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DYNAMIC MODEL FOR THE SINGLE-MODE LASER.
Regime of the stable stationary generation

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1. Goals of the work and initial dynamic model

The only reliable way of finding out the dynamic model applicability is probably its qualitative study. Indeed, the dynamic models usually have one or several parameters that are varied together with the initial conditions in the given variation region. The numeric simulation, i.e. the numeric solution of the Cauchy problem, yields the results for the countable set of parameter and initial condition values,so qualitatively different solutions are known to be able to have the initial conditions, for example, in the zero measure set among the set of possible initial conditions.In these circumstances the qualitatively differrent solution may remain unnoticed by the researcher busy with the discrete sorting of the initial conditions.

The pendulum equilibrium points are a simple example of this. The lower point of pendulum suspension is the only stable point of equilibrium, while the upper point of pendulum suspension is only unstable point of equilibrium.

On the other hand, it is desirable that the qualitative study was a rigorous one. Being approximate in any sense, the qualitative study of the dynamic model (either the study of the equations "approximating" the initial ones, or the study of solutions obtained by the pertubation theory) may yield qualitative results different from those of the rigorous and correct consideration. An example is Ref.[1], where it was rigorously shown that all solutions of a dynamic model [2], physically corresponding to superradiance, are asymptotically stable,though the study of the equations considered to be an approximaton to the initial ones showed that there were unstable solutions of physical sense [2-4].In this case the rigorous study showed that there was no "superradiant threshold"[3,4]. In this paper a dynamic model of the single-mode laser with two-level emitters and non-coherent pumping is to be considered. The model, widely known in quantum optics, is naturally obtained by averaging the operator equations in the interaction representation for a single-mode field,which interacts with two-level atoms, with relaxations taken into account in the simplest way[5]. The qualitative study of the stationary solutions of the model was carried out by Haken approximately[6]; it was widely sited in the publications dealing with self-organization processes.

The more interesting is the fact that the asymptotic stability of the stationary solutions above the generation threshold, obtained in Ref.[6], does not occur at almost all physical values of the parameters appearing in the investigated system of ordinary differential equations (see the comments). As shown below, these solutions can be either simply stable and, consequently, unstable with allowance for an external random force, or, what is even more unexpected, they can be exponentially unstable in a region of parameter values above the deneration threshold.

Let us also clarify the terminology. A specific feature of the dynamic model considered is that some of its stationary solutions do not allow study of stability by the linear approximation. The latter is due to the fact that the matrix of linear part of the given system is degenerate, i.e. its determinant is equal to zero. So there are zero roots of the characteristic equation of this matrix. In the stability theory this is called a critical case if among the non-zero roots there are no roots with the positive real part [7-9]. The physical discussion of the rigorous results obtained is to be found at the end of the paper.

A system of ordinary differential equations describing a single-mode laser with non-coherent pumping can be written down as [5-6]

$$\dot{b} = - \kappa b - ig\alpha ,$$

$$\dot{\alpha} = - \gamma \alpha + igb S, \tag{1}$$

$$\dot{S} = \frac{N d_0 - S}{T} + 2 i g (\alpha b^* - b \alpha^*).$$

In these equations $b^*$and b are the mean from the photon creation and annihilation operators, $\alpha^*$and α are the mean from the operators of the transition from the lower level to the upper one and back. The operators are taken in the interaction representation. The variable S is a difference in the number of excited and non-excited emitters. **N** is a total number of emitters, $\kappa > 0$ is the rate of the field relaxation due to radiation, $\gamma > 0$ is the rate of the transverse relaxation of the emitters, $T > 0$ is the time of longitudinal relaxation of the emitters, $g > 0$ is the atom-field interaction constant, d_0 is the pumping intensity ,which is the bifurcational parameter.

On the right-hand side of eq.(1) there can be a randomly time-dependent force. Its effect on the stability of the stationary solutions above the generation threshold will be discussed in the comments (section 4).

Equation (1) were considered by Haken (section 8.4 in Ref.[6]) in the following way. If one takes into account only the formulae, one can see that first the stationary solutions were selected. Actually, eq.(8.13) in Ref.[6] means that **bS** and, consequently, α are taken as time-independent quantities; eq.(8.17) means that the quantity S does not dependent on time, and this is also true for **b** in virtue of eq.(8.13). To consider the time evolution near the stationary solutions, the right-hand side of the rigorous relation between the stationary values of S and b^*b, which is the sum of the geometric progression, was replaced by the sum of its first two terms. An equation of the known form was obtained [6]:

$$\dot{b} = (- \kappa + \frac{g^2}{\gamma} N d_0) b - 4 \frac{g^2 \kappa T}{\gamma} b^*bb, \tag{2}$$

which has two sets of stationary solutions for real **b**, depending on the signs of the coefficients. Namely, if $d_0 < \kappa \gamma/g^2N$, there is only a zero stationary solution, which is asymptotically stable. But the stationary solution above generation threshold, when $d_0 > d_c = \kappa\gamma/g^2N$, is of main interest. As seen from (2), in this case there are two non-zero stationary solutions, which are asymptotically stable, and a zero solution, which is exponentially unstable. The Lyapunov function, well known for eq.(2) (eq.(8.23) in Ref.[6]), was given.

