ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E17-89-641

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MEAN VALUES OF FIELD AND ATOMIC OPERATOR PRODUCTS IN THE PROBLEMS OF QUANTUM OPTICS

Submitted to "International Journal of Modern Physics B"

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1989

DEDICATED TO PROFESSOR N.N.BOGOLUBOV
ON THE OCCASION OF HIS 80TH BIRTHDAY

When the model problems of quantum optics are examined, one of the important question consists in the calculation of the mean values of the field and atomic operator products [1,2]. The problem of one-photon process in a point-like two-level N-atom system in the dipolar and rotating wave approximations can be indicated as an example. The model hamiltonian has a Dicke form

$$H = H_a + H_R + H_{aR} \,, \tag{1}$$

$$H_a = \sum_k \hbar\, \omega_k\, a_k^+ a_k \quad , \qquad [\, a_k \,,\, a_{k'}^+ \,] = \delta_{kk'} \,, \qquad ,$$

$$H_R = \hbar\, \Omega\, R_3 \quad , \qquad H_{aR} = \sum_k \hbar\, g_k \,(\, R^+ a_k \; + \; a_k^+ R^- \,) \quad ,$$

$$[\, R_3 \,,\, R^\pm \,] = \pm R^\pm \quad , \qquad [\, R^+ \,,\, R^- \,] = 2\, R_3 \;.$$

Here a^+ , a are the field operators and $R = \sum_{f=1}^{N} R(f)$ are the collective operators of two-level atoms.

Let $\mathfrak{A}$ be an arbitrary dynamical variable of the system under consideration. Then its mean value in the Heisenberg representation is defined as

$$< \mathfrak{A} >_t \;=\; \mathrm{Tr}\,\{\, \mathfrak{A}(t)\, \rho(t_0)\,\} \,, \tag{2}$$

where

$$\mathfrak{A}(t) \;\equiv\; U^{-1}(t,t_0)\, \mathfrak{A}\, U\,(t,t_0) \,,$$

$$i\hbar\, \frac{d}{dt}\, U\,(t,t_0) \;=\; H\, U\,(t,t_0) \,, \qquad U\,(t_0,t_0) \;=\; 1$$

and $\rho\,(t_0)$ is the density matrix of the system at the initial time moment $t = t_0$.

Let us consider, for example, the time evolution of the mean number of photons $\langle n_k \rangle_t \equiv \langle a_k^+ a_k \rangle_t$. The Heisenberg equations for the field operators

$$i \frac{d}{dt} a_k = \omega_k a_k + g_k R^- , \quad i \frac{d}{dt} a_k^+ = - \omega_k a_k^+ - g_k R^+$$

have the following formal solutions

$$a_k (t) = \tilde{a}_k - i A_k , \quad a_k^+ (t) = \tilde{a}_k^+ (t) + i A_k^+ , \quad (3)$$

where

$$\tilde{a}_k \equiv a_k (t_0) \exp \{ - i\omega_k (t - t_0) \} ,$$

$$A_k \equiv g_k \int_{t_0}^{t} d \tau \; R^- (\tau) \exp \{ -i \omega_k (t - \tau) \} .$$

Therefore we have

$$\langle n_k \rangle_t = \langle n_k \rangle_{t_0} + \langle A_k^+ A_k \rangle_t - i (\langle \tilde{a}_k^+ A_k \rangle_t - \langle \tilde{A}_k^+ a_k \rangle_t) . \quad (4)$$

The last two terms in (4) contain the mean values we are interested. Usually, the mean values of this type are calculated for the cases of a vacuum initial state of the field or of a chaotic initial state [1-4]. Here we consider the more general case when

$$\rho (t_0) = \rho_a \otimes \rho_R$$

and the initial state of the field can be chaotic, coherent or squeezed. For this aim ρ_a should be chosen in the following form

$$\rho_a = D (\alpha) S (\xi) T (\beta) S^+(\xi) D^+(\alpha) . \quad (5)$$

For clarity, we restrict ourselves by the one-mode case. Then for the displacement operator we have

$$D (\alpha) = \exp (\alpha a^+ - \alpha^* a) .$$

$S (\xi)$ is the so-called squeezing operator

$$S (\xi) = \exp (\frac{1}{2} \xi^* a^2 - \frac{1}{2} \xi a^{+2})$$

and $T (\beta)$ describes the chaotic state with the temperature T

$$T (\beta) = e^{- \beta n} / Tr (e^{-\beta n}) , \quad \beta \equiv \hbar \omega / k_B T , \quad n \equiv a^+ a .$$

The following commutation expressions take place :

$$\begin{aligned} a S (\xi) &= S (\xi) (\mu a - \nu a^+) , \\ S^+(\xi) a &= (\mu a - \nu a^+) S^+(\xi) \\ a^+ S (\xi) &= S (\xi) (-\nu^* a + \mu^* a^+) , \\ S^+(\xi) a^+ &= (- \nu^* a + \mu^* a^+) S^+(\xi) , \\ a T (\beta) &= e^{-\beta} T (\beta) a , \\ a^+ T (\beta) &= e^{\beta} T (\beta) a^+ . \end{aligned} \quad (6)$$

Here μ and ν are complex parameters of the Bogolubov canonical transformation introducing the squeezed state ($\xi \equiv re^{i\theta}$ $\mu = \cosh r$, $\nu = e^{i\theta} \sinh r$).

