ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E17-89-650

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SOLUTION OF THE KINETIC EQUATION FOR THIN TWO- AND THREE-DIMENSIONAL METALLIC SLABS AT DIFFUSE BORDER AND ARBITRARY SCATTERING ON IMPURITIES

Submitted to "Zeitschrift für Physik B - Condensed Matter"

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1989

1. Introduction

The kinetic equation describing the response of the electron liquid in metals is solved for restricted geometries only in the relaxation time approximation, cf. e.g. [1,2,3]. The reason for this fact is the possibility to solve boundary problems for differential equations. On the other hand, such problems are much more difficult for integrodifferential equations, particularly at general or complicated integral kernels. For some effects, we exactly know which relaxation time should be used. For example, for cyclotron resonance, highly anomalous skin effect and the films thin in the scale of a mean-free path, one should use the relaxation time determined by the scattering probability density avaraged over the Fermi surface, cf. [2,3]. In the opposite limit, i.e., normal skin effect or thick films, the transport relaxation time is proper [2,3]. Unfortunately, there are almost no answers concerning proper relaxation times in the intermediate limits or in higher-order approximations to the main extremal effect.

As it will be shown, it is possible to solve, asymptotically for thin slabs, some boundary problems also for integrodifferential equations at their general kernels . Let us formulate the restrictions imposed on our model.

Calculations will be performed for the two- or three-dimensional slabs of the isotropic metal at the temperatures so low that only the impurity scattering is the active mechanism. The two-dimensional slab should not be treated as purely mathematical construction because of monoatomic metal layers deposited on the semiconductor or insulator surfaces. The Fuchs boundary conditions [4] at the border will be assumed in the purely diffusive limit. Note that it is physically clear that the specular boundary conditions lead to the solution characteristic of bulk samples whereas it is rather difficult to solve the problem for partially diffuse boundary conditions. Unfortunately, the same difficulties appear also at the boundary conditions [5] deduced from the first principles.

We shall study the response to a homogeneous electric field parallel to the border of the sample. In contrast with ref. [4], it will be not assumed that we deal with a d.c. field. For an a.c. field, one needs to introduce an effective quasiparticle interaction out of the function describing the impurity scattering probability and to distinguish the deviation from the equilibrium and from the local equilibrium [2,3,6]. As the Fermi velocity is much smaller than the velocity of light, the restriction to the homogeneous field is not serious at normal skin effect for d=3 and is not serious at all for d=2.

Let us outline the overview of our paper. The next section is devoted to the formulation of the kinetic equation in the integro-differential form. We do it also in the a.c. case $\omega \neq 0$ because all remaining transformations of this paper are then of the same complicacy as in the static case. The reader with interests restricted to d.c. fields could simply put $\omega = 0$ everywhere. It is shown in the third section that the kinetic equation at diffuse boundary conditions is equivalent to the given system of nonhomogeneous integral equations. Moreover, the current response of the system is expressed by one of the functions being the solutions of our system of integral equations. This system is solved asymptotically in the fourth section for slabs thin in the scale of the mean free path. The opposite limit is studied in the fifth section. Unfortunately, we are able there to do something only at almost isotropic scattering on impurities. The conclusion of our paper is devoted to the extension of the developed methods.

2. The kinetic equation

The probability density describing the impurity scattering of the angle ϑ dependent only on $\cos\vartheta$ will be expressed as

$$\left[1+F(\vec{n}\vec{n}')\right]\tau^{-1}, \quad \left\langle F(\vec{n}\vec{n}')\right\rangle_{\vec{n}} = 0, \tag{1}$$

where $\vec{n}$ is the unit vector directed along the electron momentum. The brackets $\langle\ldots\rangle_{\vec{n}}$ denote the average over spherical angles ($d=3$) or angles (d=2) connected with the vector $\vec{n}$.

Let us write the kinetic equation, at once for the function ψ and $\overline{\psi}$, respectively determined via the time Fourier transforms of the deviations of electron distribution functions from the local equilibrium, δf, and the equilibrium $\delta \overline{f}$, cf. [2,3] and [6]. Let us introduce the dimensionless functions ψ and $\overline{\psi}$ as follows:

$$\begin{Bmatrix} \delta f \\ \delta \overline{f} \end{Bmatrix} = e v \tau E_\omega(-\partial f_0/\partial\varepsilon) \begin{Bmatrix} \psi(\vec{n},z) \\ \overline{\psi}(\vec{n},z) \end{Bmatrix}. \tag{2}$$

Here e is the electron charge, v the Fermi velocity and E_ω the Fourier transform of the electric field. The spatial coordinate z is perpendicular to the border of the sample. The function $\overline{\psi}$ is connected with the function ψ via the formula

$$\overline{\psi}(\vec{n},z) = \psi(\vec{n},z) - \left\langle A^s(\vec{n}\vec{n}')\psi(\vec{n}',z)\right\rangle_{\vec{n}'}, \tag{3}$$

where $A^s(\vec{n}\vec{n}')$ is the dimensionless spin symmetric part of the forward scattering amplitude of quasiparticles, cf. [6] at d=3 and [7] for the base of an analogous construction at d=2. It is convenient to express the functions $F(\vec{n}\vec{n}')$ and $A^s(\vec{n}\vec{n}')$ via their Fourier (d=2) and Legendre (d=3) amplitudes. We have

$$A^s(\vec{n}\vec{n}') = \sum_{l=0} A^s_l \begin{Bmatrix} \cos l(\theta-\theta'), & d=2 \\ P_l(\vec{n}\vec{n}'), & d=3 \end{Bmatrix}, \tag{4}$$

$$F(\vec{n}\vec{n}') = \sum_{l=1} F_l \begin{Bmatrix} \cos l(\theta-\theta'), & d=2 \\ P_l(\vec{n}\vec{n}'), & d=3 \end{Bmatrix},$$

where P_l denotes the l-th Legendre polynomial.