As shown below, despite the fact that the stationary solutions of (1) and (2) coincide, the qualitative properties of these solutions are significantly different. So equation (2) can in no way serve as an approximation of initial dynamic system (1); consequently, the results obtained for (2) cannot be used in a general case for the physical discussion of dynamic system (1).

The following should also be mentioned. System (1) should not be identified with the dynamic system describing the laser and consisting of three equations, allowed with respect to the first derivatives and considered in Refs.[6,10,11]. It is seen that the system of five ordinary differential equations considered cannot be reduced by any exact transformation to the system of three differential equations [6,10,11] taken for the spatially uniform solutions (the Lorenz model [13,6]). So in a general case the results obtained for the latter system cannot be applied to system (1).

To study the stability of the stationary solutions, one must write down a system of equations in variations near these stationary solutions. Thus we obtain the stationary solutions in the explicit form. These solutions were first considered in Ref.[12].

Both for convinience of calculations and for physical discussion the real variable are more preferable. A set of these variables $\{b_1, b_2, \alpha_1, \alpha_2, S\}$ is introduced by the following evident transformation:

$$
\begin{aligned}
b_1 &= b^* + b, \\
b_2 &= i(b^* - b), \\
\alpha_1 &= \alpha^* + \alpha , \\
\alpha_2 &= i(\alpha^* - \alpha),
\end{aligned}
\qquad (3)
$$

System of equations (1) becomes of the form

$$\dot{b}_1 = -\kappa\, b_1 + g\, \alpha_2 ,$$

$$\dot{b}_2 = -\kappa\, b_2 - g\, \alpha_1 ,$$

$$\dot{\alpha}_1 = -\gamma\, \alpha_1 - g\, b_2 S,$$

$$\dot{\alpha}_2 = -\gamma\, \alpha_2 + g\, b_1 S, \tag{4}$$

$$\dot{S} = \frac{N d_0 - S}{T} + g\,(\alpha_1 b_2 - \alpha_2 b_1).$$

Let the stationary solutions of (4) be denoted by $\{ b_1^0, b_2^0, \alpha_1^0, \alpha_2^0, S^0\}$. We introduce the variations $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ near these stationary solutions by replacement of the variables:

$$\mathbf{x}_1 = b_1 - b_1^0,$$

$$\mathbf{x}_2 = b_2 - b_2^0,$$

$$\mathbf{x}_3 = \alpha_1 - \alpha_1^0, \tag{5}$$

$$\mathbf{x}_4 = \alpha_2 - \alpha_2^0,$$

$$\mathbf{x}_5 = S - S^0.$$

The equations in the variations are of the form

$$\mathbf{x} = \mathbf{L}\mathbf{x} + \mathbf{F}(\mathbf{x}) \tag{6}$$

where L is the matrix of the linear part of equations (6)

$$\mathbf{L} = \begin{pmatrix} -\kappa & 0 & 0 & g & 0 \\ 0 & -\kappa & -g & 0 & 0 \\ 0 & -gS^0 & -\gamma & 0 & -gb_2^0 \\ gS^0 & 0 & 0 & -\gamma & gb_1^0 \\ -g\alpha_2^0 & g\alpha_1^0 & gb_2^0 & -gb_1^0 & -1/T \end{pmatrix} \tag{7}$$

and $\mathbf{F}(\mathbf{x})$ is the column with nonlinear terms:

$$\mathbf{F}(\mathbf{x}) = \{ 0,\ 0,\ -g\, \mathbf{x}_2\mathbf{x}_5,\ g\, \mathbf{x}_1\mathbf{x}_5,\ g\,(\mathbf{x}_2\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_4)\}. \tag{8}$$

In the general case the matrix elements $\mathbf{L}_{ij}$ depend on the stationary solution $\{b_1^0, b_2^0, \alpha_1^0, \alpha_2^0, S^0\}$which is determined from the equations:

$$b_1^0 = \frac{g}{\kappa}\,\alpha_2^0,$$

$$b_2^0 = -\frac{g}{\kappa}\,\alpha_1^0,$$

$$\alpha_1^0\left(\frac{g^2}{\kappa}\,S^0 - \gamma\right) = 0, \tag{9}$$

$$\alpha_2^0\left(\frac{g^2}{\kappa}\,S^0 - \gamma\right) = 0,$$

$$\frac{Nd_0 - S^0}{T} = \frac{g^2}{\kappa}\left\{(\alpha_1^0)^2 + (\alpha_2^0)^2\right\} \quad .$$

