89-740

объединенный
институт
ядерных
исследований
дубна

F 33

E17-89-740

V.K.Fedyanin, V.Lisy

ON THE SCATTERING OF SOLITONS AT THE INTERFACE OF TWO SINE-GORDON SYSTEMS

Submitted to International Work Shop
"Solitons and Applications", Dubna, 1989

1989

In recent years, a great progress has been achieved in the study of one-dimensional nonlinear systems, exactly integrable by the inverse problem technique. However, considering real physical situations, there appear equations not exactly integrable. If their difference from the exactly integrable equations is small, they can be analyzed using perturbation methods. In particular, the problems of the evolution of a soliton under small perturbations are of interest. In the analysis of such problems approximate methods are used which are based both on the inverse problem technique[1] and direct perturbation methods[2,3]. The detailed study of the influence of small perturbations on a sine-Gordon (SG) kink was carried out using direct methods in a number of works beginning from [4]. In [5] there has been studied the scattering of a kink at the interface of two SG systems with a small distinction in dispersions for small vibrations. The kink is perturbed from the interface and produces small amplitude waves. The ratio of the energy of the reflected waves to the incident kink energy was calculated. In this paper we generalize these results as follows. 1.The difference of the two SG systems is taken into account both in the characteristic velocity, c, as in [5] and characteristic frequency, Ω. At some conditions this leads to a free passage of solitons from one medium to the other. 2.We consider not only kinks but also antikinks and breathers that are the exact SG solitons together with kinks.

Thus, we consider the following model system:

$$\Phi_{tt} - c^2\Phi_{xx} + \Omega^2\sin\Phi = \theta(x)\ (\varepsilon\ c^2\ \Phi_{xx} - \delta\ \Omega^2\sin\Phi), \qquad (1)$$

where $\theta(x)$ is the Heaviside step function. The point $x=0$ divides two SG regions. The first region ($x<0$) is characterized by the parameters $c_1=c$ and $\Omega_1=\Omega$, and in the second one we have $c_2^2 = c^2(1+\varepsilon)$ and $\Omega_2^2 = \Omega^2(1+\delta)$ with $|\varepsilon|, |\delta| \ll 1$. One can easily obtain the transmission and reflection coefficients for harmonic waves with frequency ω which are the solutions of the linearized equation (1) ($\sin\Phi \approx \Phi$). E.g., the coefficient of the reflection of the wave is $R=[(1-f)/(1+f)]^2$ with $f=(c_1/c_2)[(\omega^2-\Omega_2^2)/(\omega^2-\Omega_1^2)]^{1/2}$. Soliton solutions of the nonperturbed Eq. (1) are given by the formulae

$$\Phi^{\pm}(x,t)=4\tan^{-1}[\pm\gamma(x-vt)/d],\ \ [(+)\ -\ \text{kink},\ (-)\ -\ \text{antikink}], \qquad (2)$$

$$\Phi^b(x,t)=4\tan^{-1}\{(1-\nu^2)^{1/2}\cos[(t-vxc^{-2})\Omega\nu]/\nu\cosh[\gamma d^{-1}(1-\nu^2)^{1/2}(x-vt)]\},\quad \text{[breather]}. \tag{3}$$

Here $\gamma=(1-v^2c^{-2})^{-1/2}$, v is the velocity, $d=c/\Omega$ and $\nu\in(0,1)$ is the internal breather frequency. The solitons (2) and (3), beginning to travel at $t=-\infty$ in the positive direction, reach $x=0$ at the time $t=0$.

Consider first the scattering of a kink (or antikink) at the interface $x=0$. We assume a solution to Eq. (1) of the form

$$\Phi(x,t)=\Phi^s(x,t)+\Psi(x,t). \tag{4}$$

By substituting this into Eq. (1) and linearizing in Ψ, we find the following equation in the lowest order of the small quantities:

$$\Psi_{\tau\tau}-\Psi_{\xi\xi}+(1-2/\cosh^2\xi)\,\Psi=2(\pm\delta-\varepsilon\gamma^2)\Theta(\xi+v\tau/c)\,\sinh\xi/\cosh^2\xi. \tag{5}$$

