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E17-89-304

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SUPERSYMMETRIC INTEGRABLE HEISENBERG MODELS

Submitted to "Physics Letters A"

1989

A great interest of theoreticians all over the world is stimulated again to the Hubbard model [1] especially in the strong electron correlation limit [2,3] due to the discovery of Hightemperature superconductivity. In this case state space in each lattice site is defined by three components, namely: $|0\rangle$ being a hole (or vacancy) and two one-particle states with spin up $|\uparrow\rangle = C^+_\uparrow|0\rangle$ or down $|\downarrow\rangle = C^+_\downarrow|0\rangle$. Here C^+_σ, C_σ, $(\sigma=\uparrow,\downarrow)$ are the fermion creation and annihilation operators

$$\{C_\sigma, C^+_{\sigma'}\} = \delta_{\sigma\sigma'}, \{C_\sigma, C_{\sigma'}\} = \{C^+_\sigma \; C^+_{\sigma'}\} = 0 , \qquad (1)$$

and $C_\sigma|0\rangle = 0$ (two-particle states $|2\rangle = C^+_\sigma C^+_{\sigma'}|0\rangle$ are absent when electron repulsion is strong $U \to \infty$). The Hubbard operators [1] defined with these states $X^{pq}_i = |i,p\rangle\langle q,i|$ in the lattice site i form a general linear graded Lie algebra pl(2/1) [1,4]

$$[X^{pq}_i, X^{nm}_j\} = \delta_{ij}(X^{pm}_i \delta_{qn} \mp X^{nq}_i \delta_{mp}), (p,q,n,m=0,\uparrow,\downarrow). \quad (2)$$

Here anticommutator stands for two fermion operators (which change the number of electrons in the site by odd number) and commutator for other cases. The Hubbard's Hamiltonian

$$H = \sum_{i,j}\sum_{\sigma=\uparrow,\downarrow} t_{ij} C^+_{i\sigma} C_{j\sigma} + \sum_{i,j} U_{ij} n_{i\uparrow} n_{j\downarrow}, (U_{ij}=\delta_{ij}U, n_{i\sigma}=C^+_{i\sigma}C_{i\sigma}), \quad (3)$$

expressed in terms of X^{pq}_i,

$$H = \sum_{i,j}\{t_{ij}(X^{\uparrow 0}_i X^{0\uparrow}_j + X^{\downarrow 0}_i X^{0\downarrow}_j) + U_{ij} X^{\uparrow\uparrow}_i X^{\downarrow\downarrow}_j\} , \qquad (4)$$

becomes that of a generalized Heisenberg model on the superalgebra spl(2/1) [2,3]. Faithful representation of the algebra is three-dimensional and describes spinless fermions and spin waves.

Below we construct the simplest integrable Heisenberg models on the real supersubalgebra of the spl(2/1) algebra. To attain these ends (as in the quantum case) we consider a linear problem for a three-component spinor $\Phi^{T}(x,t)=(\phi_1,\phi_2\,|\,\chi)$ of the graded vector space $V(2|1)$ with two Bose and one Fermi dimensions. Compact super-subgroup $USPL(2|1)=\{M\in SPL(2|1), MM^{+}=I\}$ is generated by four Bose and four Fermi generators [5]. Let us consider matrix-valued function of two variables x and t, $S\in USPL(2|1)/H$, which is diagonalized by the matrices $g(x,t)\in USPL(2|1)$:

$$S(x,t)=g^{-1}(x,t)\Sigma\, g(x,t)\ . \tag{5}$$

Assume constraints on S to be quadratic, then there are two possibilities [6]

$$\text{I. } S^2=3S-2I \quad ,\ \text{II. } S^2=S\ , \tag{6}$$

which relate homogeneous superspaces :

$$\text{I. } USPL(2|1)/S(U(2)\otimes U(1)) \quad ,\ \text{II. } USPL(2|1)/S(L(1|1)\otimes U(1))\ . \tag{7}$$

The generators h_i of stationary subgroup H commute with Σ : $[\Sigma, h_i]=0$, i.e., under local transformations we have

$$g(x,t)\to e^{ih(x,t)}g(x,t)\ ,\ h(x,t)=\sum_i f_i(x,t)h_i:\ S\to S\ . \tag{8}$$

The sl(1,1) superalgebra is defined by the following commutation relations

$$\{T_\pm, T_\pm\}=0\ ,\{T_+,T_-\}=2\Sigma\ ,\ [T_\pm,\Sigma]=0\ , \tag{9}$$

and is isomorphic to the algebra of superquantum mechanics sqm(2). Cubic constraint

$$S^3=S \tag{10}$$

occurs on the noncompact superalgebra ospu(1,1/1) [7] , and realized on the OSPU(1,1/1)/U(1) homogeneous superspace.

