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U.Behn*, V.B.Priezzhev, V.A.Zagrebnov

ONE DIMENSIONAL RANDOM FIELD ISING MODEL: RESIDUAL ENTROPY, MAGNETIZATION, AND THE "PERESTROYKA" OF THE GROUND STATE

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*Sektion Physik, Karl-Marx-Universität Leipzig, DDR-7010 Leipzig, German Democratic Republic

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Be laconic

Platon, Protagoras, 343 BC

1. INTRODUCTION

The ferromagnetic one-dimensional Ising model in a discrete stochastic field,

$$H= -J \sum_{n=1}^{N-1} s_n s_{n+1} - \sum_{n=1}^{N} h_n s_n \quad , \; h_n = h_0 + \sigma_n h \equiv h_{\sigma_n} \qquad (1.1)$$

where $\sigma_n = \pm 1$ is an i.i.d. random sequence, shows for zero temperature a discontinuous behaviour of magnetization m and residual entropy s_{res} as functions of h_0/h and J. This was observed in [1, 2] using a finite Markov chain approach, and alternatively by energy balance arguments which explained the discontinuities by flips of microscopic spin clusters. There an understanding of the dependence on h_0/h has been obtained in *both languages*.

In the present paper we work out the energy balance arguments to explain the dependence of m and s_{res} as function of J including the continuous degeneracy of the ground state. Furthermore, we show that formula (2.14) exploited in [2] gives only an upper bound for the residual entropy.

2. ENERGY BALANCE ARGUMENTS

2.1. Outline of the idea

The discontinuous behaviour of magnetization and entropy for zero temperature can be understood as the consequence of flips of microscopic spin clusters.

There exist well defined clusters which -in an appropriate environment- become unstable at certain threshold values of the system parameters J, h_0, and h. Unstable means that no direction of the cluster is preferred energetically. *Above the threshold* one direction is preferred. Changing the parameters of the system we will observe a successive *perestroyka* (reconstruction) of the ground state.

At the threshold there is a macroscopic degeneracy of the ground state reflected by spikes of the residual entropy. Except $h_0 = 0$ (in this case we have zero magnetization) one observes in addition jumps in the magnetization.

The above arguments are only true if the cluster lives in an appropriate environment, i.e. the adjacent spins should have definite directions. The minimal configuration surviving the previous steps of perestroyka and guaranteeing in this way a definite direction (up, down) of the (right, left) adjacent spin to a given cluster is called in the following *stable* (up, down; right, left) *boundary cluster* $B^{r,l}_{\uparrow,\downarrow}$.

If *more than one type of clusters* become instable at the same threshold (this is possible for rational h_0/h, see example below) we have an additional mechanism for degeneracy: All possible combinations of those clusters contribute to the spikes.

Above threshold we then possibly face with a new phenomena: If at least two of these clusters are *antiparallel below the threshold* then above the threshold in an *antiparallel environment* they form an aligned pair, but *no* direction is preferred. This degeneracy survives at least up to the next threshold and is responsible for a plateau in the residual entropy as a function of the exchange.

The jump in magnetization passing the threshold due to flips of the corresponding clusters of only *one* type C in the appropriate environment build by the corresponding stable boundary clusters B^l and B^r is

$$\Delta m = 2\ (\Delta n_\uparrow - \Delta n_\downarrow)\ \mathrm{Prob}(\ B^l\ C\ B^r\ ) \tag{2.1}$$

where Δn_σ is the difference in the number of spins in direction σ before and after the perestroyka.

The corresponding spike of the residual entropy can be obtained by $s_{res} = k_B \ln W$, where $W = 2^{\mathrm{Prob}(\ B^l\ C\ B^r\ )}$ is the specific statistical weight of the degeneracy of the ground state, so that

$$s_{res} = k_B\ \mathrm{Prob}(\ B^l\ C\ B^r\ )\ \ln 2\ . \tag{2.2}$$

The $\mathrm{Prob}(\ B^l\ C\ B^r\ )$ is the Bernoulli measure of the cylindrical set corresponding to infinite (±) configurations with the base $B^l C\ B^r$.