The main aim of this paper is to study stationary solutions above the generation threshold

$$d_c = \frac{\varkappa\gamma}{Ng^2}$$

but for the sake of completeness we shall also briefly consider the case when $d_0 < d_c$. Below the generation threshold there is only one stationary solution of the form $(0,0,0,0,Nd_0)$, corresponding to spontaneous radiation. Substitute it in matrix (7) and denote it by L_{sp}. It can be easily seen that because of $d_0 < d_c$ the following inequality is valid:

$$\det L_{sp} = -\frac{(\kappa\gamma - N\,d_0 g^2)^2}{T} < 0, \tag{10}$$

i.e. L_{sp} is not degenerate. Since

$$\lim_{\|x\|\to 0} \frac{\|F(x)\|}{\|x\|} = 0, \tag{11}$$

the theorem of stability in the linear approximation [7-9] can be applied. A characteristic equation for the matrix L_{sp} is of the form

$$(\lambda + \frac{1}{T})(\lambda^2 + (\kappa + \gamma)\lambda + \kappa\gamma - N\,d_0 g^2)^2 = 0. \tag{12}$$

It is seen that at $-(\varkappa-\gamma)^2/4g^2N < d_0 < d_c$ this equation has three different negative roots, two of them being double degenerate. If $d_0 = -(x-\gamma)^2/4g^2N$, there are two different negative roots, one of them being fourfold degenerate. If $d_0 < -(x-\gamma)^2/4g^2N$, there are a negative root and two double degenerate complex conjugate roots with negative real parts.

Thus, according to the theorem of stability in the linear approximation, the spontaneous radiation described by the stationary solution $\{0,0,0,0,Nd_0\}$ is asymptotically stable at all values of $d_0 < d_c$.

It is seen from equations (8) that if

$$d_0 > d_c = \frac{x\gamma}{Ng^2}$$

then, besides the spontaneous radiation regime considered in subsection 2.2, a stationary solution with non-zero values of b_i^0 and α_i^0 is possible. For the latter quantities natural parametrization can be introduced:

$$\begin{aligned} \alpha_1^0 &= \alpha \cos\varphi , \\ \alpha_2^0 &= \alpha \sin\varphi , \\ \alpha &= \sqrt{(\alpha_1^0)^2 + (\alpha_2^0)^2} , \\ 0 &\le \varphi < 2\pi . \end{aligned} \tag{13}$$

Since for the stationary solution with the non-zero α_i^0 and b_i^0 the relation

$$S^0 = \frac{\kappa\gamma}{g^2} \tag{14}$$

is always valid, we obtain from (9)

$$\alpha = \sqrt{\frac{\kappa}{g^2T}\left(Nd_0 - \frac{\kappa\gamma}{g^2}\right)} . \tag{15}$$

Thus the stationary solution above the generation threshold, corresponding to the laser regime, is of the form

$$\left\{ \frac{g}{\kappa}\alpha \sin\varphi , -\frac{g}{\kappa}\alpha \cos\varphi , \alpha \cos\varphi , \alpha \sin\varphi , \frac{\kappa\gamma}{g^2} \right\} \tag{16}$$

where α is determined according to (15), and the phase φ is a free parameter determined only from the initial conditions.

Note also that above the generation threshold, i.e. under the condition $d_0 > d_c$, the stationary solution $\{0,0,0,0,Nd_0\}$, which is also possible and corresponds to the spontaneous radiation, becomes exponentially unstable.As seen from (12), in this case one of its double degenerate roots is positive.

2. **Investigation of stability above the generation threshold.**

Substitute solutions (16) in matrix (7), denote the obtained matrix by L_{las} and, using (6)-(8), write down the equations in the variations $x = (x_1, x_2, x_3, x_4, x_5)$:

$$\begin{aligned}
\dot{x}_1 &= -\kappa x_1 + g x_4 , \\
\dot{x}_2 &= -\kappa x_2 - g x_3 , \\
\dot{x}_3 &= -\gamma x_3 - \frac{\kappa\gamma}{g} x_2 + \frac{g^2}{\kappa}\alpha \cos\varphi\, x_5 - g x_2 x_5 , \\
\dot{x}_4 &= -\gamma x_4 + \frac{\kappa\gamma}{g} x_1 + \frac{g^2}{\kappa} \alpha \sin\varphi\, x_5 + g x_1 x_5 , \\
\dot{x}_5 &= -\frac{x_5}{T} + g\,\alpha \cos\varphi\, x_2 - \frac{g^2}{\kappa} \alpha \cos\varphi\, x_3 - g\,\alpha \sin\varphi\, x_1 - \\
&\quad - \frac{g^2}{\kappa} \alpha \sin\varphi\, x_4 + g x_2 x_3 - g x_1 x_4 .
\end{aligned} \qquad (17)$$

The task is to study the stability of zero solution of system (17), or, which is the same, of the stationary solution $(b_1^0, b_2^0, \alpha_1^0, \alpha_2^0, S^0) = (\frac{g}{\kappa}\alpha\mathrm{Sin}\ \varphi, -\frac{g}{\kappa}\alpha\mathrm{Cos}\ \varphi, \alpha\mathrm{Cos}\ \varphi, \alpha\mathrm{Sin}\ \varphi, \frac{\kappa\gamma}{g^2})$ of initial system (4) or (1).