Let the linear problem be of the form

$$U=i\lambda S \quad ,\quad V=i\lambda^2 S+\lambda[S,S_x]\ . \tag{11}$$

It leads to the Landau-Lifshitz equation

$$iS_t=[S,S_{xx}]\ . \tag{12}$$

Just as in the theory of spontaneous symmetry breaking the stationary subgroup H is a group of vacuum invariance; in our case it defines the symmetry of the system physical phase after transition as well as the **tangent plane** of the homogeneous superspace (7). This **tangent plane** determines gauge-equivalent NLS model, and the local symmetry group H transforms into a global symmetry of the NLS model.

In order to construct the latter we consider a current $\mathcal{J}_\mu = g_\mu g^{-1}$ $(\mu = 0, 1)$ with $g(x,t) \in USPL(2|1)$ being a diagonalizing matrix in eq.(5). Decompose superalgebra uspl(2/1) onto two ortogonal parts: $uspl(2|1) = L^{(0)} \oplus L^{(1)}$ where $[L^{(i)}, L^{(j)}\} \subset L^{(i+j)\,\mathrm{mod}(2)}$, and $L^{(0)}$ is the stationary subgroup algebra. Let $\mathcal{J}_1 \subset L^{(1)}$:

$$
\text{I.}\quad \mathcal{J}_1 = i\left(\begin{array}{cc|c} 0 & 0 & \psi_1 \\ 0 & 0 & \psi_2 \\ \hline \bar\psi_1 & \bar\psi_2 & 0 \end{array}\right) \qquad \text{II.}\quad \mathcal{J}_1 = i\left(\begin{array}{cc|c} 0 & \varphi & \psi \\ \bar\varphi & 0 & 0 \\ \hline \bar\psi & 0 & 0 \end{array}\right), \tag{13}
$$

where ψ_1, ψ_2, ψ are odd Grassmanian functions of x and t, and φ is the even Grassmanian one. The zero curvature condition $\partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu + [\mathcal{J}_\nu, \mathcal{J}_\mu] = 0$ and the equation of motion (12) allow us to construct the component $\mathcal{J}_0$. Performing gauge transformations, we have from (11)

$$
\widetilde{U} = gUg^{-1} + g_x g^{-1} = i\lambda\Sigma + \mathcal{J}_1, \quad \widetilde{V} = gVg^{-1} + g_t g^{-1} = i\lambda^2\Sigma + \lambda\mathcal{J}_1 + \mathcal{J}_0 \tag{14}
$$

and the following models :

I. A pure Grassmanian odd $U(2)$ vector NLS

$$
i\psi_{1_t} + \psi_{1_{xx}} + 2\bar\psi_2\psi_2\psi_1 = 0, \quad i\psi_{2_t} + \psi_{2_{xx}} + 2\bar\psi_1\psi_1\psi_2 = 0, \tag{15}
$$

II. Supersymmetric Bose-Fermi L(1/1) NLS

$$
i\varphi_t + \varphi_{xx} + 2(\bar\varphi\varphi + \bar\psi\psi)\varphi = 0, \quad i\psi_t + \psi_{xx} + 2\bar\varphi\varphi\,\psi = 0. \tag{16}
$$

From the equation $S_x = g^{-1}[\Sigma, \mathcal{J}_1]g$ one gets immediately a connection between the energy density of the Heisenberg supermagnet and the particle density of NLS model :

$$
\text{I.}\ \tfrac{1}{2}\,\mathrm{str}\,S_x^2 = -(\bar\psi_1\psi_1 + \bar\psi_2\psi_2), \quad \text{II.}\ \tfrac{1}{2}\,\mathrm{str}\,S_x^2 = \bar\varphi\varphi + \bar\psi\psi. \tag{17}
$$

Both models of the magnet are supersymmetric with respect to the global transformations : $S \to S' = R^{-1}SR$, $R \in USPL(2|1)$, but from the plane versions only the L(1/1) NLS model (16) posesses global supersymmetry : $\delta\varphi = i\psi\theta$, $\delta\psi = i\varphi\bar\theta$.