2. 2. Zoology of clusters for $h_0/h = 1/4$

To explain the energy balance arguments it is convenient to start with the example considered in [2]. There we studied magnetization and residual entropy as functions of J for fixed ratio $h_0/h = 1/4$.

For $J = 0$ the ground state configuration follows the direction of the field. The first cluster $C^-_\downarrow$ which becomes unstable increasing J is shown in Fig. 1a. The threshold condition is that the energy difference between initial and final configuration (binding energy of the flipping spin) becomes zero , $h_- + 2J = 0$, which defines $J_c^{(1)} = -h_-/2$.

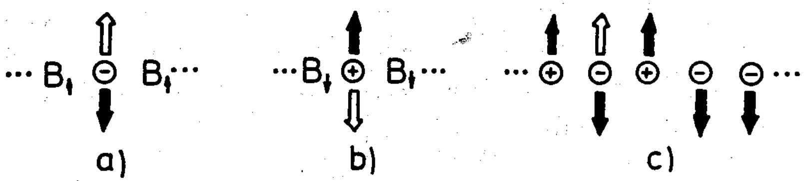

Fig. 1. Clusters $C_\downarrow^- =$ (a) and $C_\uparrow^+ =$ (b) are responsible for the 1st and the 2nd spike, respectively. The stable boundaries are $B_\uparrow =$ (a), and $B_\downarrow =$ (b). (c) illustrates the necessity to introduce the stable boundary cluster: The number of single up-spins is reduced due to the first step of the perestroyka.

For $J > J_c^{(1)}$ all $C_\downarrow^-$ disappeared (first step of the perestroyka of the ground state). The jump in the magnetization is $\Delta m^{(1)} = 2\cdot 1\cdot (1/2)^3 = 1/4$ (see (2.1)). At the threshold the $C_\downarrow^-$ don't prefer any direction, the ground state is macroscopically degenerated and the residual entropy has a spike $s_{res}^{(1)} = k_B\,(1/2)^3 \ln 2$ (see (2.2)), the magnetization is in the middle between the steps.

The thus obtained ground state is stable until reaching $J_c^{(2)} = h_+/2$ where the $C_\uparrow^+$ become unstable (see Fig. 1b). The jump in magnetization and the residual entropy are calculated as above, $\Delta m^{(2)} = 2\cdot(-1)\cdot(1/2)^5 = -1/2^4$, and $s_{res} = k_B\,(1/5)^5 \ln 2$.

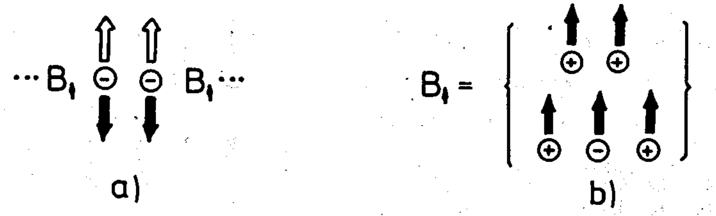

Fig. 2. The two down spin cluster $C_\downarrow^{--} =$ in an up spin environment is responsible for the 3rd threshold (a). (b) shows the set of up-spin boundary clusters which are stable for $J < J_c^{(4)} = 7/8\,h$.

The third threshold $J_c^{(3)} = -h_-$ originates from cluster $C_\downarrow^{--}$ (see Fig. 2), correspondingly $\Delta m^{(3)} = 2\cdot 2\cdot((1/2)^2+(1/2)^3)\cdot(1/2)^2\cdot((1/2)^2+(1/2)^3) = 9/2^6$, and $s_{res} = k_B\,(9/2^8)\ln 2$.

