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THE MONOPOLE-LIKE SOLUTION FOR STATIC DISCLINATIONS IN CONTINUUM MEDIA

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In solid state theory, a defect solid can be considered as an elastic medium perforated by singularities in the form of points, lines, and surfaces. The complete field theory of material bodies with dislocations and disclinations based on the Yang-Mills universal gauge construction has been presented first in [1]. As has been shown in [1], the space group $G_{sp} = SO(3) \triangleright T(3)$ may be viewed as a 6-parameter gauge group that leaves the Lagrangian of elasticity theory invariant. Breaking of the homogeneity of the action of SO(3) was shown to give rise to disclinations and rotational dislocations, while homogeneity breaking of T(3) gives rise to translational dislocations. The recent development of the gauge approach in solid and liquid defect systems has been presented in [2-4] where further references can also be found.

It should be noted that disclinations with the Frank index $N \approx 1$ are now not so well studied in solids to be compared with the case $N \ll 1$. The reason is that the investigation of disclinations with $N \approx 1$ as well as cores of disclinations meets considerable mathematical problems because the nonlinear relation between stresses and strains must be taken into account. As we shall show here, the gauge model of defects allows us to describe the disclinations with N=1 quite well. We shall restrict attention to phenomena where there is no breaking of the homogeneity of the action of the translational group.

Let us start with the disclination Lagrangian which is invariant under the inhomogeneous action of the gauge group G=SO(3) [1]

$$L = L_{\chi} + L_{W} , \qquad (1)$$

where

$$L_\chi = (\rho_0/2)B^i_4\delta_{ij}B^j_4 - [\lambda(E_{AB}\delta^{AB})^2 + 2\mu E_{AB}\delta^{AC}\delta^{BD}E_{CD}]/8 \qquad (2)$$

describes the elastic properties of the material, and

$$L_W = -(s_2/2)C_{\alpha\beta}F^\alpha_{ab}g^{ac}g^{bd}F^\beta_{cd} \qquad (3)$$

describes the disclinations. Here $E_{AB}=B^i_A\delta_{ij}B^j_B-\delta_{AB}$ and $F^\alpha_{ab}=\partial_a W^\alpha_b-\partial_b W^\alpha_a+C^\alpha_{\beta\gamma}W^\beta_a W^\gamma_b$. As has been shown in [1], in defect dynamics the deformation gradient matrix must be replaced by the distortion tensor, B^i_a. In accordance with the minimal replacement argument, we have

$$B^i_a = \partial_a\chi^i + \gamma^i_{\alpha j}\chi^j W^\alpha_a \ , \qquad (4)$$

where $\partial_a\chi^i$ describes the integrable part of the distortion, while $W^\alpha_a\gamma^i_{\alpha j}\chi^j$ describes the nonintegrable local rotation acting on the instantaneous state vector $\chi^i(X^a)=\chi^i(X^A,T)$ which characterized the configuration at time T in terms of the coordinate cover (X^A) of a reference configuration; W^α_a are the compensating gauge fields associated with disclination fields. In (2) and (3) λ and μ are the Lamé constants, ρ_0 is the mass density in the reference configuration, s_2 is the coupling constant, $C_{\alpha\beta}$ are the components of the Cartan-Killing metric of the group G, $C^\alpha_{\beta\gamma}$ are the structure constants of the Lie algebra G, and $\gamma^i_{\alpha j}$ are the generating matrices of the group G. In (3) the quantities g^{ab} are given by $g^{AB}=-\delta^{AB}$, $g^{44}=1/\zeta$, and $g^{ab}=0$ for $a\neq b$. We have used here the same notation as in [1]. The labels i,j,k,α,β, and γ are the SO(3) labels and take their values from the index set I={1,2,3}, whereas the labels a,b,c,... and A,B,C,... are the space labels and take their values from the index sets J={1,2,3,4} and I, respectively.

For the rotation group SO(3) we have $\gamma^i_{\alpha j}=\varepsilon_{i\alpha j}$, where $\varepsilon_{i\alpha j}$ is the full antisymmetric tensor, $\varepsilon_{123}=1$; $C_{\alpha\beta}=\delta_{\alpha\beta}$, and $C^\gamma_{\alpha\beta}=-C^\gamma_{\beta\alpha}$, where $C^1_{23}=1$.

Following [1], we assume disclinations being continuously distributed in materials. We also discard the assumption that the reference configuration is defect free. The Euler-Lagrange equations for (1) take the form

$$\partial_a G^{ab}_\alpha - C^\beta_{\gamma\alpha}W^\gamma_a G^{ab}_\beta = T^b_\alpha/2, \qquad (5)$$

and

$$\partial_a Z^a_i - Z^a_j W^\alpha_a \gamma^i_{\alpha j} = 0 \quad , \qquad (6)$$

where $G^\alpha_{ab}=\partial L/\partial F^\alpha_{ab}$, $T^a_\alpha=(\partial L/\partial W^\alpha_a)|_{F^\alpha_{ab}}=\gamma^i_{\alpha j}Z^a_i\chi^j$, and $Z^a_i=\partial L/\partial B^i_a$. Note that in the defect theory $Z^A_i=-\sigma^A_i$ and $Z^4_i=p_i$, where the explicit expression for the stress tensor σ^A_i is [1]

$$\sigma^A_i = \delta^A_B\delta_{ij}(\partial_C\chi^j+W^\alpha_C\gamma^j_{\alpha k}\chi^k)(\lambda\delta^{BC}\delta^{FD}E_{FD}+2\mu\delta^{RB}\delta^{SC}E_{RS}) \qquad (7)$$

and the momentum

$$p_i = \rho_0\delta_{ij}(\partial_4\chi^j + W^\alpha_4\gamma^j_{\alpha k}\chi^k). \qquad (8)$$

As can bee seen, the stresses, σ^A_i, and momenta, p_i, are not elastic as well as the strain measure E_{AB} is essentially nonlinear. The coupled nonlinear field equations (5) and (6) are difficult to solve in the general case. Usually, the linearization procedure is used, and the displacement vector u^i is introduced as follows: $\chi^i(X^b)=\delta^i_a X^a+u^i(X^b)$.