Hamiltonian structure of the plane versions (15) and (16) is generated correspondingly by the Poisson superbrackets

$$
\text{I.}\ \{A, B\}_{SP} = i\sum_\sigma \int_{-\infty}^{\infty} \left(A \frac{\overleftarrow{\delta}}{\delta\psi_\sigma} \frac{\overrightarrow{\delta}}{\delta\bar\psi_\sigma} B - (-1)^{p(A)p(B)} B \frac{\overleftarrow{\delta}}{\delta\psi_\sigma} \frac{\overrightarrow{\delta}}{\delta\bar\psi_\sigma} A \right) dx, \tag{18}
$$

$$
\text{II.}\ \{A, B\}_{SP} = i\int_{-\infty}^{\infty} \left\{ \left(\frac{\delta A}{\delta\varphi}\frac{\delta B}{\delta\bar\varphi} - \frac{\delta A}{\delta\bar\varphi}\frac{\delta B}{\delta\varphi} \right) + \left(A \frac{\overleftarrow{\delta}}{\delta\psi}\frac{\overrightarrow{\delta}}{\delta\bar\psi} B - (-1)^{p(A)p(B)} B \frac{\overleftarrow{\delta}}{\delta\psi}\frac{\overrightarrow{\delta}}{\delta\bar\psi} A \right) \right\} dx, \tag{19}
$$

and the Hamiltonian functions

$$
\text{I.}\quad H = \int_{-\infty}^{\infty} (\bar\psi_{1_x}\psi_{1_x} + \bar\psi_{2_x}\psi_{2_x} - 2\bar\psi_1\psi_1\bar\psi_2\psi_2)\, dx, \tag{20}
$$

$$
\text{II.}\quad H = \int_{-\infty}^{\infty} (\bar\varphi_x\varphi_x + \bar\psi_x\psi_x - |\varphi|^4 - 2|\varphi|^2\bar\psi\psi)\, dx. \tag{21}
$$

The global supersymmetry of the L(1/1)NLS (16) gives rise to the odd-Grassmanian integrals of motion $Q_{12}=\int\bar{\varphi}\psi\,dx$, $Q_{21}=\int\bar{\psi}\varphi\,dx$ to exist and these are the first and only local representatives of an infinite series of the odd-Grassmanian nonlocal integrals of motion **which is generated** by the elements $T_{23}(\lambda)$ and $T_{32}(\lambda)$ of the transition matrix $T(\lambda)$ (monodromy matrix). The Heisenberg model (12) is defined by the Poisson superbracket on the curve phase space [8] of the superalgebra uspl(2/1) with structure constants C_{abc} :

$$\{A,B\}=\sum_{a,b,c} C_{abc}\, A\frac{\overleftarrow{\delta}}{\delta S_a} S_c \frac{\overrightarrow{\delta}}{\delta S_b} B$$

and by the Hamiltonian function (17). For the integrals of motion which are the components of supermagnetization vector $M_a = \int_{-\infty}^{\infty} S_a(x,t)\,dx$ we get the superalgebra uspl(2/1)

$$\{M_a, M_b\} = C_{abc} M_c \quad , \tag{22}$$

then the odd-Grassmanian components $M_a, (a=5,\dots,8)$ generate the following supersymmetry transformations :

$$\delta S_1 = \frac{i}{4}(\bar{\theta}\sigma_1 f + \bar{f}\sigma_1\theta), \qquad \delta S_4 = \frac{i}{4}(-\bar{\theta} I_2 f + \bar{f} I_2\theta),$$

$$\delta S_2 = -\frac{i}{4}(\bar{\theta}\sigma_2 f + \bar{f}\sigma_2\theta), \qquad \delta f = \frac{i}{2}(\vec{S}\vec{\sigma} - S_4 I_2)\theta,$$

$$\delta S_3 = \frac{i}{4}(-\bar{\theta}\sigma_3 f + \bar{f}\sigma_3\theta), \qquad f^T = (c_1\ c_2)\ ,\ \bar{\theta} = (\bar{\theta}_1\ \bar{\theta}_2).$$