Let us calculate the determinant of the matrix L_{las} of the linear part of system (17). After simple, though long, calculations we find that at all values of κ, γ, g, $1/T$ and N

$$\det L_{las} \equiv 0 \qquad (18)$$

Equation (18) indicates the most important feature of stationary solution (16) of considered dynamic system (4) or (1). It means that among the characteristic numbers of the matrix L_{las} there is at least one zero characteristic number. Then, if among the non-zero characteristic numbers of the matrix L_{las} there is al least one with the positive real part, the zero solution of system (17) or, which is the same, stationary solution (16) of initial system (4) or (1) allows the study of the stability in the linear approximation [7-9] and is exponentially unstable.

If among the non-zero characteristic numbers of the matrix L_{las} there are no those with the positive real part, then the zero solution of system (17), or solution (16) of initial system

(4) or (1), cannot be investigated for the stability in the linear approximation, i.e. the critical case is realised.

In this paper we shall thoroughly consider the critical case for system (17), when there are a zero root and non-zero roots with the negative real part.

Let us find a characteristic equation for the matrix by calculating the following determinant:

$$\det(L_{las} - \lambda\hat{1}) = \begin{vmatrix} -(\lambda+\kappa) & 0 & 0 & g & 0 \\ 0 & -(\lambda+\kappa) & -g & 0 & 0 \\ 0 & -\frac{\kappa\gamma}{g} & -(\lambda+\gamma) & 0 & \frac{g^2}{\kappa}\alpha\cos\varphi \\ \frac{\kappa\ \gamma}{g} & 0 & 0 & -(\lambda+\gamma) & \frac{g^2}{\kappa}\alpha\sin\varphi \\ -g\alpha\sin\varphi & g\alpha\cos\varphi & -\frac{g^2\alpha\cos\varphi}{\kappa} & -\frac{g^2\alpha\sin\varphi}{\kappa} & -(\lambda+\frac{1}{T}) \end{vmatrix} \quad (19)$$

Putting determinant (19) equal to zero, after long calculations we obtain the equation

$$\lambda^5 + C_1\lambda^4 + C_2\lambda^3 + C_3\lambda^2 + C_4\lambda = 0, \quad (20)$$

which, as expected in virtue of (18), has one zero root. The coefficients entering into (20) are

$$C_1 = 2\kappa + 2\gamma + \frac{1}{T} > 0,$$

$$C_2 = (\kappa+\gamma)(\kappa+\gamma+\frac{1}{T}) + \frac{g^4}{\kappa^2}\alpha^2 > 0,$$

$$C_3 = \frac{(\kappa+\gamma)^2}{T} + \frac{g^4}{\kappa}(3+\frac{\gamma}{\kappa})\ \alpha^2 > 0, \quad (21)$$

$$C_4 = 2g^4(1+\frac{\gamma}{\kappa})\ \alpha^2 > 0.$$

All C_i are positive, so the zero root is the only one, and the necessary condition of the real parts of non-zero roots of eq. (20) being negative is satisfied. Then we demand that the Hurwitz criterion is satisfied:

$$T_0 = 1 > 0, \quad T_1 = C_1 > 0, \quad T_2 = C_1C_2 - C_0C_3 > 0, \quad (22)$$

$$T_3 = C_3T_2 - C_1^2C_4 > 0, \quad T_4 = C_4T_3 > 0 .$$

If conditions (22) are satisfied, i.e. the Hurwitz minors of the equation

$$\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_3\lambda + C_4 = 0 \quad (23)$$

are positive, the matrix L_{las} has one zero eigenvalue and four non-zero eigenvalues with the negative real part. In this case the considered zero solution of system (17), or the stationary solution of initial system (4) cannot be investigated for the stability in the linear approximation, and special methods are required [7-9].

Fultillment of conditions (22) considerably depends on the signs of the minors T_2 and T_3, which are bulky rational functions of many variables:

$$T_2 = T_2\,(d_0,\ \kappa,\ \gamma,\ T,\ g,\ N\ /) \tag{24}$$
$$T_3 = T_3\,(d_0,\ \kappa,\ \gamma,\ T,\ g,\ N\ )\ .$$

A separate paper deals with the investigation of the functions T_2 and T_3, since the situations that occur at their different values are so various and physically interesting that cannot be covered by the subject of this paper. Only one possible case is announced in this work because it is important. Meanwhile, the following should be mentioned.

