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# EXACTLY SOLUBLE CLASS OF HAMILTONIANS WITH PAIR INTERACTION

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Класс асимптотически точно решаемых частных случаев Гамильтониана с парным взаимодействием

Дан способ построения асимптотически точных (при больших значениях числа частиц n) собственных функций и собственных значений гамильтониана теории сверхпроводимости для случая, когда коэффициентная функция $c_{ij}$, входящая в парное взаимодействие, $H' = -\sum c_{ij} a_i^* a_{-i}^* a_{-j} a_j$ – постоянная, а кинетическая энергия является ступенчатой функцией импульса. Метод работы состоит в использовании спинового формализма Рака и в замене разностного уравнения Шредингера на точно решаемое /при $n \to \infty$ / дифференциальное. Работа дает адекватную математическую реализацию идеи Андерсона.

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Exactly Soluble Class of Hamiltonians with Pair Interaction

A method is proposed to construct exact wave functions and eigenvalues for a certain class of superconductivity Hamiltonian for a large number of particles. We suppose that the interaction coefficient function is a constant and that the kinetic energy is a step-like function of the momentum. For such Hamiltonians we prove in particular the existence of new excitation spectrum branches by studying the Schrödinger equation in the Reach quasispin formalism where we replace the finite difference equation by the corresponding, exactly solvable differential equation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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## 1. INTRODUCTION

We consider the superconductivity model Hamiltonian as follows[1,2],

$$\hat{H} = \sum_{i=1}^{M} T_i (a_i^* a_i + a_{-i}^* a_{-i}) - \frac{\Lambda}{n} \sum_{1 \le i, j \le M} c_{ij} a_i^* a_{-i}^* a_{-j} a_j . \tag{1.1}$$

As usual[3], we rewrite eq. (1.1) in the form

$$\hat{H} = \sum_{i=1}^{M} 2T_i \left(\frac{1}{2} + \frac{1}{2}\sigma_{zi}\right) - \frac{\Sigma}{n} \sum_{1 \le i, j \le M} c_{ij} \sigma_i^+ \sigma_j^- . \tag{1.2}$$

Here $\sigma_i^{\pm} = \sigma_{xi} \pm i\sigma_{yi}$, $\sigma$ are the Pauli matrices. The operator of the number of particles is given by

$$\hat{N} = \sum_{i=1}^{M} (a_i^* a_i + a_{-i}^* a_{-i}) = \sum_{i=1}^{M} 2\frac{1}{2}(1 + \sigma_{zi}). \tag{1.3}$$

In this wirk n always denotes the number of pairs in our system, so that the solution $\psi$ of the Schrödinger equation

$$(\hat{H} - E)\psi = 0 \tag{1.4}$$

satisfies the condition

$$(\hat{N} - 2n)\psi = 0 . \tag{1.5}$$

Up to now there exists one method of analysing Hamiltonians of the type (1.1), (1.2) for $n \to \infty$, the Bogolubov method[2,4,5]. This method is exact in the thermodynamical sense. In this paper we construct exact (if $n \to \infty$) eigenfunctions and eigenvalues of the Schrödinger equation if the kinetic energy is a step-like function of the momentum and $c_{ij} = 1$ (cf.(1.1)). Our method consists in substituting for large n the finite difference equation deduced from the Schrödinger equation by the corresponding differential equation. The article is organized as follows. In sec. 2 we study the exactly solvable double Racah model (see [6,7] ). The parameters in (1.2) are chosen as follows:

$$M = 2n$$

$$T_i = \begin{cases} 0, & 1 \le i \le n \\ 1, & n+1 \le i \le 2n \end{cases} \tag{1.6}$$

$$c_{ij} = \begin{cases} 1, & \text{if } 1 \le i, j \le n \quad \text{or if } n+1 \le i, j \le 2n, \\ 0, & \text{otherwise}. \end{cases}$$

This model is the simplest one which has a "new" branch of the excitation spectrum which is not given by the Bogolubov u-v transformation method.

In sec. 3 we consider the nontrivial model with the parameters

$$M = 2n$$

$$T_i = \begin{cases} 0, & 1 \le i \le n \\ 1, & n+1 \le i \le 2n \end{cases} \tag{1.7}$$

$$c_{ij} = 1$$

in eq. (1.2). For this case and a large number of pairs $(n \to \infty)$ we construct the wave functions and eigenvalues of the Hamiltonian which are exact if the condition (3.16) is satisfied.

