the extrapolation procedure was organized in such a way that even if there were a singularity at this point, our results for v_c would nevertheless remain true within the same accuracy.

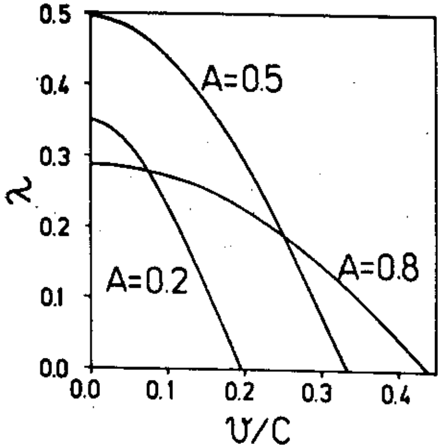

Fig. 1 *The instability growth rate λ_1 versus v/c, the velocity of the soliton in units of the sound velocity ($c = 2\sqrt{\rho_0 - A}$).*

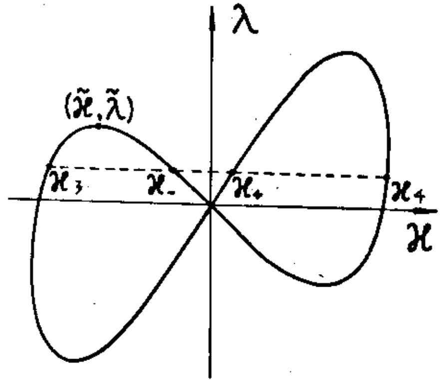

Fig. 2 *The function $\lambda(\kappa)$ defined by eq.(3.5) for $v > 0$. The left maximum, $\tilde{\lambda}$ is attained at $\tilde{\kappa} = -\left(4c^2 - v^2 - \sqrt{v^4 + 8c^2v^2}\right)^{1/2}/\sqrt{8}$ and equals $\tilde{\lambda} = \{(v^2 - c^2)\tilde{\kappa} + 2\tilde{\kappa}^3\}/v$.*

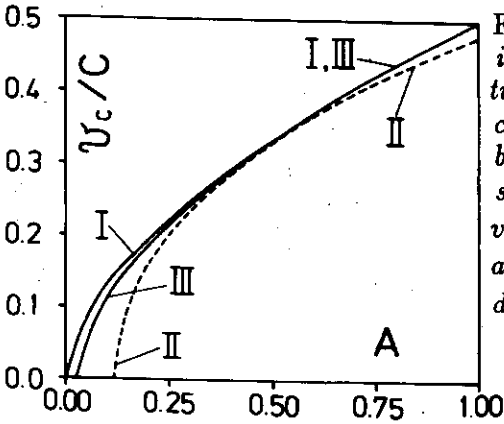

Fig. 3 *The critical velocity v_c and its one- and two-mode approximations plotted against A. The solid curve I indicates v_c as computed by solving the eigenvalue problem, sec.4. The dashed line II presents v_a defined as a root of $E_{aa}(v) = 0$, and the solid curve III shows $v_{\mathcal{D}}$ defined by $\mathcal{D}(v_{\mathcal{D}}) = 0$ (sec.5).*

Table 1

A	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v_c/c	0.1301	0.1943	0.2461	0.2911	0.3317	0.3692	0.4044	0.4377	0.4695

The critical velocities (velocities of stabilization), v_c, for different values of the parameter A. The accuracy is 10^{-4}, i.e., the last given figure differs from the true one not more than by one.

has been given so far. Also it would be very useful to realize physically why the *moving* soliton stabilizes. Here we propose a simple qualitative explanation of the both effects in the Bose gas terms.