Conditions (22) are fulfilled at the parameter values typical of the majority of practical lasers (see book [14] and table 1 on page 48 in Ref.[15]). Omitting cumbersome calculations and general expressions for T_2 and T_3, which are hard to analyse, we give experimentally natural specific cases. Namely, let

$$\frac{1}{T} = \gamma\ , \qquad \kappa = K\gamma\ , \tag{25}$$

where $K > 0$ is a number. Then

$$T_2 = g^2\gamma N\ (\ \frac{2}{K} - 1)\,d_0 + \gamma^3(2K^3+10K^2+17K+6)\ , \tag{26}$$

$$T_3 = g^4\gamma^2N^2(\ \frac{2}{K^2}+\ \frac{5}{K} - 3)\,d_0^2\ +$$
$$+\ g^2\gamma^4 N\ (\frac{6}{K} + 15 + 22K - K^2 - 2K^3)\,d_0 + \tag{27}$$
$$+\ \gamma^6(2K^5 - 16K^4 + 39K^3 + 31K^2 + 12K)\ .$$

At $d_0 > d_c = \kappa\gamma/g^2N$ and when $\kappa < \gamma$, T_2 and T_3 are evidently positive for the majority of practical lasers. Indeed, if $\kappa < \gamma$,then $K < 1$. In this case T_2 as a linear function of d_0 is positive in the region $d_0 > d_c > 0$. The quantity T_3 as a function of d_0 is a second-order polynomial whose coefficients are positive at $K < 1$. So T_3 has two negative roots and is limited from below, consequently, in the region $d_0 > d_c > 0$

the function $T_3 = T_3(d_0)$ is positive.

Thus, conditions (22) of the critical case considered are fulfilled for the majority of practical lasers [14,15].

The non-critical case, when the real part of at least one root of characteristic equation (20) is positive, is of special interest. In the non-critical case, which is not considered in detail in this paper, solution (16) is exponentially unstable. The search for these laser radiation regimes is now a very popular experimental and theoretical problem (e.g. see subsection 2.3 in Ref.[16] and additional references thereof). However, this problem was not raised for dynamic system (1), which is probably due to theoretical results (section 8.4 in Ref.[6]). Now we show that the exponentially unstable radiation regime occurs above the generation threshold for a laser described by dynamic system (1). It is enough to give an example, i.e. to show a set of values of $\mathbf{d}_0 > \mathbf{d}_c$, κ, γ, $\mathbf{T}$, $\mathbf{g}$, $\mathbf{N}$, which have physical sense, when at least one of the functions T_2 or $\mathbf{T}_3$ is negative. Then, in virtue of the Hurwitz theorem, among the roots of eq. (23) there are roots with the positive real part, and solution (16) of system (4) is exponentially instable [7-9]. The simplest way is to consider the $\mathbf{d}_0$ linear function $\mathbf{T}_2(\mathbf{d}_0)$:

$$\mathbf{T}_2(\mathbf{d}_0) = r(\kappa,\gamma,\mathbf{T},\mathbf{g},\mathbf{N})\mathbf{d}_0 + q(\kappa,\gamma,\mathbf{T}) \tag{28}$$

where the coefficients are

$$r(\kappa,\gamma,\mathbf{T},\mathbf{g},\mathbf{N}) = \frac{\mathbf{g}^2\mathbf{N}}{\kappa\mathbf{T}}\left(\gamma + \frac{1}{\mathbf{T}} - \kappa\right), \tag{29}$$

$$q(\kappa,\gamma,\mathbf{T}) = 2(\kappa+\gamma)^3 + \frac{4\kappa^2 + 9\kappa\gamma + 3\gamma^3}{\mathbf{T}} + \frac{2\kappa+\gamma}{\mathbf{T}^2}. \tag{30}$$

If the condition

$$\kappa > \gamma + \frac{1}{\mathbf{T}} \tag{31}$$

is fulfilled, then $r(\kappa,\gamma,\mathbf{T},\mathbf{g},\mathbf{N}) < 0$, and the function $\mathbf{T}_2(\mathbf{d}_0)$ becomes negative at

$$\mathbf{d}_0 > \mathbf{d}_{exp} = -\frac{q(\kappa,\gamma,\mathbf{T})}{r(\kappa,\gamma,\mathbf{T},\mathbf{g},\mathbf{N})}. \tag{32}$$