The next threshold $J_c^{(4)} = 7/8\,h$ results (in contrast to the previous one) from an *infinite* class of clusters generated by $C_\uparrow^{+-+}$ and $C_\downarrow^{--+--}$ (cf. Fig. 3) with respective

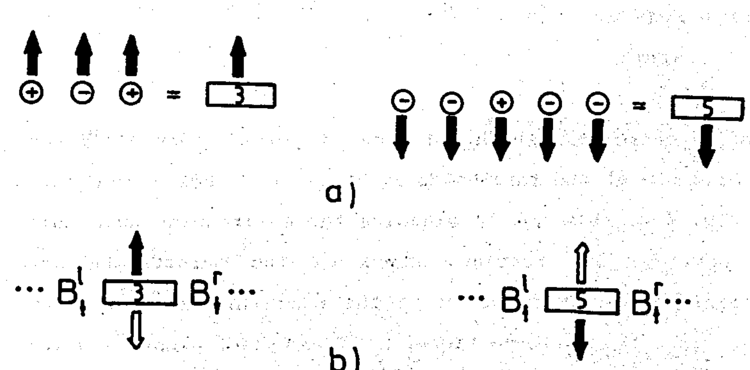

Fig. 3. Generators for composed clusters responsible for the 4th threshold (a). The 3-cluster and the 5-cluster surrounded by down (up) spins become instable at $J_c^{(4)}$ (b).

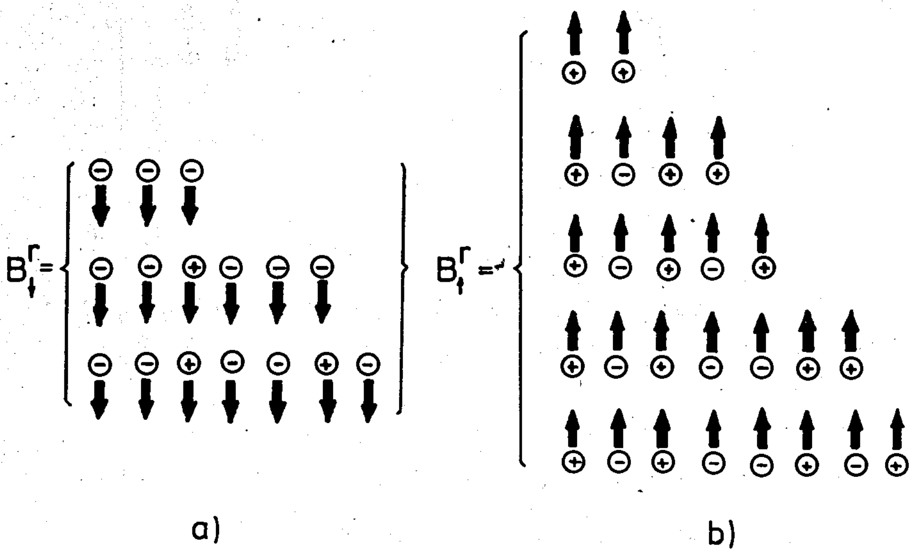

Fig. 4. The sets of down (a), respectively up (b) right boundary clusters which are stable for $J < J_c^{(5)}$. The left boundary clusters are $B^l_{\uparrow,\downarrow} = (B^r_{\uparrow,\downarrow})^M$ where M denotes mirror reflection to the left. Obviously, $\Pr(B^r_\downarrow) = 2^{-2} + 2^{-4} + 2^{-5} = 37/2^8$, and $\Pr(B^r_\uparrow) = 2^{-2} + 2^{-4} + 2^{-5} + 2^{-7} + 2^{-8} = 91/2^8$.

binding energies $-2h_+ - h_- + 2J$ and $4h_- + h_+ + 2J$. Obviously the coincidence of the thresholds is due to our choice of h_0/h. In Fig. 4 we show how to organize the environment such that it survives the previous steps of the perestroyka and remains stable at least up to the next threshold. Figs. 5 and 6 show the infinite hierarchy of composed clusters which have all the same threshold $J_c^{(4)}$.