At d=3, the amplitudes A^s_l can be expressed via the spin-symmetric Landau parameters F^s_l as follows:

$$A^s_l = F^s_l/[1+F^s_l/(2l+1)].$$

It is easy to see that at d=2 an analogous relation has the form

$$A^s_l = F^s_l/[1+(\delta_{l0}+1)F^s_l/2].$$

The stability conditions of the Fermi liquid impose the following restrictions on

$$A^s_l < 2l+1,$$

at d=3, cf. e.g. /6/, and, as it is easy to see

$$A_l^s < 2 - \delta_{l0}$$

at d=2. The inequalities

$$|F_l| \leqslant 2l+1, \quad |F_l| \leqslant 2 \quad , l \geqslant 1, \tag{5}$$

respectively at d=3 and 2, are the necessary conditions of the positivity of the function (I). One can verify that the transport mean free path l_{tr} is expressed via $l = v\tau$ as follows:

$$l_{tr} = l/(1 - F_1/d), \quad d = 2,3. \tag{6}$$

Hence, and from inequalities (5) we have

$$l_{tr} \geqslant l/2 \quad , d = 2,3. \tag{7}$$

For the transport in the homogeneous electric field, the electron density remains homogeneous and, hence, the amplitude A_0^s is cancelled in expression (3). Choosing the azimuthal axis in the z-direction at d=3, one can write the standard transport equation in the form

$$(1 + i\omega\tau)\psi(\vec{n}, z) + \cos\theta \frac{\partial \psi(\vec{n}, z)}{\partial z} = \sin\theta \sin\varphi + \tag{8}$$

$$\langle F_\omega(\vec{n}\vec{n}')\psi(\vec{n}', z)\rangle_{\vec{n}'} \quad , \quad d = 3,$$

where

$$F_\omega(\vec{n}\vec{n}') \equiv F(\vec{n}\vec{n}') + i\omega\tau\left[A^s(\vec{n}\vec{n}') - A_0^s\right], \tag{9}$$

with the coordinate z taken in the units of the mean free path, $l = v\tau$, cf. e.g. /2/, /3/ and /6/. The effects of quasiparticle interaction are important only at $\omega \neq 0$ because in the d.c. case mutual quasiparticle scattering does not lead to redistribution of the momentum. It is easy to see that an analogous equation at d=2 can also be written in the form (8),(9) at $\varphi = \pi/2$ and the angle θ varying in the interval $(0, 2\pi)$; at d=3, $0 \leqslant \theta \leqslant \pi$. The response of the system to the external field is expressed via the

conductivity depending on the frequency ω and the coordinate z perpendicular to the border of the slab. Exploiting the first of the formulae (2) we obtain for this quantity

$$\sigma_\omega(z) = (e^2 n \tau d/m^*)\langle \sin\theta \sin\varphi\, \psi(\vec{n}, z)\rangle_{\vec{n}}, \tag{10}$$

where at d=2 one should take $\varphi = \pi/2$; $n = p_F^d/d\pi^{d-1}\hbar^d$ gives the density of carriers and m^* is their effective mass.

3. Transformations of the kinetic equation

It is not difficult to verify that the solution of equation (8) can be taken in the form $\psi(\vec{n}, z) = \sin\theta \sin\varphi\, u(\cos\theta, z)$ at d=3 and, at d=2, in the same form but at $\varphi = \pi/2$ and $0 \leqslant \theta < 2\pi$. Substituting such ψ into eq. (8) and exploiting the addition theorem for spherical functions at d=3 and the definition of the Tchebyshev polynomial of the second kind at d=2 we obtain

$$c\,\frac{\partial u(c,z)}{\partial z} + a\,u(c,z) = 1 + \tag{11}$$

$$\sum_{l=1} \tilde{F}_{\omega l} R_l(c) \langle (1 - c'^2) R_l(c') u(c', z) \rangle_{c'} \quad ,$$

where $c = \cos\theta$, $c' = \cos\theta'$, $a \equiv 1 + i\omega\tau$. At d=3 $R_l(c) = P_l'(c)$, where the prime denotes the derivative, $\tilde{F}_{\omega l} = (F_l + i\omega\tau A_l^s)/l(l+1)$ and the average $\langle \dots \rangle_c$ has the form of one-half the integral over c on the interval (-1,1). At d=2, $R_l(c) = U_{l-1}(c)$ where $U_l(c)$ is the Tchebyshev polynomial of the second kind with the index l, c.f. eg. /8/, $\tilde{F}_{\omega l} = F_{\omega l}$ and the average remains in its previous form.