It has been shown first in [1] that the system (5,6) may be solved if the disclination energy density coefficient s_2 is very large. In this case the right-hand side of (5) tends to zero thus reducing (5) to the free Yang-Mills equation. The static solution (see,e.g.,[5])) was used in [1] to describe the far field of a static disclination.

As we shall show here, there exists the exact static solution of (5,6). First of all, two additional assumptions must be done in accordance with [1]. Namely, we shall satisfy both the antiexact gauge conditions, $X^a W^\alpha_a=0$, and the boundary conditions. Let us use the boundary conditions in the form:

a) the Dirichlet data for χ^1

$$\delta\chi^1|_{\partial E_4}=0 \qquad (\chi^1|_{\partial E_4} \text{ specified}), \tag{9}$$

b) the homogeneous Neumann data for W^α_a

$$G^{aB}_\alpha \mu_B|_{\partial E_3}=0\,. \tag{10}$$

Here, $\partial E_4(\partial E_3)$ are the spatial boundaries of the 4(3)-dimensional Euclidean space, respectively, and μ_B is a top-down generated basis for the $\binom{3}{2}$- dimensional space $\Lambda^2(E_3)$.

We choose the monopole-like ansatz for (5) and (6) in the form

$$\chi^1(X^A) = \delta^1_A F(r) X^A/r \tag{11}$$

and

$$W^\alpha_A(X^B) = \delta^{\alpha\beta}\varepsilon_{\beta AB}X^B/r^2, \qquad W^\alpha_4 = 0, \tag{12}$$

where $r^2=X^A X_A$. Note that (12) is the known Yang-Wu solution that is singular as $r\to 0$, while (11) is the exact monopole form analogous to that for the Higgs triplet in field theory. In accordance with (9), the function $F(r)$ in (11) must tend to the constant value, F, as $r\to\infty$. The dislocation fields W^α_A tend to zero as $1/r$.

Let us note that the solution (12) is already antiexact since $X^A W^\alpha_A + T W^\alpha_4 \equiv 0$. Using (11) and (12) we can rewrite (7) as follows:

$$\sigma^A_1 = X^1 X^A g(r)\{\lambda[g^2(r)-3]/2+\mu[g^2(r)-1]\}/r^2, \tag{13}$$

where $g(r)=\partial F(r)/\partial r$. Clearly, σ^A_1 is symmetric. Substituting (11-13) directly in (5,6), we obtain that (5) turns out into the identity, whereas (6) reduces to

$$\partial_r g(r)[3Ag^2(r)-B] = -2[Ag^3(r)-Bg(r)]/r, \tag{14}$$

where $A=\lambda/2+\mu$, and $B=3\lambda/2+\mu$. Carrying out the integration in (14) we obtain the following condition:

$$|Ag^3(r)-Bg(r)| = g_0/r^2, \tag{15}$$

where g_0 is an integration constant. The stress tensor takes now the form

$$\sigma^A_1 = g_0 X^1 X^A/r^4, \tag{16}$$

which agrees with the result of [1], but is valid for all r (with the exception of the small region near r=0 where the elasticity theory does not work). We would remind that in [1] only the region of large r has been considered. It is known that the solution (12,13) has a vortex-like behaviour. Such a behaviour is usual for disclinations and rotational dislocations. The correspondence between vortices and dislocations has been recently discussed in [6]. Due to the nonlinear character of (15) the new principal feature, as compared to the monopole solution, arises. Namely, the analysis of (15) shows two distinctly defined regions characterized by the dimensional parameter $r_0=(27g_0^2A/4B^3)^{1/4}$.

The solutions of (15) are obtained to be

$$g(r) = \begin{cases} g_1(r)=N_0\cosh[\frac{1}{3}\cosh^{-1} r^2/r_0^2], & r\le r_0 \\ g_2(r)=-N_0\cos[\frac{1}{3}\cos^{-1} r_0^2/r^2+\frac{\pi}{3}], & r\ge r_0\ , \end{cases} \tag{17}$$

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Монополеподобное решение для статических дисклинаций в непрерывных средах

В рамках калибровочной модели дефектов получено точное монополеподобное решение для статических дисклинаций. Показано, что по сравнению с известным решением т'Хофта-Полякова наше решение имеет четко определенную область, характеризующую ядро дисклинации. Определены тензор напряжений и радиус ядра дисклинации.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Osipov V.A. E17-89-749

The Monopole-Like Solution for Static Disclinations in Continuum Media

In the framework of the gauge model of defects the exact monopole-like solution for static disclinations is obtained. It is shown that in comparison with the known 't Hooft-Polyakov monopole our solution has the distinctly defined region that corresponds to the core of disclination. The stress tensor as well as the disclination core radius are determined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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