Above the threshold, in the composed clusters shown in Figs. 6 a,b , all 3- and 5- clusters point in the direction preferred by the surrounding. The pairs of 3-

and 5- clusters living in antiparallel surrounding become aligned (cf. Fig. 5) but no direction is preferred. The same mechanism works for the composed clusters of Fig. 6c and leads to a hierarchy of *kink-type configurations*, see Fig. 7. The jump in magnetization has contributions from the clusters shown in Fig. 6,

$$\Delta m_a^{(4)} = 2\Pr(B^l_\downarrow)\sum_{n=1}^{\infty}(-3n)\,2^{-3n-5(n-1)}\Pr(B^r_\downarrow) = -\frac{1}{3}\left(\frac{37}{2\cdot 5\cdot 17}\right)^2,$$

$$\Delta m_b^{(4)} = 2\Pr(B^l_\uparrow)\sum_{n=1}^{\infty}(5n)\,2^{-5n-3(n-1)}\Pr(B^r_\uparrow) = \frac{1}{5}\left(\frac{91}{2^2\cdot 3\cdot 17}\right)^2,$$

$$\Delta m_c^{(4)} = 2\Pr(B^l_\uparrow)\sum_{n=1}^{\infty}2n\,2^{-8n}\Pr(B^r_\downarrow) = \frac{37\cdot 91}{(2^3\cdot 3\cdot 5\cdot 17)^2}$$

Summing up all jumps we obtain the magnetization $m = 6/17$ for $J_c^{(4)} < J < J_c^{(5)}$ in accordance with [2]. To calculate the residual entropy we first consider the contribution to the

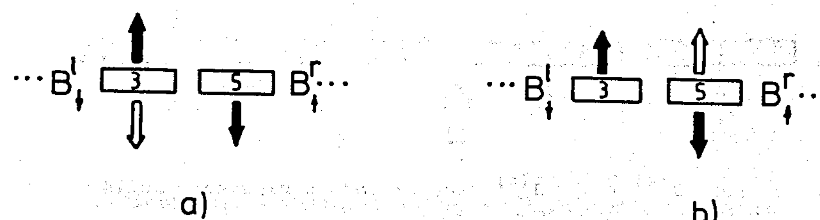

Fig. 5. Simplest composed cluster constructed from the 3- and 5-generators (see Fig. 3) which become unstable at $J_c^{(4)}$. At the threshold either the 3-cluster (a) or the 5-cluster (b) can be flipped producing three possible configurations. Also the reflected image of these configurations has to be taken into account.

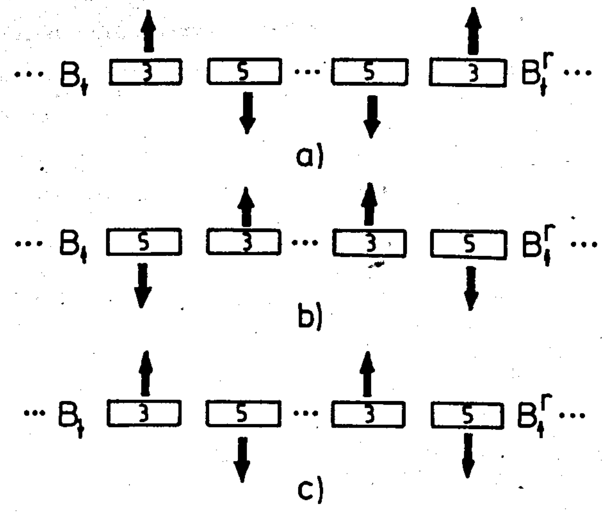

Fig. 6. Hierarchy of composed clusters which become unstable at $J_c^{(4)}$. The possible configurations at the threshold have to be constructed according to the rule described in Fig. 5. The last remark of Fig. 5 also applies. No other possibilities appear.