Let us introduce the following notation: $\delta a = a - \alpha$. Then from (5) and (6) one can obtain

$$\delta a \; \rho_a = \rho_a \{ |\mu|^2 e^{-\beta} - |\nu|^2 e^{\beta} \} \delta a - 2 \mu \nu \, sh \beta \; \delta a^+ .$$

So for the mean value (2) of the product of an arbitrary atomic operator O (t) and of δ a we have

$$\langle O \,\delta a \rangle_t = (|\mu|^2 e^{-\beta} - |\nu|^2 e^{\beta}) \langle \delta a \, O \rangle_t - 2 \mu \nu \, \mathrm{sh}\, \beta \langle \delta a^+ O \rangle_t . \qquad (7a)$$

By analogy with (7a) we get

$$\langle O \,\delta a^+ \rangle_t = 2 \mu^* \nu^* \, \mathrm{sh}\, \beta \langle \delta a \, O \rangle_t + (|\mu|^2 e^{\beta} - |\nu|^2 e^{-\beta}) \langle \delta a^+ O \rangle_t . \qquad (7b)$$

Here $\langle ... \rangle$ is defined by the expression (2).

Now solving equations (7) with respect to $\langle O \,\delta a \rangle_t$, $\langle O \,\delta a^+ \rangle_t$ and taking into account the equalities

$$\langle a^+ , a \rangle_{t_0} = |\nu|^2 + \frac{|\mu|^2 + |\nu|^2}{e^{\beta} - 1} ,$$

$$\langle a , a \rangle_{t_0} = - \mu \nu \frac{e^{\beta} + 1}{e^{\beta} - 1} ,$$

where $\langle \mathfrak{A} , \mathfrak{B} \rangle \equiv \langle \mathfrak{A} \mathfrak{B} \rangle - \langle \mathfrak{A} \rangle \langle \mathfrak{B} \rangle$, we finally obtain

$$\langle O \, a \rangle_t = \langle a \rangle_{t_0} \langle O \rangle_t + \langle a^+ , a \rangle_{t_0} \langle [a , O] \rangle_t - \langle a , a \rangle_{t_0} \langle [a^+ , O] \rangle_t ,$$

$$\langle O \, a^+ \rangle_t = \langle a^+ \rangle_{t_0} \langle O \rangle_t + \langle a^+ , a^+ \rangle_{t_0} \langle [a , O] \rangle_t - \langle a , a^+ \rangle_{t_0} \langle [a^+ , O] \rangle_t . \qquad (8a)$$

In an analogous way one can also obtain the following expressions

$$\langle a \, O \rangle_t = \langle a \rangle_{t_0} \langle O \rangle_t + \langle a , a \rangle_{t_0} \langle [O , a^+] \rangle_t - \langle a , a^+ \rangle_{t_0} \langle [O , a] \rangle_t ,$$

$$\langle a^+ O \rangle_t = \langle a^+ \rangle_{t_0} \langle O \rangle_t + \langle a^+ , a \rangle_{t_0} \langle [O , a^+] \rangle_t - \langle a^+ , a^+ \rangle_{t_0} \langle [O , a] \rangle_t . \qquad (8b)$$

These exact expressions (8) generalize the corresponding expressions of papers [1-3]. It should be noted that (8) can be easily extended to the multimode case. With the aid of expressions (8) diverse mean values of the product of the field and atomic operators can be calculated.

Let us now return to the calculation of the photon number mean value (4) for the model problem (1). From (8) we get

$$\begin{aligned} \langle n_k \rangle_t = \langle n_k \rangle_{t_0} + \langle A_k^+ A_k \rangle_t - i \big(& \langle \tilde{a}_k^+ \rangle_{t_0} \langle A_k \rangle_t - \\ & - \langle \tilde{a}_k \rangle_{t_0} \langle A_k^+ \rangle_t + \langle \tilde{a}_k^+ , \tilde{a}_k \rangle_{t_0} \langle [A_k , \tilde{a}_k^+] \rangle_t - \\ & - \langle \tilde{a}_k^+ , \tilde{a}_k \rangle_{t_0} \langle [\tilde{a}_k , A_k^+] \rangle_t - \langle \tilde{a}_k^+ , \tilde{a}_k^+ \rangle_{t_0} \langle [A_k , \tilde{a}_k] \rangle_t - \\ & + \langle \tilde{a}_k , \tilde{a}_k \rangle_{t_0} \langle [\tilde{a}_k^+ , A_k^+] \rangle_t \big) . \end{aligned} \qquad (9)$$