Consequently, if condition (31) is fulfilled, solution (16) will be exponentially unstable at least in region (32). The only thing remained is to make sure that our consideration has sense, i.e. $\mathbf{d}_{exp}$ is in the region where solution (16) exists. To be so, the following inequality must be valid:

$$d_{exp} \geq d_C = \frac{K\gamma}{g^2 N} \quad . \tag{33}$$

We just give an example in order to show that (33) is valid. Let relations (25) hold good again. Then

$$d_{exp} = \frac{\gamma^2}{g^2 N} \frac{6K + 17K^2 + 10K^3 + 2K^4}{K - 2} \quad . \tag{34}$$

At $K > 2$ condition (33) is seen to be satisfied. For example, if $K = 3$, then $d_{exp} = 603\gamma^2/g^2N = 603d_c$. Thus we have proved the existence of the threshold of the exponential instability, which was earlier unknown for this model, and have obtained approximate estimations (31) and (32) for its existence region and for its value. Evidently, (31)-(32) can be replaced by other estimations after the investigation of the function $T_3(d_0)$.

We have already shown that the fact that besides the zero root characteristic equation (20) has four non-zero roots with negative real parts is physically quite possible for practical lasers. Since in this case laser solution (16) cannot be investigated for the stability in the linear approximation, one should apply the theory of steady-state motions [7-9]. In our case system (17) must first be reduced to the form when the derivative of one of the variables has no linear part [7-9]. Omitting the calculation method [9], we introduce a new variable

$$y = b_2^0 x_1 - b_1^0 x_2 + \frac{g}{\gamma} b_1^0 x_3 + \frac{g}{\gamma} b_2^0 x_4 \quad . \tag{35}$$

The derivative of this variable is a nonlinear function of x_1:

$$\dot{y} = \frac{g^2}{\gamma} x_5 (b_2^0 x_1 - b_1^0 x_2) \tag{36}$$

Let us also assume that

$$b_2^0 \neq 0 \tag{37}$$

and replace the variable x_1 by y in equations (17). If we had $b_2^0 = 0$, we would replace x_2 by y and continued as follows (above the threshold $\alpha > 0$, so, as seen from parametrization (13), b_1^0 and b_2^0 cannot be equal to zero at the same time). System (17) gets the following form in the variables

$$\dot{y} = \frac{g^2}{K} x_5 (y - \frac{g}{\gamma} b_1^0 x_3 - \frac{g}{\gamma} b_2^0 x_4) ,$$

$$\dot{x}_2 = -\kappa x_2 - g x_3,$$

$$\dot{x}_3 = -\gamma x_3 - g S^0 x_2 - g b_2^0 x_5 - g x_2 x_5,$$

$$\dot{x}_4 = -(\kappa+\gamma) x_4 + \frac{\kappa\gamma}{g b_2^0} y + \frac{\kappa\gamma}{g} \frac{b_1^0}{b_2^0} x_2 - \kappa \frac{b_1^0}{b_2^0} x_3 + g b_1^0 x_5 + \tag{38}$$

$$+ g x_5 \left(\frac{1}{b_2^0} y + \frac{b_1^0}{b_2^0} x_2 - \frac{g}{\gamma} \frac{b_1^0}{b_2^0} x_3 - \frac{g}{\gamma} x_4 \right),$$

$$\dot{x}_5 = -\frac{x_5}{T} - g \frac{\alpha_2^0}{b_2^0} y + g\left(\alpha_1^0 - \frac{b_1^0 \alpha_2^0}{b_2^0}\right) x_2 + g\left(b_2^0 + \frac{g b_1^0 \alpha_2^0}{\gamma\, b_2^0}\right) x_3 +$$

$$+ g\left(\frac{g\alpha_2^0}{\gamma} - b_1^0\right) x_4 + g x_2 x_3 - g x_4 \left(\frac{1}{b_2^0} y + \frac{b_1^0}{b_2^0} x_2 - \frac{g b_1^0}{\gamma b_2^0} x_3 - \frac{g}{\gamma} x_4\right),$$

where the coefficients involve components of stationary solution (15)- (16).

The apply the theorem of the critical case with a zero characteristic root [7-9], it is necessary to transform equations (38) once more in order to eliminate y linear terms on their right-hand sides. Omitting the way of finding this non-linear transformation [7-9], we write down the final result. The transformation has the form

$$x_i = \xi_i + u_i(y), \quad i = 2,\ 3,\ 4,\ 5, \tag{39}$$

where the functions $u_i(y)$, $i = 2,3,4,5$, are reduced to zero at y=0 and reduce the right-hand parts of equations (38) to zero at $\xi_2 = \xi_3 = \xi_4 = \xi_5 = 0$. These functions are specified by the relations

$$u_2(y) = \frac{-A_1 - By + \sqrt{(A_1 + By)^2 - 4 A_2 (D_1 y + D_2 y^2)}}{2 A_2}$$

$$u_3(y) = -\frac{\kappa}{g} u_2(y),$$

$$u_4(y) = \frac{\kappa\gamma}{g b_2^0 (\kappa+\gamma)} y + \frac{\kappa b_1^0}{g b_2^0} u_2(y), \tag{40}$$

$$u_5(y) \equiv 0.$$

The parameters A_1, A_2, B, D_1 and D_2 depend on the parameters of initial system (4) or (1) and stationary solution (15)-