In sec. 4 we discuss the general case where the interaction coefficient function is a constant and the kinetic energy is a step-like function of the momentum. The parameters in eq.(1.2) are chosen as follows:

$$M = n \cdot \frac{d}{b},$$

$$T_i = L_k, \quad \text{if } \frac{n}{b}(k-1) < i \le \frac{n}{b} \cdot k, \; k = 1, 2, \ldots, d, \tag{1.8}$$

$$c_{ij} = 1.$$

Here the numbers $d$, $b$, $n/b$ are integers, $d > b$. Since every continuous function can be approximated by a step-like function as exact as necessary the only assumption of our method is $c_{ij} = 1$.



One can prove that in the limit, where the kinetic energy becomes a continuous function, the new excitation spectrum branches (with the frequences $\omega_a$, $a = 1, 2, \ldots d-1$ cf.(4.34)) will be confluent with the old ones (with the frequences $r_k$, $k = 1, 2, \ldots d$ cf.(4.12)). Note that our method may be generalized to the case of an interaction function of the form

$$c_{ij} = \sum_a g_a c_{ia} c_{ja}.$$

In sec. 5 we compare our results with some results of the u-v transformation method. Our exact (asymptotically exact) equations for the spectrum lead to new branches besides those branches of the spectrum which are given by the u-v transformation method (see the dependence on the parameters $m, k$, $k_a$ in equation (2.8), (3.15), (4.41), respectively).

It is necessary to remark that these new branches of the spectrum do not contribute essentially to the statistical sum

$$z(\theta, n, M) = \sum_\beta \langle \beta, n | e^{-\frac{1}{\theta}\hat{H}} | \beta, n \rangle \tag{1.9}$$

(the summation runes over all states with n pairs, M is the number of vacancies in eq.(1.2)) in the sense that they do not contribute to the quantity

$$\lim_{\substack{n \to \infty \\ M = \gamma n}} \ln z(\theta, n, M)/n. \tag{1.10}$$

The reason is that the multiplicity of the state with quantum numbers $\ell_k$, $k_a$, $k = 1,2,\ldots,d$, $a = 1,2,\ldots,d-1$, $k_a = 1,2,3,\ldots$ does not depend on $k_a$ [7]:

$$\rho(\ell_1, \ldots, \ell_d, k_1, \ldots, k_{d-1}) = \rho(\ell_1, \ldots, \ell_d) = \prod_{p=1}^{d} f(\ell_p), \tag{1.11}$$

$$f(\ell_p) = \frac{n'!(n'-2q+1)}{q!(n'-q+1)!}, \quad \ell_p = \frac{n'}{2} - q. \tag{1.12}$$

Here $n' = \frac{n}{b}$ is the number of added spins; it is supposed to be even so that all numbers $\ell_p$ are integer. Since the multiplicity (1.11) is independent of $k_a$, the summation over $k_a$, $a = 1,2,\ldots, d = 1$ gives us some factor in eq. (1.9), weakly dependent on $n$.

This work is related in a way with refs.[8-10] devoted to the problem of error of the $u-v$ transformation method.

## 2. TRIVIAL EXAMPLE

Here we study the Schrödinger equation (1.4) with the Hamiltonian (1.2), (1.6). Let us introduce spin-vectors $\vec{S}_k$, $k = 1, 2$

$$\frac{1}{2}\sum_1^n \vec{\sigma}_i = \vec{S}_1, \quad \frac{1}{2}\sum_{n+1}^{2n} \vec{\sigma}_i = \vec{S}_2. \tag{2.1}$$

Spin $\vec{S}_k$ is described by the quantum numbers $\ell_k$, $m_k$,

$$\ell_k \le n/2, \quad |m_k| \le \ell_k. \tag{2.2}$$

Now the Hamiltonian (1.2), (1.6) looks like

$$\hat{H} = n + 2S_{z2} - \frac{\Lambda}{n}(S_1^+S_1^- + S_2^+S_2^-), \tag{2.3}$$

where

$$\begin{aligned}(\ell'm'|S_z|\ell m) &= m\delta_{\ell\ell'}\delta_{mm'},\\ (\ell'm'|S^+|\ell m) &= \sqrt{(\ell+m+1)(\ell-m)}\,\delta_{\ell\ell'}\delta_{m+1,m'},\\ S^- &= (S^+)^*.\end{aligned} \tag{2.4}$$

