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SOME REMARKS ON THE STABILITY OF KINKS AND BUBBLES

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1. The nonlinear Schroedinger equation (NLS),

$$i\phi_t + \phi_{xx} + F(|\phi|^2)\phi = 0 \tag{1}$$

has received a considerable attention when ϕ is subject to the vanishing boundary conditions, $\phi \to 0$ as $|x| \to \infty$. The case of the nonvanishing conditions,

$$|\phi|^2 \to \rho_o \quad \text{as} \quad |x| \to \infty \tag{2}$$

is much less explored though it also has plenty of applications in diverse areas of physics. In the present note* we analyze stability of the soliton solutions of (1)-(2) i.e., stability of the "dark" solitons.

Our treatment will be based on the integrals of motion of eqs. (1)-(2) which are the energy, momentum and number of particles. In terms of r and θ where $\phi = (\rho_o - r)^{1/2} e^{i\theta}$, these can be written as

$$E = \int \{ \frac{1}{4} r_x^2 (\rho_o - r)^{-1} + \theta_x^2 (\rho_o - r) + U(r) \} dx, \tag{3}$$

where U is defined by $F(\rho) = -dU(\rho_o - \rho)/d\rho$;

$$P = -\int r\theta_x dx, \tag{4}$$

$$N = -\int r dx, \tag{5}$$

resp. The travelling waves $\phi(x - vt)$ satisfy

$$-iv\phi_x + \phi_{xx} + F(|\phi|^2)\phi = 0 \tag{6}$$

whence we have, by integration,

$$\theta_x = -\frac{1}{2} vr(\rho_o - r)^{-1} . \tag{7}$$

Using this in eq. (4) we obtain

$$Pv = 2\int \theta_x^2 (\rho_o - r)\, dx. \tag{8}$$

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* *This communication is an extended abstract of an article to be submitted elsewhere.*

2. It is straightforward to see that eq.(6) arises as the condition $(\delta E)_P = 0$, with N not necessarily being preserved by the variations. The number of particles is also irrelevant for the stability analysis. Indeed, the condition $\delta N = 0$ amounts to the requirement that the variations $\delta\phi = f + ig$ belong to the kernel of the functional

$$\mathfrak{F}[f, g] = 2\int(\phi' f + \phi'' g)\,dx,$$

where $\phi = \phi' + i\phi''$. Since ϕ', ϕ'' are not square integrable, this linear functional is not continuous in L^2; still less in the Sobolev space and accordingly, its kernel is everywhere dense in W^1. Hence, if some perturbation $\delta\phi$ not preserving N provides $(\delta^2 E)_P$ with a negative value, the same value will be attained by $(\delta^2 E)_{P,N}$ i.e., by $\delta^2 E$ with both P and N fixed. So the condition for stability is $(\delta^2 E)_P > 0$.

3. Now let us apply a composite perturbation consisting of a simple scaling $\theta(x) \to \alpha\theta(x)$ preceded by the velocity shift. This shift, $v \to \tilde v$ produces variations in P and E: $P(v) \to P(\tilde v)$, $E(v) \to E(\tilde v)$, while the scaling leads to additional changes: $P(\tilde v) \to P'(\tilde v)$, $E(\tilde v) \to E'(\tilde v)$. The aim of the velocity shift is to compensate the change in P caused by the scaling, i.e. we require $P'(\tilde v) = P(v)$.

Infinitesimally, one obtains $\tilde v = v + dv$,

$$P(\tilde v) = P(v) + P_v dv + \frac{1}{2} P_{vv}(dv)^2, \tag{9}$$

$$E(\tilde v) = E(v) + E_v dv + \frac{1}{2} E_{vv}(dv)^2. \tag{10}$$

On the other hand, $\alpha = 1 + d\alpha$ and

$$P'(\tilde v) = P(\tilde v) + d\alpha\cdot P(\tilde v), \tag{11}$$

$$E'(\tilde v) = E(\tilde v) + [2d\alpha + (d\alpha)^2]\int\theta_x^2(\rho_0 - r)\,dx, \tag{12}$$

with θ and r corresponding to $\tilde v$. Now using (8),

$$E'(\tilde v) - E(\tilde v) = [d\alpha + \frac{1}{2}(d\alpha)^2]\,P(\tilde v)\,\tilde v. \tag{13}$$

Next, the equality $P'(\tilde v) = P(v)$ yields

$$d\alpha\cdot P(\tilde v) + P_v dv + \frac{1}{2} P_{vv}(dv)^2 = 0, \tag{14}$$

defining $d\alpha$ in terms of dv. Consequently, eq. (13) becomes

$$E'(\tilde v) - E(\tilde v) = -\tilde v P_v dv + \frac{1}{2}\tilde v[P_v^2/P(\tilde v) - P_{vv}](dv)^2. \tag{15}$$

Finally, comparing (15) to (10) and recalling that $E_v = vP_v$, we arrive at

$$(\delta^2 E)_P = 2[E'(\tilde v) - E(v)] = (v^2/P)\,P_v\,(P/v)_v(dv)^2. \tag{16}$$

Thus the stability criterion is $P_v(P/v)_v > 0$.

4. As an example, we discuss the so-called $\psi^3 - \psi^5$ NLS, i.e. eq. (1) with $F(\rho) = (\rho - 1)(2A + 1 - 3\rho)$. This equation arises in a large variety of contexts [1] and possesses both kink and bubble solutions [2] which can be written in a unified way:

$$\phi = \sqrt{2}\cosh(\xi/2 - i\mu)[(2 - A)(A^2 + v^2)^{-1/2} + \cosh\xi]^{-1/2}, \tag{17}$$

with $\xi = (c^2 - v^2)^{1/2}(x - vt)$, $c^2 = 4(1 - A)$, $\cos 2\mu = (A + v^2/2)(A^2 + v^2)^{-1/2}$. For $A < 0$ eq. (17) describes a kink, for $0 < A < 1$ a bubble. The momentum (4) is readily computed to be

$$P = vN/2 + 2\mu, \quad N = \mathrm{Arcosh}[(2 - A)(A^2 + v^2)^{-1/2}] < 0. \tag{18}$$

A direct numerical simulation has shown [3] that certain critical velocity exists, v_c such that the bubble is stable for $v \geq v_c$ and unstable otherwise (this conclusion has been corroborated later [4] in the precise study of the associated eigenvalue problem). Next M.M.Bogdan, A.S.Kovalev and A.M.Kosevich observed (private communication; to appear in Fiz. Nizk. Temp.) that $E(P)$ is a double-valued function whose two branches meet at a cusp at some $P = P_c$. Guided by the proximity of this P_c to $P(v_c)$, where v_c is the numerical result of [3], they have conjectured that the two in fact coincide. Speaking otherwise, BKK's conjecture is that v_c is the root of $P_v(v) = 0$.

Now differentiating (18) we have that $(P/v)_v < 0$ holds for all A and v. Consequently, the stability condition for the "dark" solitons (17) is merely $P_v < 0$. For $A < 0$ eq. (18) yields $P_v < 0$ for all v, so that the kinks are always stable. As regards the bubbles, the above conclusion justifies BKK's conjecture and provides a natural interpretation of the results of [3,4].

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