Our treatment will be based on the integrals of motion of eq.(1.2). In the Bose gas terminology, these are energy:

$$E = \int \{ |\phi_x|^2 + (|\phi|^2 - \rho_0)^2 (|\phi|^2 - A)\} dx \qquad (5.1)$$

$$= \int \{ |\phi_x|^2 + |\phi|^6 - (A + 2\rho_0) |\phi|^4 + \rho_0(2A + \rho_0) |\phi|^2 - A\rho_0^2 \} dx; \qquad (5.2)$$

momentum:

$$P = \tfrac{1}{2} i \int (\phi_x^* \phi - \phi_x \phi^*)(1 - \rho_0 / |\phi|^2) dx$$

$$= \tfrac{1}{2} i \int (\phi_x^* \phi - \phi_x \phi^*) dx - \rho_0 Arg\phi \Big|_{-\infty}^{+\infty}, \qquad (5.3)$$

and the total number of particles lacking in the condensate:

$$N = \int (|\phi|^2 - \rho_0) dx. \qquad (5.4)$$

The last term in eq(5.3) is essential for solutions with non-trivial topology[6] i.e.,for those obeying $\phi(+\infty) \neq \phi(-\infty)$. If we use eq.(5.3) to evaluate the momentum of the soliton, we ensure that the relation of the Hamilton mechanics holds: $dE/dP = v$. This fact has been observed by Ishikawa and Takayama [19] in the case of the moving kink of the repulsive ϕ^3 NLS equation . (Kulish *et. al.* [18] also use the definition (5.3) but their motivation is presumably different.) In fact, it is not difficult to demonstrate that adopting this definition for the momentum, $dE/dP = v$ holds in NLS equations with *any* nonlinearity.

We shall show that at $v \geq v_c$ the "bubble" realizes a minimum of E under P and N fixed, while at $v < v_c$ the minimum is changed into a saddle. Identifying the unstable mode (i.e., the direction of the energy decrease) we shall become able to understand both the nature of instability and the mechanism of the stabilization at higher velocities.

2. So, let us pass to the analysis of the energy minima under the condition that both P and N be fixed, $\delta P = \delta N = 0$. First of all, we

[6]The idea that it is just the momentum integral with this additional term that is relevant for stability analysis, has been advanced first in the paper by Bogdan, Kovalev and Kosevich [6]. We are indebted to M.M.Bogdan for drawing it to our attention.

observe that in the case of bubbles the second condition can be ignored i.e., the minimum of E under both $\delta P = \delta N = 0$ can be always attained by a configuration with only $\delta P = 0$. The proof is a simple generalization of the one given in Ref.[13] for the $v = 0$ case, and we only sketch it here.

For travelling waves $\phi = \phi(x - vt)$ eq.(1.2) amounts to the requirement $(\delta E)_P = 0$, with N not necessarily being preserved by the variations. Next, the second variation is given by

$$\delta^2 E = \int y^T(x) H_v y(x) dx \ , \qquad (5.5)$$

where H_v and y are as in (2.3-2.4), $y^T = (f, g)$, $f = Re\delta\phi$, $g = Im\delta\phi$. The constraint $\delta N = 0$ is equivalent to $\mathcal{F}[f, g] = 0$, where

$$\mathcal{F}[f,g] \equiv 2 \int \{\phi_R(x) f(x) + \phi_I(x) g(x)\} dx \qquad (5.6)$$

is a linear functional in the space of vector-functions (f, g). [We recall that $\phi_R(\tilde{x})$ and $\phi_I(\tilde{x})$ are the real and imaginary part of the "bubble" (1.4), respectively]. In view of the non-vanishing asymptotics of $\phi_s(\tilde{x})$, we have by the Riesz theorem that $\mathcal{F}$ is discontinuous both in $\mathcal{L}^2$ and in Sobolev's space $\mathcal{W}^1$. This fact implies, in turn, that the kernel space of $\mathcal{F}$ (defined by the condition $\mathcal{F}[f, g] = 0$), is dense in $\mathcal{W}^1$ (see e.g. [20]). Speaking otherwise, for any (f_0, g_0) there exists a sequence (f_n, g_n) obeying $\mathcal{F}[f_n, g_n] = 0$ and converging to (f_0, g_0) in Sobolev's norm,

$$\|(f,g)\|_{\mathcal{W}} \equiv \int \{f^2 + g^2 + f_x^2 + g_x^2\} dx.$$

Since the quadratic functional (5.5) is continuous in $\mathcal{W}^1$, $\delta^2 E[f_n, g_n]_{n\to\infty} \to \delta^2 E[f_0, g_0]$ follows. Thus $min(\delta^2 E)_{P,N} = min(\delta^2 E)_P$.