We will look for the function $u(c, z)$ in the form

$$u(c, z) = \left[A(c,z)e^{-za/c} + 1\right](a - F_{\omega 1}/d)^{-1}. \tag{12}$$

It is worth mentioning that the denominator $a - F_{\omega 1}/d$ can also be written in the form $l/l_{tr} + i\omega\tau m_c/m^*$ at d=2 and 3, where m_c denotes the "crystal" mass of the electron and m^* is its full effective mass containing also the contribution of the interelectron interaction. Substituting the function (12) into eq. (11) one finds

$$c\, e^{-za/c}\,\partial A/\partial z = \sum_{n=1}\tilde{F}_{\omega n} R_n(c)\left\langle (1-c'^2) R_n(c') e^{-za/c'} A(c',z)\right\rangle_{c'} \equiv \sum_{n=1}\tilde{F}_{\omega n} R_n(c) B_n(z), \tag{13}$$

where the last equality serves as a definition of the functions $B_n(z)$

It is easy to see that the specular reflection at the border of the slab, $z=\pm b$, is fulfilled at $A(c,z)=0$. In such a case, $\sigma_\omega(z)=e^2 n\tau/m^*(a-F_{\omega 1}/d)$, is z - independent and coincides with the conductivity of the bulk sample at d=2 and 3. This fact is independent of the ratio $2b/\ell$ and has a very simple physical meaning because the mirror is not an obstacle for the component of motion parallel to it.

Let us impose the boundary conditions corresponding to diffuse scattering on the function $A(c,z)$. We have: $U(b,c)=0$ at $c<0$ and $U(-b,c)=0$ at $c>0$. Taking into account formula (12) one can verify that the suitable function A, satisfying eq. (13) has the form

$$A(c,z)=e^{-ba/|c|}+\sum_{n=1}\tilde{F}_{\omega n} c^{-1} R_n(c)\int_{-bc/|c|}^{z} dy\, B_n(y) e^{ay/c} . \tag{14}$$

Substituting the function (14) into the definitions of the functions $B_n(z)$, cf. the last formula (13), one finds

$$B_k(z)=\sum_{\alpha=\pm 1}\alpha^{k-1} V_k(a(b+\alpha z)) + \sum_{n=1}\tilde{F}_{\omega n}\int_{-b}^{b} dy\, \mathrm{sign}(z-y)^{n+k} W_{kn}(a|z-y|) B_n(y). \tag{15}$$

Here, at d=3 we have

$$V_k(b)=\frac{1}{2}\int_0^1 dc\,(1-c^2) P'_k(c)\, e^{-b/c}, \tag{16}$$

$$W_{kn}(b)=\frac{1}{2}\int_0^1 dc\,(1-c^2)c^{-1} P'_k(c) P'_n(c)\, e^{-b/c}.$$

At d=2 one obtains

$$V_k(b)=\frac{1}{\pi}\int_0^1 dc\,(1-c^2)^{1/2} U_{k-1}(c)\, e^{-b/c}, \tag{17}$$

$$W_{kn}(b)=\frac{1}{\pi}\int_0^1 dc\,(1-c^2)^{1/2} c^{-1} U_{k-1}(c) U_{n-1}(c)\, e^{-b/c} .$$

Now, let us express the conductivity $\sigma_\omega(z)$ in terms of the functions $B_k(z)$. Substituting the function $\sin\theta \sin\varphi\, U(\cos\theta, z)$ into expression (10) and then exploiting eqs. (15), at k=1, $R_1(c)$ = =1, after rather simple transformations one finds

$$\sigma_\omega(z)=e^2 n\tau\left[1+d\, B_1(z)/(d-1)\right]/m^*(a-F_{\omega 1}/d). \tag{18}$$

It is easy to see that $B_n(z)$ are odd functions at n=2k and even functions at n = 2k+1. The reason for this fact is very simple because of invariance of equation (8) under simultaneous z -reflection in the momentum and configuration space.

The next transformation of the system (15) facilitates its solution, at least at $2b \ll 1$, i.e. at mean free path much bigger than the width of the slab. Note that b, consequently, is taken in the units 1, analogously to z . Let us introduce the new funĉtions $D_k(z)$ determined as follows:

$$B_k(z)=(d-1)\left[(a-F_{\omega 1}/d) D_k(z)-\delta_{k1}\right]/d . \tag{19}$$

In terms of these functions $\sigma_\omega(z)=e^2 n\tau\, D_1(z)/m^*$.

Exploiting the formula

$$\int_{-b}^{b} dy\, \mathrm{sign}(z-y)^{k-1} W_{k1}(a|z-y|) =$$

$$a^{-1}\Big[(d-1)\delta_{k1}/d - \sum_{\alpha=\pm 1} \alpha^{k-1} V_k(a(b+\alpha z))\Big], \tag{20}$$

obtained via the direct integration, one can rewrite the system of equations (15) in the form

$$D_k(z) = a^{-1}\Big[\delta_{1k} - d\sum_{\alpha=\pm 1}\alpha^{k-1} V_k(a(b+\alpha z))/(d-1)\Big] + \tag{21}$$

$$\sum_{n=1}' \tilde{F}_{\omega n} \int_{-b}^{b} dy\, \mathrm{sign}(z-y)^{n+k} W_{kn}(a|z-y|) D_n(y) .$$