Here, the transformation has been used

$$\tau=\Omega\gamma(t-vxc^{-2}),\quad \xi=\Omega\gamma(x-vt)/c. \tag{6}$$

Eq. (5) differs from the analogous equation in [5] only by the multiplier in the right-hand side (we note also that $\Omega=c=1$ and $\delta=0$ in [5]). Thus, we can completely follow [5] and not give here the detailed calculations. The new feature appearing here is that the soliton travelling with velocity v_0 passes the interface without any change (in our approximation):

$$(v_0/c)^2=1\mp\varepsilon/\delta\ . \tag{7}$$

This is possible if, for kink, ε and δ have the same, and for antikink, different signs. Eq. (5) is to be solved using the initial condition that at $t=-\infty$ we have only a soliton at $x=-\infty$. After obtaining Ψ the result (4) can be represented as follows:

$$\Phi(x,t)=\Phi^s(\xi\pm A(\tau)\pm\xi\alpha/2)+\Phi'(x,t)+o(\alpha^2),\quad \alpha=\pm\delta-\varepsilon\gamma^2, \tag{8}$$
$$A(\tau)=\alpha c[\tau+cv^{-1}\ln\cosh(v\tau c^{-1})+\ln 2]/4v.$$

The argument of the first (soliton) term in Eq. (8) as $t\to-\infty$ will be $(\gamma/d)[x-t(v\mp\alpha c^2/2v\gamma^2)]$. This means that the soliton after passing the interface acquires an additional velocity

$$\Delta v=c^2[\pm\varepsilon-\delta(1-v^2c^{-2})]/2v, \tag{9}$$

different for kinks and antikinks. The second (so-called continuum) term Φ' at $\tau<0$ decreases rapidly if $|\tau|$ increases. For $\tau<0$, Φ' contains terms decreasing with τ and waves moving to the left and right directions. Taking the wave Φ' moving to the left, we determine the reflection constant R:

$$R=E_l/E_s,\quad E_l=(A/2)\int_{-\infty}^{\infty}dx[\Phi'^2_{lt}+c^2\Phi'^2_{lx}+\Omega^2\Phi'^2_l], \tag{10}$$

where E_l is the energy of the wave and $E_s=8A\Omega c\gamma$ is the incident soliton energy (A is a constant different for various concrete physical models). The result is as follows:

$$R=\frac{\pi}{2^5\gamma^4}\left(\frac{c}{v}\right)^3(\pm\delta-\varepsilon\gamma^2)^2\int_0^{\infty}dx/\cosh^2\{\pi[x^2+(c/v\gamma)^2]^{1/2}/2\}$$
$$\sim\begin{cases}\pi(\pm\delta-\varepsilon)^2(c/v)^{7/2}\exp(-\pi c/v)/8.2^{1/2}, & v\to 0\\ \varepsilon^2/16, & v\to c.\end{cases} \tag{11}$$

Now, consider the scattering of breathers. The complex expression (3) makes the analysis more difficult than for kinks. So, we will consider here only the low-energy (small-amplitude) breathers with $\sigma=(1-\nu^2)^{1/2}\ll 1$. Such breathers play a significant role in the thermodynamic and dynamic properties in a number of systems described by the SG model [6-9]. Using Eq. (4) with Φ^b instead of Φ^s and substituting it into Eq. (1), we obtain the following equation in the lowest order of the small quantities:

$$\Psi_{\tau\tau}-\Psi_{\xi\xi}+\Psi=-4\Theta(\xi+v\tau/c)[\delta+\varepsilon(v/c)^2]\cdot X, \tag{12}$$

where $X=\sigma\cos(\tau/\gamma)/\cosh\sigma\xi$. If ε and δ are of different signs and $|\delta/\varepsilon|<1$ the breather with the velocity v_0,

$$(v_0/c)^2=-\delta/\varepsilon, \tag{13}$$

freely passes the interface. Eq. (12) is solved using the Fourier transform and the initial conditions at $t=-\infty$. Due to the paper being short, we omit the terms vanishing as $t\to\infty$:

$$\Psi(\tau,\xi)=\Psi_1+\Psi_2,\qquad \Psi_1=-4D\cos(\tau/\gamma)\int_0^{\infty}dk[k^2+(v/c)^2]^{-1}\cos k\xi\,\cosh^{-1}(\pi k/2\sigma),$$
$$\Psi_2=-D(v/c)\int_{-\infty}^{\infty}dk\;w_k^{-1}\cos(k\xi-w_k\tau)\;f(k). \tag{14}$$

Here $D=\sigma+\varepsilon(v/c)^2$, $w_k=(k+1)^{1/2}$ and $f(k)=g_-(k)-g_+(k)$, where $g_\mp(k)=\{(\gamma^{-1}\mp w_k)\cosh[\pi(k+cw_k/v\mp c/v\gamma)/2\sigma]\}^{-1}$. The main contribution to Ψ_1 for small σ is

$$\Psi_1(\tau,\xi)\approx-4D\cos(\tau\gamma^{-1})(c/v)^2\sigma/\cosh\sigma\xi. \tag{15}$$