Equation (2.4) implies that the Hamiltonian (2.3) is diagonal. The condition (1.5) assumes the form

$$(S_{z1} + S_{z2})\psi = 0$$

and gives

$$m_2 = -m_1 = m. \tag{2.5}$$

Thus, the eigenvalues of the Hamiltonian are

$$H_{\ell_1\ell_2 m} = n + 2m - \frac{\Lambda}{n}\{\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - 2m^2\}. \tag{2.6}$$



Let us denote

$$m_0 = -\frac{n}{2\Lambda}, \qquad \ell_k = \frac{n}{2} - p_k, \quad k = 1, 2. \tag{2.7}$$

Then eq. (2.6) turns into

$$H_{\ell_1\ell_2 m} = E_0 + 2\frac{\Lambda}{n}(m - m_0)^2 + \Lambda(p_1 + p_2 - \frac{p_1^2 + p_2^2}{n}), \tag{2.8}$$

$$0 \le p_k \le \frac{n}{2}, \quad k = 1, 2,$$

where

$$E_0 = -\frac{\Lambda\cdot n}{2}(1 - \frac{1}{\Lambda})^2 - \Lambda. \tag{2.9}$$

If $\Lambda > 1 = \Lambda_0$ then $|m_0| < n/2$ (see eqs. (2.7), (2.2)) and $E_0$ is the ground state energy. On the other hand, if $\Lambda < 1$, to the ground state there corresponds $\ell_1 = \ell_2 = n/2 = -m$; then the ground state energy is $(-\Lambda)$.

## 3. THE SIMPLEST NONTRIVIAL HAMILTONIAN

Here we consider the Hamiltonian (1.2), (1.7) in the notation of sec. 2

$$\hat{H} = n + 2S_{z2} - \frac{\Lambda}{n}(S_1^+ + S_2^+)(S_1^- + S_2^-). \tag{3.1}$$

Later on, the number $n$ will be considered to be large; the numbers $\ell$, $m$, $\ell \pm m$ will be supposed to be of an order of magnitude $n$; correspondingly, we substitute $\sqrt{\ell^2 - m^2}$ for the radical in eq. (2.4). Taking into account (2.5) the Schrödinger equation (1.4) has the form

$$V(m)\psi_{\ell m} - \frac{\Lambda}{n}(\ell^2 - m^2)\{\psi_{\ell m+1} + \psi_{\ell m-1} - 2\psi_{\ell m}\} = 0, \tag{3.2}$$

$$V(m) = n + 2m - 4\Lambda(\ell^2 - m^2)/n - E. \tag{3.3}$$

here we have taken $\ell_1 = \ell_2 = \ell$ (cf sec. 2) for simplicity. Like in sec. 2 we represent $V(m)$ as

$$V(m) = A + \frac{4\Lambda}{n}(m - m_0)^2 , \tag{3.4}$$

$$m_0 = -\frac{n}{4\Lambda}. \tag{3.5}$$

Suppose

$$4\Lambda > n/\ell , \quad 2\Lambda > 1, \quad \Lambda = 1/2, \tag{3.6}$$

so that $|m_0| < \ell$ (cf. eq. (2.2)). We have

$$A = n - \frac{4\Lambda}{n}\ell^2 + \frac{4\Lambda}{n} m_0^2 - E + 2m_0. \tag{3.7}$$

The crucial point of this and of the next section is the transformation of the difference equation (3.2) into a differential equation for large $n$ (as $n \to \infty$). We have

$$\psi_{\ell m+1} + \psi_{\ell m-1} - 2\psi_{\ell m} = \left(\frac{d^2}{dm^2} + \frac{1}{12}\frac{d^4}{dm^4} + \dots\right)\psi_{\ell m} . \tag{3.8}$$

The substitution

$$m - m_0 = y\sqrt{n} \tag{3.9}$$

in eqs. (3.2), (3.3), (3.4) leads to

$$\left[F(y,n) - \frac{d^2}{dy^2} - \frac{1}{12n}\frac{d^4}{dy^4} - \dots\right]\Phi(y) = 0, \tag{3.10}$$

where

$$F(y,n) = \frac{A + 4\Lambda y^2}{\Lambda(B + cy/\sqrt{n} + y^2/n)} \equiv (\omega^2 y^2 - 2\epsilon)\left(1 - \frac{c}{B}\frac{y}{\sqrt{n}} + \dots\right), \tag{3.11}$$



$$\omega^2 = 4/B, \quad 2\epsilon = -A/(\Lambda \cdot B), \qquad B = (\ell^2 - m_0^2)/n^2, \quad c = -2m_0/n. \tag{3.12}$$