4. The asymptotic solution for thin slabs

Let us solve the system of integral equations (21) at $g \equiv 2b \ll 1$ in this case the integral term is small and the system can be solved asymptotically exactly by iteration. We proceed with the case d=3. Let us find the asymptotics of the following integral:

$$S_n(b) \equiv \int_0^1 c^n e^{-b/c} dc = \frac{1}{2i}\oint_L d\varsigma \int_0^1 dc \frac{b^{\varsigma} c^{n-\varsigma}}{\sin\pi\varsigma\, \Gamma(\varsigma+1)} =$$

$$-\frac{1}{2i}\oint_L d\varsigma\, b^{\varsigma}/\sin\pi\varsigma\,\Gamma(\varsigma+1)(\varsigma-n-1), \tag{22}$$

at n= -1,0,1, ... In the transformation of the preliminary integral

(22) Mellin's representation of the inverse exponent with the anticlockwise contour L around the points 0,1,2,3,... of the complex plane was exploited. Taking into account that the subintegral function in the last integral (22) has the first order poles at all natural numbers k including zero if $k \neq n+1$ and the second order pole at k = n+1, one finds

$$S_n(b) = \sum_{k=0}' \frac{(-b)^k}{(n+1-k)k!} - \frac{(-b)^{n+1}}{(n+1)!}\Big(\ln b + C - \sum_{k=1}^{n+1} 1/k\Big), \tag{23}$$

where C is the Euler constant and the prime above the sum denotes that the term with k = n+1 should be there omitted. Particularly, at n = -1

$$S_{-1}(b) = -(\ln b + C) - \sum_{k=1}^{\infty}{}' (-b)^k / k\, k! \tag{24}$$

Exploiting the well-known properties of the Legendre polynomials and formula (23), one obtains

$$Q_{2n+1}(z) \equiv a^{-1}\Big[\delta_{n0} - d\sum_{\alpha=\pm 1} V_{2n+1}(a(b+\alpha z))/(d-1)\Big] =$$

$$-(-1)^n \frac{3(2n+1)!!}{4(2n)!!}\sum_{\alpha=\pm 1}(b+\alpha z)\Big[\ln(b+\alpha z) + C - 1/2 + \ln a\Big] + O(b^2), \tag{25}$$

$$\sum_{\alpha=\pm 1} \alpha V_{2n}(a(b+\alpha z)) = O(b), \quad d=3.$$

We have also

$$W_{1,2n+1}(a|z|) = -(-1)^n \frac{(2n+1)!!}{2(2n)!!}\big(\ln|z| + C + 1/2 + \ln a\big) + O(|z|) \tag{26}$$

$$W_{1,2n}(a|z|) = O(1), \quad d=3.$$

Exploiting the formula

$$\int_{-b}^{b} dy\, \mathrm{sign}(z-y)^{k-1} W_{k1}(a|z-y|) =$$

$$a^{-1}\Big[(d-1)\delta_{k1}/d - \sum_{\alpha=\pm1} \alpha^{k-1} V_k(a(b+\alpha z))\Big], \qquad (20)$$

obtained via the direct integration, one can rewrite the system of equations (15) in the form

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$$-\frac{1}{2i}\oint_L d\varsigma\, b^\varsigma / \sin\pi\varsigma\, \Gamma(\varsigma+1)(\varsigma-n-1), \qquad (22)$$

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$$S_n(b) = \sum_{k=0}' \frac{(-b)^k}{(n+1-k)k!} - \frac{(-b)^{n+1}}{(n+1)!}\Big(\ln b + C - \sum_{k=1}^{n+1} 1/k\Big), \qquad (23)$$

where C is the Euler constant and the prime above the sum denotes that the term with k = n+1 should be there omitted. Particularly, at n = -1

$$S_{-1}(b) = -(\ln b + C) - \sum_{k=1}^{\infty} (-b)^k / k\, k! \qquad (24)$$

Exploiting the well-known properties of the Legendre polynomials and formula (23), one obtains

$$Q_{2n+1}(z) \equiv a^{-1}\Big[\delta_{n0} - d\sum_{\alpha=\pm1} V_{2n+1}(a(b+\alpha z))/(d-1)\Big] =$$

$$-(-1)^n \frac{3(2n+1)!!}{4(2n)!!} \sum_{\alpha=\pm1}(b+\alpha z)\Big[\ln(b+\alpha z) + C - 1/2 + \ln a\Big] + O(b^2), \qquad (25)$$

$$\sum_{\alpha=\pm1} \alpha V_{2n}(a(b+\alpha z)) = O(b), \quad d=3.$$

We have also

$$W_{1,2n+1}(a|z|) = -(-1)^n \frac{(2n+1)!!}{2(2n)!!}\big(\ln|z| + C + 1/2 + \ln a\big) + O(|z|) \qquad (26)$$

$$W_{1,2n}(a|z|) = O(1), \quad d=3.$$

Let us make analogous estimations at d=2. Exploiting Mellin's representation of the inverse exponent and the definition of Euler's B -function one gets

$$Z_n(b) \equiv \frac{1}{\pi}\int_0^1 (1-c^2)^{1/2} c^n e^{-b/c}\, dc = \frac{1}{8\sqrt{\pi}\, i}\oint_L \frac{d\xi\, b^{\xi}\, \Gamma\left(\frac{n-\xi+1}{2}\right)}{\sin\pi\xi\, \Gamma(\xi+1)\Gamma\left(\frac{n-\xi+4}{2}\right)}, \qquad (27)$$