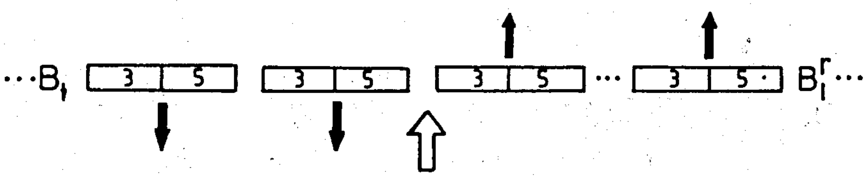

Fig. 7. For $J_c^{(4)} < J < J_c^{(5)}$ the 3- and 5-clusters build up rigid pairs (8-clusters) which try to align. As in Fig. 5 we have to spend the energy for one unsatisfied bond (kink) denoted by the arrow. Thus n pairs produce n+1 equivalent configurations. The energy to flip one pair in an antiparallel environment (shift of the kink) $3h_+ + 5h_-$ vanishes due to our specific choice $h_0/h=1/4$.

specific statistical weight W from all the clusters C^n of n pairs shown in Fig. 7 which is $W_n = (n+1)^{p_n}$ where $p_n = 2\ \mathrm{Prob}\ (\ B^l_\downarrow C^n B^r_\uparrow\ ) = 2\cdot\frac{37}{2^8}\cdot 2^{-8n}\cdot\frac{91}{2^8} = 37\cdot 91\cdot 2^{-8(n+2)+1}$.

Thus we obtain $W = \prod_{n=1}^{\infty} W_n$ and the residual entropy

$$S_{res} = k_B\,\frac{37\cdot 91}{2^{15}}\sum_{n=1}^{\infty} 2^{-8n}\cdot\ln(n+1) = k_B\ 0.00041\ldots\cdot\ln 2.$$

At the fourth threshold, we face with a new quality: We have two types of instable clusters, which correspond to the situation that a spike in the residual entropy touches with a plateau. Combinatorical calculations similar to the one above show a discrepance with [2]. Therefore, situation of this type call for further investigations of the formula for magnetization exploited there. The next threshold will be $J_c^{(5)} = h$. To find the responsible clusters is a nontrivial exercise, the simplest one is shown in Fig. 8.

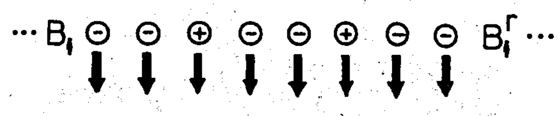

Fig. 8. The simplest cluster responsible for the 5th threshold.

2.3. Rational vs. irrational values of h_0/h and asymptotics for large J

We consider a rigid aligned cluster living in antiparallel environment (For example take the 8-cluster of Fig. 7). A continuous degeneracy is observed if the binding energy of both orientations is the same, namely zero

$$n_+h_+ + n_-h_- = (n_+ + n_-)h_0 + (n_+ - n_-)h = 0\ . \qquad (2.3)$$

n_+ and n_- are the numbers of the sites where the external field points up or down, respectively. Since the n_σ are natural numbers the zero result (2.3) is only possible if h_0/h is *rational*. For *irrational* h_0/h we have no continuous degeneracy, the residual entropy as a function of J exhibits only spikes.

Thus, for a given J, the residual entropy as a function of h_0/h feels very sensitively whether this ratio is rational or irrational. $s_{res}(h_0/h)$ behaves similar to a Dirichlet function. Modifications are due to the dependence on J, and the possibility of spikes for irrational values of h_0/h, see Fig. 9.

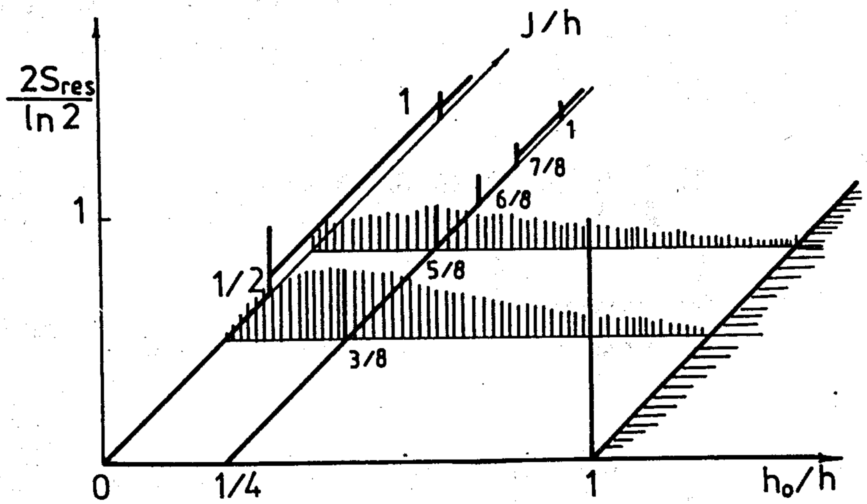

Fig. 9. Schematic picture of the residual entropy as a function of h_0/h and J. The two "fractal fences" are the Dirichlet type functions mentioned in the text. The case h_0/h =1 is a simple exercise, whereas h_0/h =0 can be analyzed similarly to the case h_0/h =1/4.