Defining r and θ by $\phi = \sqrt{\rho} e^{i\theta}$ and $r = \rho_0 - \rho$, eqs.(5.3),(5.1-5.2) read

$$P = -\int r\theta_x dx, \qquad (5.7)$$

$$E = E^0 + E^{int},$$

$$E^{int} = \int \{(\rho_0 - r)^3 - (A + 2\rho_0)(\rho_0 - r)^2 + \rho_0^2 (A + \rho_0)\} dx,$$

$$E^0 = \int \{\tfrac{1}{4} r_x^2 (\rho_0 - r)^{-1} + \theta_x^2 (\rho_0 - r) + \rho_0 (2A + \rho_0)(\rho_0 - r) - \rho_0^2 (2A + \rho_0)\} dx. \qquad (5.8)$$

In eq.(5.8) E^0 stands for the contribution of the dispersion and E^{int} for that of the nonlinearity. In other words, E^0 corresponds to the kinetic energy of bosons while E^{int} to their interaction. Now consider r and θ pertaining to the "bubble" (1.4): $\phi_s = \sqrt{\rho_s} exp(i\theta_s), r_s = \rho_0 - \rho_s$ and introduce a simple perturbation of the form

$$r_s(x) \to r(x) = ar_s(bx), \quad \theta_s(x) \to \theta(x) = a^{-1}\theta_s(bx) \tag{5.9}$$

preserving the momentum (5.7). The energy (5.8) becomes then the function of a and b :

$$E^0(a,b) = \tfrac{1}{4}a^2 b \int r_x^2(\rho_0 - ar)^{-1} dx + ba^{-2} \int \theta_x^2(\rho_0 - ar)dx$$
$$-ab^{-1}\rho_0(2A + \rho_0) \int r dx,$$

$$E^{int}(a,b) = b^{-1} \int \{(\rho_0 - ar)^3 - (A + 2\rho_0)(\rho_0 - ar)^2$$
$$+\rho_0^2(A + \rho_0)\} dx, \tag{5.10}$$

where the subscript "s" has been omitted at r and θ. The condition of that $E(a,b)$ considered as a function of a have a minimum at $a = b = 1$, is $(\partial^2 E/\partial a^2)|_{a=b=1} > 0$ while the minimum w.r.t. both variables occurs when

$$\mathcal{D} = (E_{aa}E_{bb} - E_{ab}^2)\Big|_{a=b=1} > 0. \tag{5.11}$$

From (5.10) we obtain the necessary derivatives:

$$E_{aa}^0 = \tfrac{1}{2}\rho_0^2 \int r_x^2(\rho_0 - r)^{-3} dx + 2\int \theta_x^2(3\rho_0 - r)dx \tag{5.12}$$

$$E_{aa}^{int} = 6\int r^2(\rho_0 - r)dx - 2(A + 2\rho_0)\int r^2 dx \tag{5.13}$$

$$E_{ab}^0 = \tfrac{1}{4}\int r_x^2(2\rho_0 - r)(\rho_0 - r)^{-2} dx - \int \theta_x^2(2\rho_0 - r)dx$$
$$-\rho_0(2A + \rho_0)\int r dx \tag{5.14}$$

$$E_{ab}^{int} = \int \{3r^3 - 2(\rho_0 - A)r^2 + \rho_0(2A + \rho_0)r\} dx \tag{5.15}$$

$$E_{bb}^{int} = 2\int\{-r^3 + (\rho_0 - A)r^2 + \rho_0(\rho_0 + 2A)r\}dx \tag{5.16}$$

$$E_{bb}^{0} = -2\rho_0(2A + \rho_0)\int r dx. \tag{5.17}$$

The expressions (5.12), (5.14) can be simplified by means of the relation $\theta_x = -\frac{1}{2}vr/(\rho_0 - r)$ which follows from eq.(1.2) in the case of $\phi(x,t) = \phi(x - vt)$.