where the contour L coincides with that in the integral (22). Now we have second order poles at $\xi = 2k+n+1$, $k = 0,1,2,\ldots$ and the first order poles at $\xi = 0,1,\ldots n$ and $\xi = 2k+n+2$. It leads to a more complicated functional form of these integrals, the analogue of the expansion (23) has now the form $f(b)+g(b)\ln b$ with the $f(b), g(b)$ - functions expandable into the power series at b=0. Due to complicacy of the full expansion formulae for $Z_n(b)$ $n = -1,0,1,\ldots$, we will write them only with the accuracy sufficient for us. We have

$$Z_{-1}(b) = W_{11}(b) = -\frac{1}{\pi}(\ln b + C - \ln 2 + 1) + O(b),$$
$$Z_0(b) = V_1(b) = \frac{1}{4} + \frac{b}{\pi}(\ln b + C - \ln 2) + O(b^2), \qquad (28)$$
$$Z_m(b) = \Gamma\left(\frac{m+1}{2}\right)/4\pi^{1/2}\Gamma\left(\frac{m}{2}+2\right) + O(b), \quad m > 0.$$

Taking into account that Tchebyshev polynomials of the second kind, $U_n(c)$, determined as $\sin[(n+1)\varphi]/\sin\varphi$ expressed in terms of the polynomials of the variable $c = \cos\varphi$ can be written as

$$U_{2m}(c) = \sum_{k=0}^{m}(-1)^{m+k}\binom{m+k}{m-k}(2c)^{2k}, \quad U_{2m+1}(c) = \sum_{k=0}^{m}(-1)^{m+k}\binom{m+k+1}{m-k}(2c)^{2k+1},$$

one can obtain exploiting eqs. (28)

$$Q_{2m+1}(z) \equiv a^{-1}\Big[\delta_{m0} - d\sum_{\alpha=\pm 1} V_{2m+1}(a(b+\alpha z))/(d-1)\Big] =$$
$$\frac{2(-1)^m}{\pi}\sum_{\alpha=\pm 1}(b+\alpha z)\left[\ln(b+\alpha z) + C - \ln 2 + \ln a\right] + O(b^2),$$
$$\sum_{\alpha=\pm 1}\alpha V_{2m}(a(b+\alpha z)) = O(b), \qquad (29)$$
$$W_{1,2m+1}(a|z|) = -\frac{(-1)^m}{\pi}(\ln|z| + C - \ln 2 + 1 + \ln a) + O(|z|),$$
$$W_{1,2m}(a|z|) = O(1), \quad d = 2.$$

Let us solve the system of equations (21) at d=3 and 2 if $2b \equiv q \ll 1$. Since in the cases $d=2$ or 3, $D_{2m}(z) = O(q)$ and $W_{1,2m}(|z|) = O(1)$; hence, the term $D_1(z)$ is influenced by the functions D_{2m} only with the accuracy $O(q^2)$. Moreover, the character of the kernels $W_{1,2m+1}$ and the functions $Q_{2m+1}(z)$ allows one to conclude that $B_{2m+1}(z) = Q_{2m+1}(z) + O(q^2\ln q)$ at d=2 and 3. This is the reason why two first terms of the expansion over the small variable $b+\alpha z$ are free from influence of the parameters $F_{\omega l}$, l = 1,2,3,... . To calculate the corrections $O(q^2\ln^2 q)$ and $O(q^2\ln q)$ to the "dimensionless conductivity" $D_1(z)$ it is sufficient to substitute $D_{2m+1}(z)$ in the form $Q_{2m+1}(z)$ into the integral term of equations (21) neglecting the quantities $D_{2m}(z)$. By means of formulae (25), (26) and (29), after elementary integration and disregarding the $O(q^2)$ terms, we have

$$D_1(z) = -\frac{3}{4}\sum_{\alpha=\pm 1}(b+\alpha z)\left[\ln(b+\alpha z) + C - 1/2 + \ln a\right] + \qquad (30)$$
$$\frac{3}{16}\phi_3(z)\sum_{n=0}F_{\omega,2n+1}\left[(2n+1)!!\right]^2/(2n+1)(n+1)\left[(2n)!!\right]^2 + O(q^2), \quad d=3,$$

where

$$\Phi_3(z) = \phi(z) + 2(C + \ln a) g^2 \ln g \quad , g = 2b \ . \tag{31}$$

The function $\phi(z)$ is determined as follows:

$$\phi(z) = \sum_{\alpha=\pm 1} \int_{-b}^{b} dy \, \ln|z-y| \, f(b+\alpha y) =$$
$$\sum_{\alpha=\pm 1} \sum_{\beta=\pm 1} \int_0^{b+\beta z} dx \, \ln x \, f(b+\alpha z - \alpha\beta x) \tag{32}$$

at $f(x) = x \ln x$. Changing the integration variable x = bt, disregarding the terms $O(g^2)$ and expressing the result via the variable $z' = z/b$ one finds

$$\phi(z) = g^2 \ln^2 g - (3/2 + \ln 2) g^2 \ln g + \frac{1}{2} g^2 \ln g \sum_{\alpha=\pm 1} (1+\alpha z') \ln(1+\alpha z') =$$
$$- \frac{3}{2} g^2 \ln g + g \ln g \sum_{\alpha=\pm 1}' (b+\alpha z) \ln(b+\alpha z). \tag{33}$$

