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EXACT CRITICAL EXPONENT FOR THE N-VECTOR MODEL WITH ARBITRARY N AND D

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1. Critical exponents are among the most characteristic parameters of phase transitions and critical phenomena[1]. Calculation of critical exponents has been performed by various methods for a wide variety of models.[2,3,4] The obtained results, based on the renormalization group approach,[1,5,6] provide us with a set of successful quantitative predictions. The critical exponents exhibit a universal and model-independent behaviour. What is essential is the dimension of space D and the number of physical degrees of freedom. For some particular values of these parameters the results are known exactly. This is true e.g. for the D = 2 Ising model.[1] However, usually, the exact solution is absent. The most accurate estimates of the exponents have been obtained up to now in the field theoretical formulation of the renormalization group, involving the $g(\varphi^a)^4$ field theory, where φ^a is an n-component isovector scalar field.[7]

2. We now present the results of these calculations for the critical exponent ν as a function of two variables: the number of components n and space dimension D. They are summarized in Table 1

Some comments are in order:

1) When $n = -2$ $\nu = 1/2$ for any D as a result of perturbation theory, as far as all the corrections are proportional to $(n+2)$ and hence vanish when $n = -2$.

2) When $D = 4$ $\nu = 1/2$ for any n as far as $D = 4$ is the critical dimension where the critical charge vanishes and so do the perturbative corrections.

3) For $n = 0$ one has the Flory conjecture[8]

$$\nu_F = 3/(D+2) \quad , \tag{1}$$

which is not proved yet, but seems rather probable[9] and is in a good agreement with numerical data (see a discussion below).

4) The $n = 1$ case corresponds to the Ising model, which is exactly solved for $D = 2$ with the result $\nu_I = 1$[1].

Table 1

Critical exponent ν (exact and numerical results)

n \ D	1	2	3	4	
-2	1/2	1/2	1/2	1/2	
-1		5/8		1/2	
0	1	3/4 b).76±.03	3/5 a).588±.002 b).589±.003	1/2	$\frac{3}{D+2}$
1	∞	1 b).99±.04	a).630±.002 b).631±.003 .628±.002 c).630±.003 d).625±.005	1/2	
2	x	x	a).669±.002 b).671±.005 .666±.004 d).675±.001	1/2	
3	x	x	a).705±.003 b).710±.007 .700±.007 c).715±.025	1/2	
		1/(4-2t)			

5) For D = 2 the Nienhuis conjecture[10] gives the result

$$\nu_N = 1/(4-2t) \quad , \tag{2}$$

where t is connected with n by the equation

$$n = -2\cos(2\pi/t) \quad . \tag{3}$$

A solution of eq.(3) is ($|n| \leqslant 2$)

n	-2	-1	0	1	2
t	1	6/5	4/3	3/2	2

(4)

The Nienhuis formula is in agreement with the Flory result and the Ising model for D = 2 and n = 0,1.

6) The absence of a phase transition for the Ising model when D = 1 corresponds to $\nu = \infty$ [7].

The numerical values in Table 1 are given according to:

a) perturbation theory in the coupling constant for a fixed dimension[2];

b) ε - expansion method, where $\varepsilon = 4 - D$ [3,7];
c) high temperature expansion[4];
d) experiment[11].

In the first two cases a special technique for summing asymptotical series was applied. It is based on the Borel summation method plus some improvements due to the asymptotic estimates for high order perturbative coefficients and analytical properties (for details see ref.[2,3,7]).

It should be noted, that the errors presented in the table for numerical results are not properly proved mathematically and should not be taken too seriously. Because of the asymptotical character of the original series the method of summation itself contains some arbitrariness thus making the estimation of the errors not clear enough.

3. The existence of some analytical results, though may be not proved yet, suggests an attempt to find a general analytical expression which covers all Table 1 for ν as a function of n and D and fits all the exact values. Furthermore the analytical form of ν_F and ν_N , eqs.(1) and (2), suggests it to be a linear fractional function of t and D :

$$\nu(t,D) = \frac{AtD + BD + Ct + E}{PtD + SD + Qt + M} \quad ,$$

where A, B, etc. are some numbers. If we now impose the constraints

$\nu(1,D) = \nu(t,4) = 1/2$, $\nu(4/3,D) = \nu_F$, $\nu(t,2) = \nu_N$, $\nu(3/2,1) = \infty$

we get

$$\nu(t,D) = \frac{1}{2}\,\frac{(D-2)(3t-4) + 2}{(D-2)(3t-4) + 2 - (t-1)(4-D)} \quad . \tag{5}$$

To find the values of ν for n>2, we have to modify eq.(3) somehow. Our hypothesis is that to get one to one correspondence between t and n for |n|>2 we have to reflect the cosine function with respect to x-axis as shown in Fig.1. Then eq.(3) is modified in a cumbersome way

$$n = -2\,(-)^{[x-1]}\cos(\pi x) - 4\,[x-1]\,, \qquad x = 2/t\,, \qquad (6)$$

while the solution for integer n looks like

$$x = \frac{6-n}{4} \qquad \text{for } n \text{ even}\,,$$

$$x = \frac{5 - n + [n/4]}{3} \qquad \text{for } n \text{ odd}\,. \qquad (7)$$

With account taken of eqs.(5) and (7) the exponent ν can be calculated for arbitrary n and D. The results are summarized in Table 2.