So, for large $n$, eq. (3.10) reduces to the quantum hadronic oscillator equation. Cobsequently, the eigenfunctions and eigenvalues are

$$\Phi_k(y) = \exp\{-\omega/2 \cdot y^2\} H_k(\sqrt{\omega}\cdot y) + \dots , \tag{3.13}$$

$$\epsilon_k = (k + \tfrac{1}{2})\omega + \dots . \tag{3.14}$$

Using eqs. (3.12), (3.7) we get

$$E = n - \frac{4\Lambda}{n}\ell^2 - \frac{n}{4\Lambda} + 4\Lambda\cdot k\sqrt{\frac{\ell^2 - (n/4\Lambda)^2}{n^2}} + \dots . \tag{3.15}$$

Note that eq. (3.14) is valid only in a certain region

$$k < g\cdot n \ll n . \tag{3.16}$$

If the difference $2\Lambda - 1$, $2\Lambda > 1$, tends to zero, the number $g$ also tends to zero.

## 4. THE HAMILTONIAN WITH A STEP-LIKE KINETIC ENERGY

4.1. Let us study the Schrödinger equation (1.4) with the Hamiltonian (1.2), (1.8). Using the notation of sec. 2 we have

$$\hat{H} = \sum_{k=1}^{d} L_k(n/b + 2S_{zk}) - (\Lambda/n)\sum_{k,p=1}^{d} S_k^+ S_p^- \tag{4.1}$$

(this time $\vec{S}_k = \frac{1}{2}\Sigma\,\vec{\sigma}_i$, $n(k-1)/b < i \le nk/b$) ) and the Schrödinger equation assumes the form

$$\{V(m_1, m_2, \dots, m_d) - (\Lambda/n)\sum_{1\le k,p\le d} R_k R_p \frac{1}{2}(\partial/\partial m_k - \partial/\partial m_p)^2 + \dots\}\psi_{\ell_1 m_1 \dots \ell_d m_d} = 0, \tag{4.2}$$

where

$$R_k = \sqrt{\ell_k^2 - m_k^2}\,, \tag{4.3}$$

$$V(m_1,\ldots,m_d) = \sum_{k=1}^{d} L_k\,(n/b + 2m_k) - \Lambda f^2/n - E\,, \tag{4.4}$$

$$f = \sum_{k=1}^{d} R_k \tag{4.5}$$

(cf. eqs. (3.2), (3.8)).

The numbers $L_k$ and b will be supposed to be fixed and as in sec. 3 we suppose further $n/b$ to be large. Remark that the numbers $m_k$, $k = 1, 2, \ldots, d$ are connected by the condition (1.5)

$$\sum_{k=1}^{d} (n/2b) + m_k) = n\,. \tag{4.6}$$

4.2. To calculate the minimum of the function (4.4) under condition (4.6) we consider the equations for the absolute extremum of the function $V - 2a\sum m_k$

$$\frac{\partial V}{\partial m_k} - 2a = 0\Big|_{m=m^\circ} \tag{4.7}$$

where a is a constant.

Denoting by

$$a_k = L_k - a$$
$$\Delta = f\cdot\Lambda/n \tag{4.8}$$

we rewrite eq. (4.7) as

$$a_k + \Delta m_k^\circ / R_k^\circ = 0\,. \tag{4.9}$$

Then, we have

$$m_k^\circ = - a_k\,\ell_k/r_k \tag{4.10}$$

$$R_k^\circ = \Delta\,\ell_k/r_k \qquad \text{(cf. eq. (4.3))} \tag{4.11}$$

$$r_k = \sqrt{a_k^2 + \Delta^2}\,. \tag{4.12}$$



From eqs. (4.5), (4.8) and (4.11) it follows

$$\Delta = \Delta\,\frac{\Lambda}{n}\sum_{k=1}^{d}\frac{\ell_k}{r_k}\,, \tag{4.13}$$

and eq. (4.6) gives

$$\sum_{k=1}^{d}\Big(\frac{n}{2b} - a_k\cdot\ell_k/r_k\Big) = n\,. \tag{4.14}$$

In principle, equations (4.13) and (4.14) are sufficient to determine both the parameters $\Delta$ and a .