In an analogous way, we obtain for d=2

$$D_1(z) = -\frac{2}{\pi} \sum_{\alpha=\pm 1} (b+\alpha z) \left[\ln(b+\alpha z) + C - \ln 2 + \ln a \right] +$$
$$\frac{2}{\pi^2} \Phi_2(z) \sum_{n=0} F_{\omega, 2n+1} + O(g^2), \quad d=2 , \tag{34}$$

where

$$\Phi_2(z) = \phi(z) + 2 (C - \ln 2 + 1/2 + \ln a) g^2 \ln g \ . \tag{35}$$

The expression in the brackets of the last term of the solution (34) can be expressed as $(F_\omega(1) - F_\omega(-1))/2$, cf. formula (I), i.e. via the difference of forward and backward scattering probability. The result for d=3 is not so easily interpretable but it is easy to see that $[(2n+1)!!]^2/(2n+1)(n+1)[(2n)!!]^2 = O(1)$ and, hence, the series appearing in formula (30) is convergent provided that the series for $F(\vec{n}\vec{n}')$ is convergent too.

Let us calculate the average values of $\sigma_\omega(z)$ over z in the interval (-b,b) because it is a directly measurable quantity. It is to see that the averaged quantity $\phi(z)$ has the form

$g^2(\ln g - 1)^2 + O(g^2)$. Calculating also the remaining elementary integrals, one finds at d=3

$$\sigma_{\omega, av} = \frac{e^2 n \tau}{m^*} \Big\{ -3g(\ln g + C - 1 + \ln a)/4 +$$
$$3g^2 \ln g (\ln g - 2 + 2C + 2\ln a) \sum_{n=0} F_{\omega, 2n+1} [(2n+1)!!]^2 / 16(2n+1)(n+1)[(2n)!!]^2 \Big\}$$
$$+ O(g^2). \tag{36}$$

Analogous procedures at d=2 give

$$\sigma_{\omega, av} = \frac{e^2 n \tau}{m^*} \Big[-2g(\ln g + C - \ln 2 - 1/2 + \ln a)/\pi +$$
$$(F_\omega(1) - F_\omega(-1)) g^2 \ln g (\ln g - 1 - 2\ln 2 + 2C + 2\ln a)/\pi^2 \Big] + O(g^2). \tag{37}$$

Note that the term $O(g^2 \ln g)$ at $\omega = 0$ and d=2 and 3 always has the same sign as the term $O(g^2 \ln^2 g)$. These terms disappear in the relaxation time approximation. The sign of these terms is intuitively clear at d=2 and the static limit; prevailing forward scattering leads to growing conductivity, prevailing backward scattering leads to diminishing conductivity. For d=3, disregarding the terms $O(g^2 \ln^2 g)$ we come to the result /4/. The lack of influence of the parameters $F_{\omega l}$ on the term $O(g \ln g)$ in (36), (37) is quite clear because the main nonanalyiticty is weakened under the integral. On the other hand, an analogous effect for the $O(g)$ term of eqs. (36) and (37) cannot be explained so simply.

5. About the asymptotic solution for thick slabs

Let us now discuss our problem at $b \gg 1$; in this limit the conductivity $\sigma_\omega(z)$ should be close to the conductivity of the bulk sample, $\sigma_{\omega,\infty} = e^2 n \tau / (a - F_{\omega 1}/d) m^*$ unless $\min_\alpha (b+\alpha z) \lesssim 1$. Now, because the role of the singular term of the kernels (16),(17) is weakened, it is impossible to obtain an analytic approximate solution for $B_1(z)$ even at

$F_{\omega l} \sim \delta_{1l}$. It is easy to find the behaviour of the functions $V_k(b)$ and kernels W_{kn} (b) at b ≫ 1. If d=2, then $V_k(b) \sim W_{kn}(b) = O(e^{-b}/b^{3/2})$ at d=3 we have $V_k(b) \sim W_{kn}(b) = O(e^{-b}/b^2)$; the concrete forms of the asymptotic expansions are easy to obtain. The same behaviour should have the functions $B_k(z)$ at $\underset{\alpha}{Min}(b+\alpha z) \gg 1$. On the other hand, at small values $b+\alpha z$, $b, |z| \gg 1$, the functions $B_k(z)$ still should have the asymptotics similar to those expressed via eqs. (30) and (34), taking into account the relation between the functions $B_k(z)$ and $D_k(z)$ (19). It should be expansized that still the near-border region of the variable y, i.e. $b+\alpha y \lesssim 1$, $b, |y| \gg 1$ and small values of the modulus $|z-y|$ will play the most important role in the integration in eqs. (15). This is the reason why it is impossible to write the asymptotic formula for $B_1(z)$ valid in the whole domain of the variable z and, hence, also the expression for the averaged conductivity.

In the limit b ≫ 1, it is possible to simplify the problem by replacing it by the problem for the half-space (d=3) or the half-plane (d=2) neglecting exponentially small mutual influence of the functions $B_k(z)$ at $b+z \lesssim 1$ and $b-z \lesssim 1$. The equations for the simplified problem by means of eqs. (15), can be written in the form

$$B_k(z) = -V_k(az) + \sum_{n=1} \tilde{F}_{\omega n} \int_0^\infty dy\, sign(z-y)^{n+k} W_{kn}(a|z-y|) B_n(y), \tag{38}$$

where now the metal is contained in the half-space or half-plane $z, y \geq 0$. Note that the border $Z=-b$ in the previous notation, corresponds to $Z=0$ now. Taking into account formula (18) one can write that

$$\sigma_{\omega, av} = \sigma_{\omega,\infty}\left[1 + d\int_0^\infty dz\, B_1(z)/b(d-1)\right]. \tag{39}$$

It is worth remarking that this formula does not contain any exponentially small terms. The second term in the square brackets, diminishing the average conductivity at $\omega = 0$, can be treated as a result of additional resistance of the border, almost independent of the width of the slab at thick slabs and, hence, giving the contribution $O(b^{-1})$ to the average conductivity.