To resolve constraint (6) let us consider the representation $S = \left(\begin{array}{c|c} \tilde{S}+I_2 S_4 & \frac{f}{2S_4} \\ f^+ & \end{array}\right)$, where $\tilde{S} = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}$, $f^+ = (c_1^+\ c_2^+)$ are elements of two-dimensional algebra with generators θ and $\bar{\theta}$:

$$S_a(x,t) = S_a^{(0)}(x,t) + S_a^{(1)}(x,t)\bar{\theta}\theta\ ,\ c_{1,2}(x,t) = f_{1,2}(x,t)\theta\ , (a=1,\dots,4),$$

where $S_a^{(0,1)}$, $f_{1,2}$ are c-number functions of x , t . Then one gets

I. $S \in USPL(2|1)/S(U(2)\otimes U(1))$; $S_i^{(0)}=0$, $(i=1,2,3)$, i.e., in such a system there is no pure bosonic limit(neither as for the plane version of NLS(15)) and dynamics of the model is reduced to spin wave dynamics

II. $S \in USPL(2|1)/S(L(1|1)\otimes U(1))$. There is a bosonic limit here (as well as in the plane L(1/1) version of NLS(16)) :

$$S_-^{(0)} = S_1^{(0)} - iS_2^{(0)} = -\zeta/(1+|\zeta|^2)\ ,\ S_3^{(0)} = -\frac{1}{2}(1-|\zeta|^2)/(1+|\zeta|^2), (\zeta = \frac{f_2}{f_1}), \tag{23}$$

so that $(S_1^{(0)})^2 + (S_2^{(0)})^2 + (S_3^{(0)})^2 = \frac{1}{4}$ and f_1, f_2 is the fermionic components.

Hamiltonian of the Heisenberg model in the first case is

$$H = -\frac{1}{2}\int_{-\infty}^{\infty}(c_{1x}^+ c_{1x} + c_{2x}^+ c_{2x})\,dx \quad ,$$

that is the Hamiltonian of the model of free nonrelativistic fermions. The latter is related to a continuum limit of "classical" Hubbard model (4) at $U \to \infty$, $(U_{ij} = \delta_{ij} U)$, expressed in terms of the Hubbard operators. In terms of the Fermi operators (1) model (3) may be reduced in the same approximation to $U(2)$ NLS(15) (e.g. by averaging over grassmanian coherent states $|\Psi\rangle = e^{-\frac{1}{2}\sum_\sigma \bar{\Psi}_\sigma \Psi_\sigma} e^{\sum_\sigma c_\sigma^+ \Psi_\sigma} |0\rangle$. But for a macroscopic (classical) state of the condensate type to exist a bosonic limit (c-number limit) should exist as well. In our model it is only possible in the second case $S \in USPL(2|1)/S(L(1|1) \otimes U(1))$, when the Hamiltonian is

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\{(\vec{S}_x^{(0)}\vec{S}_x^{(0)}) + 4\vec{S}_x^{(0)}\vec{S}_x^{(1)}\bar{\theta}\theta - (S_+^{(0)}c^+)_x(S_-^{(0)}c)_x - ((S_3^{(0)}+\tfrac{1}{2})c^+)_x((S_3^{(0)}+\tfrac{1}{2})c)_x\}\,dx ,$$

then its fermionic part corresponds to the continuum limit of the effective Hamiltonian

$$H = -\sum_{i,j} t_{ij}\,\alpha_i^+\alpha_j\{S_i^+S_j^- + (S_{i3}+\tfrac{1}{2})(S_{j3}+\tfrac{1}{2})\} , \qquad (24)$$

proposed in [9] for the Hubbard model in the limit $U \to \infty$. In Eq.(24) $S_i^- = \alpha_i c_{i\downarrow}$, $S_i^+ = c_{i\downarrow}^+ \alpha_i^+$ are the pseudospin operators, $\alpha_i = c_{i\uparrow}^+$, $\alpha_i^+ = c_{i\uparrow}$ are the vacation operators. In the ground state there are no vacations and all the pseudospins are directed up. By analogy with the case I, one can assume the plane version of this model ($L(1|1)$NLS (16)) to describe another representation of model (24) in the same phase with the macroscopic quantum average value.

Ultimately we note that for the noncompact supergroup OSPU(1,1/1) more complicated equations occur with higher nonlinear terms [7] due to the cubic constraint $S^3 = S$, which are similar in the form to those of the su(3) generalized Heisenberg model [10] . The extension of the model is possible including superalgebra spl(N | M) in order to take into account p , d and other electron states [11] . Two-dimensional models of Heisenberg supermagnet are also possible to find both in the form of either nonlinear σ model or the Ishimori one [11] .

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Received by Publishing Department on April 28, 1989.