(16) in the following way:

$$A_1 = \frac{2g\alpha^2}{\alpha_1^0},$$

$$A_2 = -\frac{\kappa\alpha^2}{(\alpha_1^0)^2},$$

$$B = -\frac{2\kappa^2\gamma\ \alpha_2^0}{g(\kappa+\gamma)(\alpha_1^0)^2},$$

$$D_1 = \frac{2\kappa\gamma\alpha_2^0}{(\kappa+\gamma)\ \alpha_1^0}, \tag{41}$$

$$D_2 = -\frac{\kappa^3\gamma^2}{g^2(\kappa+\gamma)^2(\alpha_1^0)^2}.$$

After transformations (35) and (39)-(41) system (17) takes the form

$$\begin{aligned}
\dot{y} &= Y(y,\xi_3,\xi_4,\xi_5),\\
\dot{\xi}_2 &= -\kappa\ \xi_2 - g\ \xi_3,\\
\dot{\xi}_3 &= -\gamma\ \xi_3 - gS^0\xi_2 - gb_2^0\xi_5 - \Xi_3(y,\xi_2,\xi_5),\\
\dot{\xi}_4 &= -(\kappa+\gamma)\xi_4 + \frac{\kappa\gamma b_1^0}{gb_2^0}\xi_2 - \frac{\kappa b_1^0}{b_2^0}\xi_3 + gb_1^0\ \xi_5 + \Xi_4(y,\xi_2,\xi_3,\xi_4,\xi_5),\\
\dot{\xi}_5 &= -\frac{\xi_5}{T} + g(\alpha_1^0 - \frac{b_1^0\alpha_2^0}{b_2^0})\xi_2 + g(b_2^0 + \frac{gb_1^0\alpha_2^0}{\gamma b_2^0})\xi_3 +\\
&\quad +g\ (\frac{g\alpha_2^0}{\gamma} - b_1^0)\xi_4 + \Xi_5(y,\xi_2,\xi_3,\xi_4,\xi_5).
\end{aligned} \tag{42}$$

Omitting the form of the functions Y and Ξ_i, $i = 3,4,5$ which are easy to calculate, we point out their main feature: at $\xi_2 = \xi_3 = \xi_4 = \xi_5 = 0$ these functions become equal to zero:

$$Y(y,0,0,0,0) = \Xi_3(y,0,0) = \Xi_4(y,0,0,0,0) = \Xi_5(y,0,0,0,0) = 0. \tag{43}$$

In this case system (42) allows a particular solution:

$$y = \text{Const},\ \xi_2 = \xi_3 = \xi_4 = \xi_5 = 0. \tag{44}$$

According to relations (43) and the terminology adopted in the theory of stability [7-9], system (17) belongs to a "special case" of the critical case with a zero root of the characteristic equation. Then, according to the relevant Lyapunov theorem [7-9], the trivial solution of system (17) or, which is the same, stationary solution (15)-(16) of initial system (4) or (1) is stable, but not asymptotically stable. According to the corollary of the theorem, any non-trivial solution of system (17) sufficiently close to the trivial one or, which is the same, any solution of initial system (4) or (1) sufficiently close to the stationary one tends to a solution determined from relations (44).

3. Investigation of stability at the generation threshold.

But what does happen at $d_0=d_c=\kappa\gamma/g^2N$? It is seen from (16) that under this condition matrix L_{sp} has double zero eigenvalue and negative other eigenvalues. Consequently, stability problem is not solved by linear part of equations - the critical case with double zero root is realised.

To solve this problem let us construct explicitly corresponding Lyapunov function. In this case system of equations in variations (13)-(15) is written down as

$$\dot{x}_1= -\kappa\, x_1+ g\, x_4;\quad \dot{x}_2= -\kappa\, x_2- g\, x_3;\quad \dot{x}_3= -\gamma\, x_3- \frac{\kappa\,\gamma}{g} x_2- g\, x_2x_5;$$

$$\dot{x}_4 = -\gamma\, x_4+ \frac{\kappa\,\gamma}{g} x_1+g\, x_1x_5;\quad \dot{x}_5= - \frac{x_5}{T} + g(x_2x_3-x_1x_4). \tag{45}$$