Equations of the type (38) are usually solved, if they are solved, only at one nonvanishing parameter $F_{\omega n}$, by the Wiener-Hopf method, cf. e.g. /9/. If $F_{\omega n} \sim \delta_{1n}$ then the Fourier

transforms of the integral kernels will have the form at a = 1, i.e., $\omega = 0$

$$W_{11}(k) = F_1\left[(1+k^{-2})k^{-1} arctg\, k - k^{-2}\right]/(8\pi)^{1/2}, \; d=3, \tag{40}$$

$$W_{11}(k) = F_1\left[(1+k^2)^{1/2} - 1\right]/(2\pi)^{1/2}k^2, \; d=2.$$

These functions, independently of their relatively simple form, do not allow one to perform the integral characteristic for the Wiener - Hopf technique and, hence, to obtain $\sigma_\omega(z)$ at b ≫ 1 and $F_{\omega n} \sim \delta_{1n}$. Note that at d=3, this problem is very familiar though more complicated than Milne's problem, cf. e.g. /9/.

If we treat the integral term in eq. (8) as perturbation, which corresponds to almost isotropic scattering, then one can write

$$B_1(z) = -\sum_{\alpha=\pm1} V_1(a(b+\alpha z)) - \sum_{n=1,\alpha=\pm1} \tilde{F}_{\omega n}\alpha^{n-1}\int_{-b}^{b} dy\, sign(z-y)^{n-1} W_{1n}(a|z-y|) V_n(a(b+\alpha y)) + \dots \tag{41}$$

Even this formula and, moreover, even at $F_{\omega n} = 0$, does not allow for additional simplification at $b, |z| \gg 1$ since then $b+\alpha z \lesssim 1$ is also possible. Let us obtain the average value of the function $B_1(z)$, $B_{1,av}$ at B_1 given by formula (41). After long but rather simple calculations, disregarding the terms $o(e^{-g})$ and explciting formulae (16) and (17) one finds

$$B_{1,av} = -(ga)^{-1}\left\{\frac{1}{4} + \sum_{n=1}\frac{F_{\omega n}a^{-1}}{n(n+1)}\left[\delta_{n1}/b + (-1)^n\int_0^1 dx\int_0^1 dy\, xy(1-x^2)(1-y^2)P_n'(x)P_n'(y)/2(x+y)\right]\right\} + O(e^{-g}g^{-3}) + O(F^2), \tag{42}$$

for d = 3 and

$$B_{1,av} = -(\pi g a)^{-1}\Big\{\frac{2}{3} + \sum_{n=1}(F_{\omega n}/a)\Big[\delta_{n1}/6 + \qquad (43)$$

$$2(-1)^n\int_0^1 dx\int_0^1 dy\, xy(1-x^2)^{1/2}(1-y^2)^{1/2}U_{n-1}(x)U_{n-1}(y)/\pi(x+y)\Big]\Big\} + O(e^{-g}g^{-\frac{5}{2}}) + O(F^2),$$

for d=2. It is not difficult to see that in the next order of the perturbation theory with respect to $F_{\omega n}$ there appear triple integrals, etc. but, according to formula (39), $B_{1,av}$ will not contain the terms $O(g^{-k})$ at $k > 1$.

Double integrals (42) at d=3 have the form $a_n + b_n \ln 2$ with a_n and b_n rational numbers. All these integrals can be obtained by the formula

$$\int_0^1 dx\int_0^1 dy\, x^n y^m/(x+y) = [(-1)^n + (-1)^m]\ln 2/(n+m+1) + \qquad (44)$$

$$[(-1)^n + (-1)^m]/(n+m+1)^2 + (-1)^n\sum_{k=1}^{n+m} 1/k(n+m+1) +$$

$$\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{-1}\sum_{\ell=1}^{k}\binom{k}{\ell}(m+n-\ell-1)^{-1} -$$

$$\sum_{k=0}^{n+m}(-1)^{n-k}\binom{n+m}{k}2^{k+1}(k+1)^{-2},$$

which can be obtained in an elementary way. The right-hand side of equality (44) is not manifestly symmetric under transition of n and m; it leads to some identity for the sums of binomial coefficients. It is worth mentioning that because of definite parity of every Legendre polynomial, formula (44) is useful for us only at $(-1)^n = (-1)^m$. As Tchebyshev polynomials of the second kind satisfy analogous property, a two-dimensional analogue of formula (44) is sufficient only at $(-1)^n = (-1)^m$. In such a case, after long but elementary transformations one gets

$$\int_0^1 dx\int_0^1 dy\, x^n y^m(1-x^2)^{1/2}(1-y^2)^{1/2}/(x+y) = (-1)^m(\pi/2)\Big[(n+m)!!/(n+m+3)!! -$$

$$(n+m-1)!!(n+m-1)/(n+m)!!(n+m+1)(n+m+3) + \qquad (45)$$

$$\frac{1}{8}\sum_{k=1}^{m}(-1)^k\binom{m}{k}\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\Gamma\Big(\frac{n+m+\ell-k+1}{2}\Big)\Gamma\Big(\frac{k-\ell}{2}\Big)\Big/\Gamma\Big(\frac{n+m+\ell-k+4}{2}\Big)\Gamma\Big(\frac{k-\ell+3}{2}\Big)\Big].$$