To estimate the asymptotic behaviour of $s_{res}(J \rightarrow \infty)$ it is sufficient to restrict ourselves to the spikes since they have higher values than the plateau. Let us consider a rigid cluster which is responsible for the spike. It's binding energy is zero,

$$-2J + n_+h_+ + n_-h_- = -2J+(n_+ + n_-)h_0 + (n_+ - n_-)h = 0.$$

To fulfill this equation, for large J also the $n_{+/-}$ have to be large, and $n_+ - n_-$ cancels with Prob 1 for $J \rightarrow \infty$.

Therefore, asymptotically the length of the cluster $n = n_+ + n_- \sim J/h_0$. To estimate s_{res} from above we replace the probabilities of the stable boundaries by 1 and obtain $s_{res}(J)/(k_B \ln 2) < 2^{-2J/h_0}$.

3. COMPARISON WITH FINITE MARKOV CHAIN ANALYSIS

The magnetization $m(h_0/h=1/4, J)$ obtained above reproduces the results of the finite Markov chain analysis (cf. Table 1 in [2]).

The residual entropy $s_{res}(h_0/h=1/4, J)$ however differs. To solve this puzzle we reconsider the derivation of the formula for s_{res}.

3. 1. Residual entropy revised

For the sake of brevity we introduce $\zeta_n^\sigma = \beta(\zeta_n + \sigma J)$. where ζ_n is the auxiliary random field recursively generated by $\zeta_n = h_n + A(\zeta_{n-1})$, cf. [2]. From

$$f^N = -(\beta N)^{-1}\ln Z^N = -(\beta N)^{-1}\Big(\sum_{\substack{n=1 \\ \sigma=\pm}}^{N-1} \ln(2\mathrm{ch}\zeta_n^\sigma)^{1/2} + \ln 2\mathrm{ch}\beta\zeta_N \Big)$$

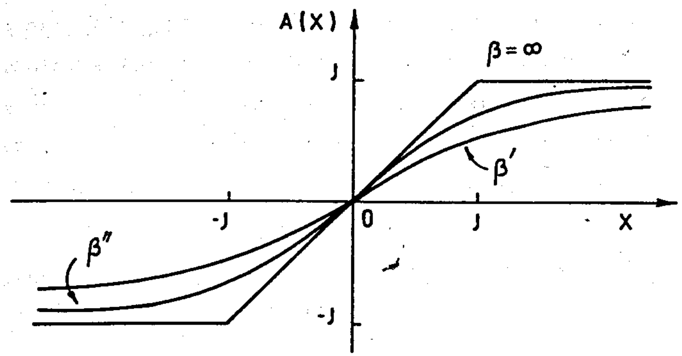

Fig. 10. Shapes of the function $A(x) = (2\beta)^{-1}\ln[\mathrm{ch}\beta(x+J)/\mathrm{ch}\beta(x-J)]$ for $\beta' > \beta$.

we obtain

$$s^N = -\partial f^N/\partial T =$$

$$=(k_B/2N)\sum_{\substack{n=1\\ \sigma=\pm}}^{N-1}\left(\ln(2\mathrm{ch}\zeta_n^\sigma) - \zeta_n^\sigma\,\mathrm{th}\zeta_n^\sigma - \beta^2\,\partial\zeta_n/\partial\beta\ \mathrm{th}\zeta_n^\sigma\right) +$$

$$+(k_B/N)(\ln 2\mathrm{ch}\beta\zeta_N - \zeta_N\mathrm{th}\beta\zeta_N - \beta^2\,\partial\zeta_N/\partial\beta\ \mathrm{th}\beta\zeta_N)\ . \qquad (3.1)$$