Thus, the point $m_k = m_k^\circ$, $k = 1,2,\ldots,d$ is the conditional minimum of the function (4.4). Now, using the Taylor expansion and eqs. (4.4) and (4.10) we have in the neighbourhood of the minimum

$$V = A + \frac{\Lambda}{n}\sum_{k,p}\Big[\frac{r_k^3}{\Delta^2\ell_k}(m_k - m_k^\circ)^2\,\frac{\ell_p}{r_p} - \tag{4.15}$$
$$- \frac{a_p a_k}{\Delta^2}(m_k - m_k^\circ)(m_p - m_p^\circ) + O((m - m^\circ)^3)\Big]\,.$$

$$A = \sum_k 2L_k\Big(\frac{n}{2b} - \frac{a_k\ell_k}{r_k}\Big) - \Delta^2\cdot n/\Lambda - E\,. \tag{4.16}$$

Let us study the term with derivatives in eq. (4.2), $T_2$. Substituting there $R_k^\circ$ (see eq. (4.11)) for $R_k$, one gets

$$\hat{T}_2 = -\frac{\Lambda\Delta^2}{n}\sum_{k,p}\frac{\ell_k\ell_p}{r_k r_p}\Big(\frac{\partial^2}{\partial m_k^2} - \frac{\partial^2}{\partial m_k\partial m_p}\Big)\,. \tag{4.17}$$

4.3. In the following, we use the standard procedure for the simultaneous diagonalization of two quadratic forms (4.17) and (4.15). Denote

$$Z_k = (m_k - m_k^\circ)\sqrt{r_k/\ell_k/\Delta}\,, \tag{4.18}$$

denote further

$$D^2 = \Sigma \frac{\ell_k}{r_k} \tag{4.19}$$

and introduce vectors $h$, $h_a$ with the components

$$h_k = \sqrt{\ell_k / r_k}\,/D, \quad \Sigma h_k^2 = 1, \tag{4.20}$$

$$h_{ak} = a_k \cdot h_k. \tag{4.21}$$

Then, eq. (4.17) transforms into

$$\hat{T}_2 = \frac{\Lambda}{n} D^2 [ (h \frac{\partial}{\partial Z})^2 - \sum_k \frac{\partial^2}{\partial Z_k^2} ]. \tag{4.22}$$

With the notation (4.18) and (4.20) we rewrite conditions (1.5), (4.6) in the form

$$(h, Z) = 0. \tag{4.23}$$

If

$$\Phi(Z) = \Phi_1(Z)\Phi_2((h,Z)), \tag{4.24}$$

then

$$\hat{T}_2\, \Phi(Z) = \Phi_2((h,Z))\, \hat{T}_2 \Phi_1(Z). \tag{4.25}$$

Taking

$$\Phi_2((h, Z)) = \delta((h, Z)) \tag{4.26}$$

we ensure eq. (4.23) to be fulfilled.

4.4. Now let us consider the quadratic form $V_2$ eq. (4.15). In veriables $Z$, compare (4.18), it looks as follows

$$V_2 = \frac{\Lambda}{n} D^2 [ -(h_a Z)^2 + \Sigma r_k^2 Z_k^2 ]. \tag{4.27}$$

We have to check the extremum of the expression in the brackets under additional conditions (4.23) and

$$\Sigma Z_k^2 = 1. \tag{4.28}$$



Let us substract from the expression in the brackets the function $2(h, Z)\delta + \omega^2 \sum_k Z_k^2$. In this way we derive our conditions of the extremum of the expression (4.27)

$$(r_k^2 - \omega^2) z_k - h_{ak}(h_a, Z) - h_k \delta = 0. \tag{4.29}$$

Whence it follows that

$$Z_k = (h_{a_k} \gamma + h_k \delta)/(r_k^2 - \omega^2), \tag{4.30}$$

where

$$\gamma = (h_a, Z) = \sum_k (h_{ak}^2 \gamma + h_{ak} h_k \delta)/(r_k^2 - \omega^2). \tag{4.31}$$

Cobdition (4.23) implies

$$\sum_k (h_{ak} h_k \gamma + h_k^2 \delta)/(r_k^2 - \omega^2) = 0. \tag{4.32}$$

The system of equations (4.31), (4.32) has a nontrivial solution $(\gamma, \delta)$ if its determinant is equal to zero. This means

$$(\sum_k \frac{h_k h_{ak}}{(r_k^2 - \omega^2)})^2 = \sum_k \frac{h_k^2}{r_k^2 - \omega^2} (\sum_k \frac{h_{ak}^2}{r_k^2 - \omega^2} - 1). \tag{4.33}$$