Using here expressions (3) for the commutators of the atomic and field operators we obtain

$$< n_k >_t = < n_k >_{t_0} + < A_k^+ A_k >_t + < \tilde{a}_k^+ \tilde{a}_k >_{t_0} <[A_k^+ , A_k]>_t -$$

$$- i(< \tilde{a}_k^+ >_{t_0} < A_k >_t - < \tilde{a}_k >_{t_0} < A_k^+ >_t) +$$

$$+ < \tilde{a}_k^+ , \tilde{a}_k^+ >_{t_0} <[A_k^> , A_k^<]>_t + < \tilde{a}_k , \tilde{a}_k >_{t_0} <[A_k^{+<} , A_k^{+>}]>_t , \qquad (10)$$

where

$$<[A_k^> , A_k^<]>_t = <[A_k^{+<} , A_k^{+>}]>_t^* =$$

$$= - g_k^2 \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\sigma \, e^{-i\omega_k(2t - \tau - \sigma)} <[R^-(\tau) , R^-(\sigma)]> .$$

This expression contains only the mean values of the field operators at the initial time moment $t = t_0$ and the mean values of various combinations of the atomic operators. To calculate $< n_k >_t$ one ought to complete (10) by suitable equations for the mean values of the atomic operators. For an arbitrary atomic operator $O(t)$ we have

$$i\hbar \frac{dO}{dt} = [O , H] .$$

Averaging this expression in compliance with (2) and (5), using equalities (3) and (8) we obtain

$$\frac{d}{dt} < O >_t + i\Omega <[O , R_3]>_t = \sum_k g_k \{ <[R^+, O] A_k >_t +$$

$$+ < A_k^+ [O , R^-] >_t - i< \tilde{a}_k >_{t_0} <[O , R^+]>_t +$$

$$- i< \tilde{a}_k^+ >_{t_0} <[O , R^-]>_t - < \tilde{a}_k , \tilde{a}_k >_{t_0} <[[O , R^+] , A_k^+]>_t -$$

$$- < \tilde{a}_k^+ , \tilde{a}_k^+ >_{t_0} <[[O , R^-] , A_k]>_t -$$

$$- < \tilde{a}_k^+ , \tilde{a}_k >_{t_0} <[[O , R^+] , A_k]>_t -$$

$$- < \tilde{a}_k^+ , \tilde{a}_k >_{t_0} <[[O , R^-] , A_k^+]>_t \} . \qquad (11)$$

This expression represents a hierarchic equation for the mean values of the atomic operators and depends only on the initial mean values of the field variables. Thus the application of the exact equalities (8) permits us to eliminate the Bose-operators from the dynamical equations which describes the time evolution of system (1). It should be emphasised that the expressions (8) can be also applied for the examination of the dynamical behaviour of more complicated model problems then (1).

In the particular case of the chaotic initial state ($\rho_a = T(\beta)$) expressions (8) coincides with the result of the so-called Lemma of Bogolubov [3,5]. In the simplest case of $T = 0$ using the Born-Markoff approximation together with the decoupling $< R^+R^-R_3 > \simeq < R^+R^- >< R_3 >$ from (10) and (11) for the radiation intensity

$$\mathcal{I}_t = \sum_k \hbar\omega_k \frac{d}{dt} < n_k >_t$$

one can obtain the known Rehler-Eberly equation (e.g.see [4,6]).

A far more complicated situation takes place in the case of the coherent initial state of the field, when $< \tilde{a} >_{to} = \alpha \, e^{-i\omega(t-to)}$, $< \tilde{a}^+ >_{to} = \alpha^* \, e^{i\omega(t-to)}$. Then instead of (10) and (11) we have

$$< n_k >_t = |\alpha_k|^2 + < A_k^+ A_k >_t - i\{ \alpha_k^* \, e^{i\omega_k(t-t_0)} < A_k >_t -$$

$$- \alpha_k \, e^{-i\omega_k(t-t_0)} < A_k^+ >_t \} , \qquad (12)$$

$$\frac{d}{dt} < O >_t + i\Omega <[O , R_3]>_t =$$

$$= \sum_k g_k \{<[R^+, O] A_k >_t + < A_k^+ [O , R^-]>_t +$$

$$-i \alpha_k e^{-i\omega_k(t-t_0)} <[O , R^+]>_t -i \alpha_k^* e^{i\omega_k(t-t_0)} <[O , R^-]>_t \} .$$

Examination of equations (12) and of the time behaviour of the corresponding radiation intensity presents an original problem and needs a special investigation.

Expressions (8) can also be used for the calculation of diverse field correlation functions, such as the degrees of coherence of any order etc., in the multi-atom system with squeezed vacuum.

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Received by Publishing Department
on September 7, 1989.

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E17-89-641

Средние значения произведений операторов поля и атомов в задачах квантовой оптики

Найдены точные выражения для средних значений от произведений операторов поля и атомов в общем случае хаотического, когерентного или сжатого начального состояния поля. Эти соотношения позволяют исключить переменные поля из динамических уравнений.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

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E17-89-641

Mean Values of Field and Atomic Operator Products in the Problems of Quantum Optics

Exact expressions for the mean values of the products of field and atomic operators are obtained in the general case of a chaotic, coherent or squeezed initial state of the field. These expressions permit one to eliminate the field variables from the dynamical equations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989