For $\beta \to \infty$ the first two terms under the sum give ln2 for each n with $\zeta_n + \sigma J = 0$, and zero otherwise. Thus for $N \to \infty$ we find the contribution already obtained in [2] (cf. (2.14) there)

$$(k_B/2)(\,w^*_J + w^*_{-J}\,)\ln 2\ , \qquad (3.2)$$

where the $w^*_{\pm J}$ are the weights of the invariant measure for the states $\zeta_n = \pm J$, respectively .
To estimate the contribution from the third term we observe that $\mathrm{sgn}\sum_{\sigma=\pm}\mathrm{th}\zeta_n^\sigma = \mathrm{sgn}\ \mathrm{sh}2\beta\zeta_n = \mathrm{sgn}\ \zeta_n$, and $\mathrm{sgn}\ \partial\zeta_n/\partial\beta = \mathrm{sgn}\ \zeta_{n-1}$, because for $\beta' > \beta$ we have $\zeta_n^{\beta'} > \zeta_n^{\beta}$ if

$\zeta_{n-1}^{\beta'} = \zeta_{n-1}^{\beta} > 0$ and $\zeta_n^{\beta'} < \zeta_n^{\beta}$ if $\zeta_{n-1}^{\beta'} = \zeta_{n-1}^{\beta} < 0$ (cf. Fig. 10). The sign of this contribution is nonpositive since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}\mathrm{sgn}\ \zeta_n\ \mathrm{sgn}\ \zeta_{n-1} \geq 0 \quad \text{with Prob 1}\ . \qquad (3.3)$$

Therefore (3.2) is only an upper bound for s_{res}.

3. 2. Explanation in the language of energy balance

Comparing the results for the spikes of $s_{res}(J)$ obtained by the energy balance arguments with those of Table 1 in [2] we observe that (3.2) corresponds to taking Prob (B^l C) instead of Prob(B^l C B^r) in (2.2). A careful analysis of the discrete stochastic mapping $\zeta_n = h_n + A(\zeta_{n-1})$ shows that the sequence of signs (characterizing the driving process h_n) corresponding to B^l leads the driven process ζ_n to the fixed point. Then the signs corresponding to C lead in the degenerate situation the driven process ζ_n to J or -J .This two sequences contribute to (3.2) . The further history corresponding to B^r as well as the possibility of kinks is not reflected in (3.2). This is a second (and independent) argument that (3.2) is only an upper bound for s_{res}.
For the case of *continuous* degeneracy (3.2) neglects not only Prob B^r but also the additional degeneracy due to kinks (cf. Fig. 7).
It is clear that for $J < h/2$ $s_{res}(h_0/h)$ is correctly predicted by (3.2) because the stable boundaries are up spins which are not affected by the perestroyka . Therefore Prob B^l = Prob B^r = 1.

4. CONCLUDING REMARKS

We have obtained a physical understanding of the discontinuous behaviour of m and s_{res} at zero temperature as a result of the perestroyka of the ground state. By energy balance arguments the (possibly infinite) class of clusters which are responsible for a given jump are selected for some examples. The thus calculated magnetization reproduces the results obtained by the finite Markov chain approach [1, 2].

The picture of flipping clusters makes the physical mechanism transparent but the selection of the relevant clusters is a rather tedious task. The finite Markov chain approach, on the other hand, does not reflect this picture in an obvious way, but gives a clear and constructive formalism which finally reduces to linear algebra.

For the calculation of the residual entropy within the Markov chain formalism some formal questions still remain open. For instance, it seems difficult to evaluate the thermodynamic formula (3.1) for zero temperature with the exception of the contribution given by (3.2). The derivation of the formula for magnetization (2.10) in [2] from the thermodynamic expression $m = -\partial f/\partial h_0$ seems similar difficult . In both cases the derivations of the driven process (or the invariant measure) with respect to the physical parameters β and h_0 are involved.

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