Table 2
Critical exponent ν according to eqs.(5) and (7)

n \ D	1	2	3	4	
-2	1/2	1/2	1/2	1/2	1/2
-1	2/3	5/8	4/7	1/2	(7-D)/(10-D)
0	1	3/4	3/5	1/2	3/(D+2)
1	x	1	5/8	1/2	(D+2)/4(D-1)
2	x	x	2/3	1/2	(D-1)/3(D-2)
3	x	x	7/10	1/2	(5D-8)/2(7D-16)
	$\frac{2-t}{2(3-2t)}$	$\frac{1}{2(2-t)}$	$\frac{3t-2}{4t-2}$	$\frac{1}{2}$	

The comparison of the Tables 1 and 2 is shown in Figs.2-4.

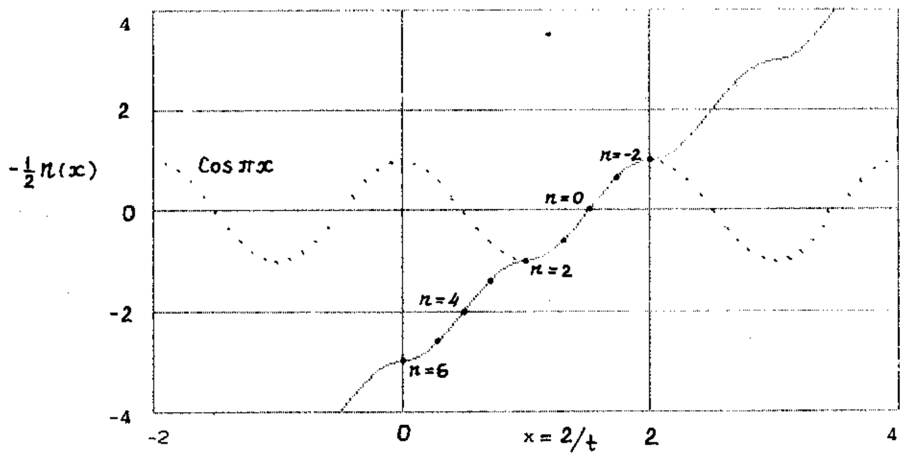

Fig. 1.

Dotted and solid lines correspond to eqs. (3) and (6), respectively

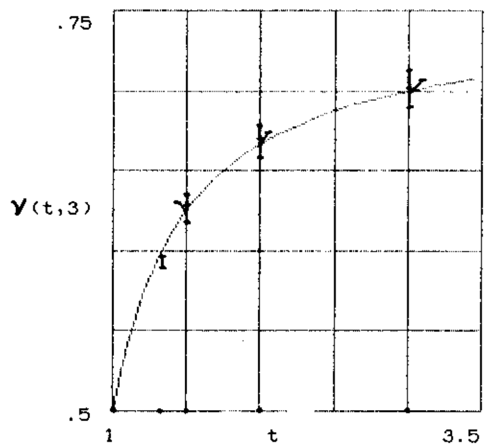

Fig. 2.

Solid lines correspond to eq. (5), the error bars are taken from Table 1

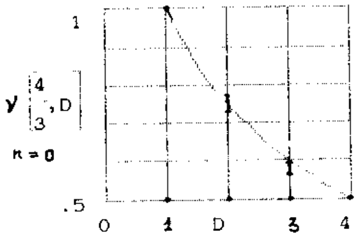

Fig.3

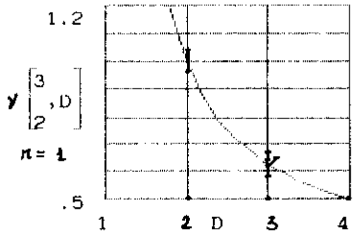

Fig.4

Solid lines correspond to eq.(5), the error bars are taken from Table 1

One can observe a remarkably good agreement of data having in mind the note concerning the errors of numerical calculations made above. An important particular case is $n = \infty$ ($t = 0$, according to eq.(6)), which corresponds to the spherical model. The latter is exactly solved with the result[12]: $\nu = 1$ for $D = 3$ and $\nu = 1/2$ for $D = 4$ in agreement with eq.(5). Anyhow, even if eq.(5) is not the true exact solution (if the latter exists), it can serve as a very accurate approximation and probably can be derived in a rigorous way, unlike the one presented here.

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