Having set $\rho_0 = 1$ in (1.4), we evaluated the integrals (5.12)-(5.17). At small velocities, both E_{aa} and $\mathcal{D}$ were found to be negative. However, there are certain velocities v_a and $v_{\mathcal{D}}$, $v_a < v_{\mathcal{D}} < v_c$ such that $E_{aa} > 0$ for $v > v_a$, and $\mathcal{D} > 0$ for $v > v_{\mathcal{D}}$. The curves $v_a(A)$ and $v_{\mathcal{D}}(A)$ are depicted in Fig.3. It is seen that for not too small A's, $v_{\mathcal{D}}$ and even v_a provide rather good approximations for v_c. Accordingly, the instability of the "bubble" can be interpreted qualitatively as the instability w.r.t. the a mode.

The negative contribution to E_{aa} comes exclusively from the part of the energy corresponding to interaction of bosons. More precisely, it comes from the second item in (5.13) arising from the term $-(A + 2\rho_0) \mid \phi \mid^4$ in the Hamiltonian (5.2). Physically, this term corresponds to the two-body attraction so that it is just this part of interaction that is responsible for the slow bubbles' instability.

4. In order to illustrate the latter statement, it proves helpful to consider the following toy model of the bubble. Let $\phi(x)$ be a real static configuration with

$$\phi^2(x) = \rho(x) = \begin{cases} \rho_0, & \mid x \mid > L/2 \\ \tilde{\rho}, & \mid x \mid < L/2 \end{cases}, \tag{5.18}$$

$\tilde{\rho} = const, 0 < \tilde{\rho} < \rho_0$. The energy of bosons' interaction is, by eq.(5.8), $E^{int} = L\{\tilde{\rho}^3 - (A + 2\rho_0)\tilde{\rho}^2 + A\rho_0^2 + \rho_0^3\}$. On the other hand, the number of particles (5.4) is $N = (\tilde{\rho} - \rho_0)L$ so that eliminating $\tilde{\rho}$ we have

$$E^{int}(L) = N^2L^{-1}(\rho_0 - A + N/L) - \rho_0(2A + \rho_0)N. \tag{5.19}$$

The force acting on the bubble's wall consists of two parts. The first part, $f^0 = -dE^0/dL$ corresponds to the kinetic energy of bosons. For a bubble of any form this part is obviously positive, since it tries to disperse the bubble. The second part originates from the interaction of bosons inside the bubble:

$$\begin{aligned} f^{int} = -dE^{int}/dL &= N^2L^{-2}(2NL^{-1} - A + \rho_0) \\ &= (\tilde{\rho} - \rho_0)^2(2\tilde{\rho} - \rho_0 - A). \end{aligned} \tag{5.20}$$

For $\tilde{\rho} > (\rho_0 + A)/2$, i.e., when the density of the gas inside the bubble is high, the three-body repulsion dominates and we have $f^{int} > 0$. At the densities lower than $(\rho_0 + A)/2$ the gas inside is, conversely, self-gravitating so that the pressure is negative: $f^{int} < 0$. In order to ensure $f^0 + f^{int} = 0$, the internal density must therefore be lower than at least $(\rho_0 + A)/2$. Hence, only high-rarefaction bubbles can exist.