Suppose the numbers $r_k$, $k = 1,2,\ldots,d$ are different in pairs. Then, one can reduce eq. (4.22) to the following form:

$$\sum_k \frac{\ell_k \phi_k}{r_k(r_k^2 - \omega^2)} = 0, \tag{4.34}$$

$$\phi_k = \sum_p \frac{\ell_p a_k}{r_p (a_k + a_p)}. \tag{4.35}$$

We rearrange the numbers $r_k$ into a monotonously increasing sequence $r_k \to p_k(r)$, $p_{k+1}(r) > p_k(r)$. If all the numbers $\phi_k$ have the same sign (we did not succeed in proving this conjecture), equation (4.34) has just one root $\omega_k$ in-between every two numbers $p_k(r)$, $p_{k+1}(r)$ and $(d-1)$ roots altogether.

4.5. Our system has $(d-1)$-degrees of freedom ( cf. eqs. (4.6) and (4.23)); thus, it is quite natural that it has (d − 1) modes. In a usual way, (with eq. (4.23)), one can prove that the eigenvectors $Z(\alpha) = (Z_k(\alpha))$, , $k = 1,2,\ldots, d$, $\alpha = 1,2,\ldots, d-1$ form the orthogonal basis

$$(Z(\alpha), Z(\beta)) = \delta_{\alpha\beta} \qquad \alpha, \beta = 1,2,\ldots, d-1. \tag{4.36}$$

Instead of variables $Z_k$ we introduce now new variables $y_\alpha$,

$$Z_k = \sum_{\alpha=1}^{d-1} Z_k(\alpha) y_\alpha . \tag{4.37}$$

With this notation the quadratic forms (4.27) and (4.22) look like

$$V_2 = \frac{\Lambda}{n} D^2 \sum_{\alpha=1}^{d-1} \omega_\alpha^2 y_\alpha^2 , \tag{4.38}$$

$$\hat{T}_2 = -\frac{\Lambda}{n} D^2 \sum_{\alpha=1}^{d-1} \frac{\partial^2}{\partial y_\alpha^2} . \tag{4.39}$$

Using the condition $\frac{\Lambda}{n}D^2 = 1$ (cf. eqs. (4.13) and (4.19)) and eqs. (4.15), (4.27), (4.38), (4.17) and (4.39) for $\Lambda > \Lambda_0$ (cf. eq. (4.43)), the Schrödinger equation may be transformed into the equation

$$\frac{1}{2}\Big(\sum_{\alpha=1}^{d-1}\Big(-\frac{\partial^2}{\partial y_\alpha^2} + \omega_\alpha^2 y_\alpha^2\Big) + A + \ldots\Big)\Phi(y) = 0 . \tag{4.40}$$

This is the equation of the system of quantum hadronic oscillators weakly perturbed. Thus, we get the eigenfunctions as some products of functions (3.13) in every variable $y_\alpha$, and the admissible values of the energy

$$E = \Big\{ \sum_{k=1}^{d} 2L_k\Big(\frac{n}{2b} - \frac{a_k \ell_k}{r_k}\Big) - n\frac{\Delta^2}{\Lambda}\Big\} + 2\sum_{\alpha=1}^{d-1} k_\alpha \omega_\alpha + \ldots . \tag{4.41}$$

Here, the values of the parameters $k_\alpha$ are restricted analogously to eq. (3.16).

Note if $\ell_k \approx n/2b$ for all $k$, $k = 1,2,\ldots, d$, then the transformation (4.18), (4.37) is similar to transformation (3.9). Regarding to parameters $\ell_k$, $k = 1,2,\ldots, d$, the r.h.s. of eq. (4.41) is exaxt only for large values of these parameters.