This means in our approximation that Ψ_1 gives only the change in the breather amplitude:

$$\Phi^b+\Psi_1\approx 4X[1-D(c/v)^2]\approx[1-\varepsilon-\delta(c/v)^2]\Phi^b. \tag{16}$$

The term Ψ_2 describes the transmitted (Ψ_r) and reflected (Ψ_l) waves. For the wave moving to the left the condition $k<-v\gamma/c$ must be satisfied. Substituting Ψ_l into Eq. (10) instead of Φ'_l, we find the energy of the reflected wave, E_l:

$$E_1 = \pi A\Omega\gamma(vD)^2 \int_{-\infty}^{-v\gamma/c} dk\ (w_k + vk/c) f^2(k)/cw_k. \qquad (17)$$

The main contribution to E_1 for small σ is given by k close to $-v\gamma/c$. Then, the integral in (17) can easily be estimated. We divide E_1 by the breather energy $E_b = 16A\Omega c\gamma\sigma$ to obtain the reflection constant, R:

$$R \approx KD^2\sigma^{-1/2}(v/c)^{13/2}\gamma^{-5/2} \quad (\text{moderate } v \text{ and } v\to 0), \qquad (18)$$

$$R \approx 4K(\delta+\varepsilon)^2\sigma^{-1/2}\gamma^{-5/2} \quad (v\to c). \qquad (19)$$

Here $K=(\sqrt{8}-1)\zeta(3/2)/16\sqrt{2}$ and $\zeta(x)$ is the Riemann function. The calculations were carried out in the lowest order of σ. The natural requirement $R<1$ will be accomplished if $\sigma<D^4$.

Thus, the ratio of the energy of the reflected waves to the incident small-amplitude breather energy has been obtained. For small initial velocities we find $R \sim v^{13/2}$ and for $v\to c$, $R\to 0$ as $[1-(v/c)^2]^{5/4}$. Such a behaviour differs considerably from the above results for kinks and antikinks.

REFERENCES

1. Karpman, V.I. and Maslov, E.M., JETP 73, 537 (1977).
2. Nayfah, A.H., "Perturbation Methods", Mir, Moscow (1976).
3. Longren, K. and Scott, A.C., eds., "Solitons in Action", Academic Press, N.Y.-S.Francisco-London, (1978).
4. Fogel, M.B., Trullinger, S.E., Bishop, A.R. and Krumhansl, J.A., Phys.Rev. B15, 1578 (1977).
5. Yamamoto, H., Progr.Theor.Phys., 78, 1209 (1987).
6. Makhankov, V.G. and Fedyanin, V.K., Phys.Rep. 104, 1 (1984).
7. Makhankov, V.G., JINR report P2-82-248, Dubna (1982).
8. Lisy, V. and Fedyanin, V.K., Sov.J.Low Temp.Phys 11, 166 (1985).
9. Lisy, V., Makhankov, V.G. and Fedyanin, V.K., JINR report P-17-85-410, Dubna (1985).

Received by Publishing Department
on October 30, 1989.

Федянин В.К., Лисы В. E17-89-740
О рассеянии солитонов на границе раздела
двух систем синус-Гордона

Изучается рассеяние солитонов модели синус-Гордона на границе раздела двух сред, незначительно отличающихся как в характеристической частоте, так и в предельной скорости распространения солитонов. Рассмотрены кинки, антикинки, а также низкоэнергетические бризеры. В рамках теории возмущений найдены условия беспрепятственного прохождения солитонов из одной среды в другую. Солитоны при рассеянии излучают проходящие и отраженные волны. Получены коэффициенты отражения для рассеянных волн, исследовано изменение скорости и формы солитонов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

Fedyanin V.K., Lisy V. E17-89-740
On the Scattering of Solitons
at the Interface of Two Sine-Gordon Systems

The scattering of solitons at the interface of two sine-Gordon (SG) systems is studied. The systems differ slightly both in characteristic frequencies and upper velocities of the solitons. Kinks, antikinks and low-energy breathers are taken into account. Using direct perturbation methods, conditions of the free passage of the solitons through the interface are found. The solitons at the scattering produce reflected and transmitted waves. The reflection constants of the scattered waves have been obtained, and changes of the soliton velocity and form are studied.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989