Now let us proceed to stability properties. The necessary condition for stability is $d^2E/dL^2 > 0$. For the interaction part of the energy we have

$$d^2E^{int}/dL^2 = 2(\rho_0 - \tilde{\rho})^2(3\tilde{\rho} - 2\rho_0 - A). \tag{5.21}$$

We observe that $d^2E^{int}/dL^2 > 0$ holds only for sufficiently *high* densities, $\tilde{\rho} > \frac{2}{3}\rho_0 + A/3$. This means that the repulsive interaction tries to stabilize the configuration, whereas the attraction of bosons produces instability. So our toy model helps to realize qualitatively why the static bubbles are always unstable. Namely, in order to exist the bubble must be sufficiently rarefied so that the attraction of bosons can compensate the dispersive "dissolution". On the other hand, it is just the bubbles of the rarefied, self-gravitating gas that appear to be unstable.

5. Finally, it remains to explain in qualitative terms the critical velocity occurrence. As is seen from Fig.3, it is quite sufficient to take into account only the a mode.

When v is increased, the width of the soliton increases as well, while the amplitude [i.e., the maximum of $r(\tilde{x})$] decreases. The latter means that the density of the boson gas inside the bubble grows, the three-body repulsive interactions become dominating and as a result, E^{int}_{aa} grows from negative to positive values. On the other hand, both two items that form the dispersion part of E_{aa} [eq.(5.12)] always stay positive. Therefore, *the very fact* of the stabilization can be explained just by the density growth; however if we want to find a qualitative estimate of the *critical velocity*, the change of the dispersion part should be also taken into account.

The velocity increasing, the first item in (5.12) diminishes while the second one grows. This latter item, $2\int \theta_a^2(3\rho_0 - r)dx$ is related to the "twisting" of the soliton, i.e., it has a topological origin. The total increase in the topological item over the range $(0, v_a)$ is about half of the corresponding change of E^{int}_{aa}. Consequently, we may conclude that the stabilization occurs mainly due to the density growth inside the bubble. The topological effects also contribute but to a less extent.

As can be seen from Fig.3, when $A \to 0$ the above choice of perturbations does not provide a faithful description of the instability. The nature of the instability is presumably the same in this case, but a different type of distortions is required to demonstrate this.

6. CONCLUDING REMARKS AND OPEN PROBLEMS

1. So we have shown that the *one-dimensional* bubble-like solitons stabilize when moving sufficiently fast. It would be interesting to find out whether analogous critical velocities exist for two- and three-dimensional "bubbles" which are also known to be unstable at rest. In this connection it is worthwhile to recall [11] that propagation of *transonic* "bubbles" is governed approximately by the Korteweg - de Vries equation at $D = 1$, and by the two- and three-dimensional Kadomtsev-Petviashvili equations at $D = 2$ and 3. On the other hand, the KdV soliton as well as the two-dimensional KP lump is known to be stable while the three-dimensional lump unstable. This suggests that the critical velocity v_c such that the $\psi^3 - \psi^5$ NLS "bubble" is unstable for $v < v_c$ and stable at $v \geq v_c$, exists only for $D = 1$ and 2. The three-dimensional "bubble", conversely, is expected to be unstable at *any* velocity.

2. For A close to zero we have not succeeded in describing the instability of the bubble as instability w.r.t. simple scaling distortions. From the interpretational viewpoint it would be desirable therefore to find an analitically expressed perturbation such that it could describe the instability threshold for all A universally.

3. Finally, let us remark that when one analyses NLS equations, the dependence of the soliton's stability on its velocity is inherent only for solitons with non-vanishing boundary conditions. Really, consider the case of the *vanishing* conditions $\phi(x,t)_{|x|\to\infty} \to 0$ and suppose we have a lump $\phi_s(x,t)$ moving with velocity v. The Galilean transformation

$$\phi(x,t) \to \tilde{\phi}(x,t) = e^{-\frac{1}{2}iv(x+\frac{1}{2}vt)}\phi(x+vt,t) \tag{6.1}$$

takes then the soliton ϕ_s to the rest frame. Furthermore, this transformation can be applied to any nearby solution, so that the stability problem will be reduced to the one for $v = 0$. On the other hand, in the case of the boundary conditions (1.7) the transformation (6.1) does not take the travelling soliton to the static one; the dependence on the velocity is more complicated here, see e.g. eq.(1.4).

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