Note further that it is impossible to compare eqs. (3.15) and (4.41) because in eq. (3.15) we have taken $b = 1$, $d = 2$, $L_1 = 0$, $L_2 = 1$ and then eqs. (4.13), (4.14) give $a = 1/2$, $r_1 = r_2$ ; on the contrary, when deriving eq. (4.41) we have proposed that all numbers $r_k$, $k = 1,2,\ldots, d$ are different. Denote by $B(\ell_1, \ell_2, \ldots, \ell_d)$ the expression in braces in eq. (4.41). One can prove that

$$\begin{aligned} &B(\ell_1, \ldots, \ell_{k-1}, \ell_k - 1, \ell_{k+1}, \ldots, \ell_d) - \\ &- B(\ell_1, \ldots, \ell_{k-1}, \ell_k, \ldots, \ell_d) = 2r_k + O(1/n). \end{aligned} \tag{4.42}$$

One has to take into account the variation of the parameters $\Delta$ and $a$ in eq. (4.41) indiced by variation of the parameters $\ell_s$, $s = 1,2,\ldots, d$ in eq. (4.42) (all these parameters are connected by eqs. (4.13), (4.14)).

The critical value of the parameter $\Lambda$, $\Lambda = \Lambda_0$ as mentioned begore is determined by the condition

$$\frac{\Lambda_0}{n} \sum_k \frac{\ell_k}{|L_k - a|} = 1 , \quad \Delta = 0 \tag{4.43}$$

(see eq. (4.13)) and by eq. (4.14). If $\Lambda < \Lambda_0$, then $\Delta = 0$; if $\Lambda > \Lambda_0$, then $\Delta = \Delta(\ell_1, \ell_2, \ldots, \ell_d; \Lambda) > 0$, $D^2\Lambda/n = 1$. Consider formally the limit where the kinetic energy becomes a continuous function of the momentum. From eq. (4.43) it follows that

$$\Lambda_0( n/(2b), \ldots, n/(2b) ) \to 0$$

$$\text{if} \quad d, b \to \infty$$

$$\text{and} \quad |L_k - L_{k-1}| \to 0 .$$

## 5. CONCLUSION

Here we shall compare our formulas for the allowed energy values (eq. (2.8), (2.9), (4.41), (4.42)) with analogous formulas of the Bogolubov method.

For the ground state energy the latter method gives

$$E_0 = \sum_{i=1}^{M} 2T_i v_i^2 - \frac{\Delta}{n} \sum_{i,j=1}^{M} c_{ij} u_i v_i u_j v_j . \qquad (5.1)$$

The excitation spectrum, when the term of the fourth power of the creation and annihilation operators $\alpha^*$, $\alpha$ are neglected, is determined by the operator

$$H_2' = \sum^{M} \Omega_i (\alpha_i^* \alpha_i + \alpha_{-i}^* \alpha_{-i}), \qquad (5.2)$$

where (we have taken $c_{ij} = 1$)

$$\Omega_i = \sqrt{\Delta^2 + e_i^2}, \quad e_i = T_i - a ; \qquad (5.3)$$

so that $\Delta$, $a$, $u$, $v$ are defimed by

$$\Delta = \Delta \frac{\Lambda}{2n} \sum_{j=1}^{M} \Omega_j^{-1}, \quad \sum_{1}^{M} v_i^2 = n , \qquad (5.4)$$

$$2u_i^2 = (1 + e_i/\Omega_i), \qquad 2v_i^2 = (1 - e_i/\Omega_i). \qquad (5.5)$$

At first, let us consider the formulae in the fourth section. Similarly to eq. (1.8) we transform in eq. (4.8) the quantities $e_i$ into $a_k$. We are convinced of the fact that eq. (5.4) agrees to eqs. (4.13), (4.14) if only all numbers $\ell_k$, $k = 1,2,\ldots,d$ are equal or close to the maximum $\ell_k = n/(2b)$. This result together with eqs. (4.41) and (4.42) shows the $\ell_k$-dependence of eq. (4.41), $k = 1,2,\ldots,d$, to coincide, if $(n/(2b)) - \ell \ll n/b$, with the excitation spectrum wrich is given by the u-v transformation method.

On the contrary, the $k_\alpha$- dependent part of eq.(4.41) $\alpha = 1,2,\ldots, d-1$ represents new branches of spectrum which are not given by the u-v transformation method. The dependence on the parameters $p_k$, $k = 1,2$, in eq.(2.8), if $p_k \ll n$, coincides with the excitation spectrum (5.2). The m-dependence in eq. (2.8) can also be given by the method of u-v transformations; one has to apply this method separately to both the independent Hamiltonians which constitute the Hamiltonian (2.3). Finally we should like to remark that our method and the Bogolubov method give an n-independent ground state energy difference.

We shall conclude by the remark that our work gives the implementation of the idea by Anderson [11], §3.



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