Using non-degenerate linear transformation

$$z_1= \frac{\gamma}{g} x_1+x_4;\quad z_2= -\frac{\gamma}{g} x_2+x_3;\quad z_3=x_3+\frac{\kappa}{g} x_2;\quad z_4=x_4-\frac{\kappa}{g}x_1;\quad z_5=x_5, \tag{46}$$

we can lead system (45) to the form

$$\dot{z}_1= \frac{g^2}{\kappa+\gamma} z_5\ (z_1-z_4);\quad \dot{z}_2= \frac{g^2}{\kappa+\gamma} z_5(z_2-z_3);$$

$$\dot{z}_3= -(\kappa+\gamma)z_3 + \frac{g^2}{\kappa+\gamma} z_5(z_2-z_3);\quad \dot{z}_4= -(\kappa+\gamma)z_4+ \frac{g^2}{\kappa+\gamma}z_5(z_1-z_4);$$

$$\dot{z}_5= - \frac{z_5}{T} - \frac{g^2}{(\kappa+\gamma)^2}\left\{(z_1-z_4)(\kappa\, z_1+\gamma\, z_4) + (z_2-z_3)(\kappa\, z_2+\gamma\, z_3)\right\}. \tag{47}$$

Defining the Lyapunov fuction as

$$V = \frac{1}{2}\left\{ \kappa(z_1^2+ z_2^2) + \gamma(z_3^2 + z_4^2) + (\kappa+\gamma)z_5^2 \right\} \geq 0. \tag{48}$$

and differentiating it over the time, we obtain that in virtue of (47)

$$\frac{d\,V}{d\,t} = -\ (\kappa+\gamma)\ \left\{\gamma(z_3^2 + z_4^2) + \frac{z_5^2}{T}\right\} \le 0. \qquad (49)$$

Since the stability problem for the z_i variables coincides with the stability problem for the original x_i variables we can conclude [7-9] that stationary soluiton x_{sp} describing spontaneous emission is only stable but not asymptotically stable at $d_0 = d_c$.

4. Comments

We have not considered the other critical case possible for system (4) or (1). It is the case when characteristic equation (20)-(21) has one zero and two pure imaginary roots. This case is realised if $T_3(d_0, \kappa, \gamma, T, g, N) = 0$. Let us note that fourth order equation (23) can only have the imaginary roots if they are double degenerate. According to the analytical computer calculations, the discriminant [17] differs from zero at non-zero parameters κ, γ, T, g, N, consequently, the critical case with one zero and four imaginary roots takes no place for the system considered. Above mentioned possible critical cases require special methods of investigation [7-9] that differ from the applied one, that is why they were not included in this paper.

In the case considered allowance for a random external force leads to instability of the solution, since only asymptotical stability is a sufficient condition of stability of the solution of equations with permanently acting perturbation (see the main theorem in section 74 of Ref. [9]), and the solution investigated is not asymptotically stable.

One can look for an asymptotically stable laser solution only in the critical case with one zero and two imaginary roots of the characteristic equation. Even if this solution is found (the author thinks it is unlikely), it follows from the condition $T_3 = 0$ that this solution is realised at the only value of $d_0 > d_c$ for the fixed parameters of the problem (e.g. see (27)).

The known approximate result (Haken [6]) was that the laser radiation described by system (1) is asymptotically stable at $\kappa \ll \gamma, \frac{1}{T}$. Consequently, one can ignore fluctuations of, for example, temperature, at least for $\kappa \ll \gamma, \frac{1}{T}$, which would make the model suitable for description of lasers with a sufficiently stable phase and amplitude of radiation.

Here the rigorous investigation was done for all physical values of the parameters. It was found out that there is no place for the asymptotical stability: the system can be either simply stable or exponentially unstable.

The former means that the random force, i.e. fluctuations, cannot be ignored. Allowance for fluctuations leads to instability. It will be the instability of the radiation phase φ, since in virtue of (13) it is this phase that parameterizes family of solutions (44), to one of which tends any slightly perturbed solution in virtue of the corollary of the theorem of stability in the critical case. So in this case the model describes a laser with the unstable phase and stable amplitude of radiation.

The latter means that at least under condition (31) a higher pumping intensity leads to the exponentially unstable regime of such a laser. The existence of the exponential instability threshold was not earlier known for the given model. The analysis of the parameters of practical lasers [14,15] shows that the most suitable object for the experimental check of the model predictions under condition (31) is a CO_2 laser. The experiment is probably possible for other gas lasers if their resonators can be changed in length as necessary. Note that condition (31) coincides with the similar one for the Lorenz model (see relations (12.10) in Ref.[6]), which indicates a qualitative similarity of the models in the given region of parameters.

Finally, remember the analogy between the generation threshold in the laser and the 2nd order phase transition [18,15]. This analogy is based on the Lyapunov function for the laser system being similar to the Landau expansion for the thermodynamic potential near the critical point. It follows from the stability of the laser solution that for this case there is a Lyapunov function [19], but it is not quite clear if its expansion at small field amplitudes will have the form similar to the above-mentioned Landau expansion. So this question remains open for the system considered.

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