This formula is not manifestly symmetric under transposition of n and m. It is easy to see that at $(-1)^n = (-1)^m$ the expression in the square brackets of formula (45) is a rational number. Taking into account that $R_0(c) = 1$, $R_1(c) = dc$, d=2,3 via equalities (44),(45) one finds

$$\sigma_{\omega,av}/\sigma_{\omega,\infty} - 1 = -(ga)^{-1}\Big[\frac{3}{8} + (17+16\ln 2)F_{\omega 1}/210a + \qquad (46)$$

$$(48\ln 2 - 19)F_{\omega 2}/1260 +$$

$$\frac{3}{4}\sum_{n=3}(-1)^n\frac{F_{\omega n}}{an(n+1)}\int_0^1 dx\int_0^1 dy\, xy(1-x^2)(1-y^2)P_n'(x)P_n'(y)/(x+y)\Big] + O(e^{-g}g^{-3}) +$$

$$O(F^2),$$

at d=3 and

$$\sigma_{\omega,av}/\sigma_{\omega,\infty} - 1 = -(\pi g a)^{-1}\Big[\frac{4}{3} + F_{\omega 1}/5a + 16F_{\omega 2}/105a + \qquad (47)$$

$$4\sum_{n=3}(-1)^n(F_{\omega n}/\pi a)\int_0^1 dx\int_0^1 dy\, xy(1-x^2)^{1/2}(1-y^2)^{1/2}U_{n-1}(x)U_{n-1}(y)/(x+y)\Big]$$

$$+ O(e^{-g}/g^{5/2}) + O(F^2),$$

where $\sigma_{\omega,\infty}$ is the conductivity of the bulk sample. Note that the coefficients at $F_{\omega n}/g a^2\pi$ in formula (47) will be rational numbers. The coefficients at $-F_{\omega n}/g a^2$ are relatively small at d=2,3 and n=1,2. At d=3 we have 0.1338 and 0.0113 for n=1 and 2, respectively. At d=2 analogously we have 0.0637 and 0.0485. At $F_{\omega n}=0$ formula (46) coincides with that of ref. [4] and [1].

7. Conclusions

As one can verify, the perturbation procedure applied directly to the kernel F_{ω} in equation (8) leads for diffuse borders to serious mathematical difficulties, at least for thin slabs. Namely, even in the first order perturbation term there appear singular double integrals. The natural way of their regularization by their symmetrization with respect to variables x,y, cf. eqs. (46) and (47), leads unfortunately to false results for thin slabs. Hence, the system of the integral equations (15) shows its usefulness even in simplest calculations beyond the relaxation time approximation.

It is worth emphasizing the analogue between the electrical transport in isotropic metals and the one-velocity approach to the neutron transport in isotropic media, cf. e.g. [10]. In this last case, the theory is richer because of possibilities of absorption as well as the production of neutrons. Moreover, the form of the source term, corresponding to the free term of eq. (8), should be restricted for neutrons to non-negative functions and, on the other hand, in this case $A^{\wedge}(\vec{n}\vec{n}')=0$. Our boundary conditions correspond to the walls absorbing neutrons. It seems that the technique developed here can also be modified for neutrons provided that their sources will be spatially homogeneous in the volume of the slab.

The collision term characteristic of the kinetic equation (8) can contain also the effect of interelectron scattering. It is connected with the idea to represent two-body collision integral for degenerate Fermi liquid in the reduced form appearing in eq. (8). This idea has been introuced by Abrikosov and Khalatnikov [11] and has been developed in the papers by Brooker and Sykes [12] and Wölfle [13].

In this case the collision integral becomes also temperature-dependent, in the manner characteristic of degenerate Fermi system, $a+b(T/\varepsilon_F)^2,\ k_B=1$. According to the analysis by Kagan and Zhernov [14], the scattering of electrons with the oscillations of impurities leads to the enhancement of the residual resistivity by the term $O(T^2)$. Mechanisms like that are not taken into account in our collision integral. This is the reason why one can include the interelectron collision into our scheme but only at T=0 and in the elastic limit.

Even amplitudes of the density of scattering probability (I) disappear in $\sigma_{\omega}(z)$ till the terms $O(g^2 \ln g)$ because of z-inversion invariance of equation (8). It is possible to break this symmetry by solving the boundary problem with one diffuse and one specular border. It is not only mathematical construction because of a possibility of preparation of samples. It seems that even in this case two biggest terms of $\sigma_{\omega}(z)$, $O(g\ln g)$ and $O(g)$, remain unaffected for thin slabs by the amplitudes $F_{\omega\ell}$. The solution of this problem will be obtained in the near future.

Acknowledgements. The author is greatly indebted to M.I.Kaganov, who is the spiritus movens of an enterprise out- of-relaxation-time--calculations, for multiple stimulating discussions and valuable comments. The instructive role of G.V.Efimov in asymptotics of integrals at $g \ll 1$ is also gratefully acknowledged. The author is also greatly indebted to D.E.Khmelnitsky who critically read the manuscript and suggested some improvements.

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Received by Publishing Department
on September 13, 1989.