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SELF-LOCALIZATION OF THE SCHRÖDINGER WAVE PACKETS AND QUASI-SOLITON STABILITY

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In the present work the expression for wave packet mean-square radius is obtained and investigated for quite a general U(1) nonlinear Schrödinger equation (NSE).

Earlier investigating similar expression V.I.Talanov /1/ has obtained the condition of self-focusing laser beam propagating in nonlinear media, and V.E.Zakharov /2/ has get the condition for collapse of spherically symmetric Langmuir wave packets in plasma.

In what follows our concern will be with the behaviour of "Schrödinger type systems" near the steady states, under both trivial (drop like) and condensate boundary conditions.

I. Let us consider NSE

$$i\Psi_t + \Delta\Psi - \frac{dU}{d\rho}\Psi = 0, \tag{1}$$

where $\rho = |\Psi|^2$ is the density of "particles".
Being U(1) symmetrical this equation, in general case, possesses the following integrals of motion

$$\begin{aligned} E &= \int \{ |\Psi_{\vec{x}}|^2 + U \} d\vec{x}, \\ N &= \int |\Psi|^2 d\vec{x}, \\ P &= \int \frac{1}{2i} \{ \bar{\Psi}\cdot \operatorname{grad}\Psi - \operatorname{grad}\bar{\Psi}\cdot\Psi \} d\vec{x} = \int j\, d\vec{x}. \end{aligned} \tag{2}$$

We shall focus on the dynamics of well-localized (in space) "wave packets", so as

$$B = \int \vec{x}^2 \rho\, d\vec{x} < \infty. \tag{3}$$

Obviously, functional B is proportional to wave packet mean-square radius $B = \langle r^2 \rangle N$. To study its behaviour in time we shall find the second time derivative of B and two conservation laws are enough for that

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0,$$
$$\frac{\partial j_k}{\partial t} + \frac{\partial}{\partial x_\ell} I_{k\ell} = 0, \tag{4}$$

where $I_{k\ell} = 4\left(\frac{\partial}{\partial x_k}\bar{\psi}\right)\left(\frac{\partial}{\partial x_\ell}\psi\right) + 2\left(\frac{dU}{d\rho}\rho - U\right)\delta_{k\ell}.$

Differentiating B once over t

$$\dot{B} = \int \vec{x}^2 \dot{\rho}\, d\vec{x} = -\int \vec{x}^2 \operatorname{div} \vec{j}\, d\vec{x} = 2\int (\vec{x}, \vec{j})\, d\vec{x}$$

Analogously, using (4) we obtain

$$\ddot{B} = -2\frac{d}{dt}\int (\vec{x}, \vec{j})\, d\vec{x} = -2\int x_k \frac{\partial}{\partial x_\ell} I_{k\ell}\, d\vec{x} = 2\int \delta_{k\ell} I_{k\ell}\, d\vec{x}$$

or

$$\ddot{B} = 8\int |\psi_x|^2 d\vec{x} + 4D\int \left(\frac{dU}{d\rho}\rho - U\right) d\vec{x}. \tag{5}$$

Another form for $\ddot{B}$ will be also of use in terms of energy integral

$$\ddot{B} = 8E + 4\int \left(D\frac{dU}{d\rho}\rho - (D+2)U\right) d\vec{x}, \tag{6}$$

where D is space dimensions.

II. Drop-solitons

For more visual presentation of wave packet behaviour we apply to mechanical analogy, i.e. we treat the time behaviour of functional B as the behaviour of a "particle" in an external field, the radius of "wave packet" as the distance between the "particle" and the origin. Then $\ddot{B}$ is proportional to the acceleration of the particle*).

*) When boundary conditions are trivial. In this case one can choose "initial velocity" r be zero because of the Galilean invariance (see also /3/).

We call $\ddot{B}=0$ the "stationary point" (equilibrium position), and let the particle be there at t=0. Deviation of the particle from this position leads to the change of acceleration sign (Fig.1).

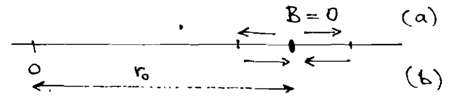

Fig.1

If at the moment $t_1=0$ $\ddot{B} > 0$ ($\ddot{B} < 0$) when $r > r_0$ ($r < r_0$) and sign of inequality concerves in the process of particle motion (Fig.1a), then the position $\ddot{B}=0$ proves to be unstable. In other words "external" force takes particle away from equilibrium position. If on the contrary in the process of the particle motion the sign of acceleration changes, i.e. "external" force returns it backward ($\delta\ddot{B}/\delta r < 0$), then the position $\ddot{B}=0$ is stable (Fig.1b). In the language of "wave packet", in the former case it either collapses or disperses depending on the initial conditions. In the latter case packet is stable (remains localized near $r = r_0$).

1. $U = -\frac{1}{n}\Phi^n$ (see also /3/), where $\Phi = |\psi|^2$. In this case expression for $\ddot{B}$ has the form

$$\ddot{B} = 8E + \frac{4}{n}\{2 - D(n-1)\}2^{D-1}\pi \int \Phi^n r^{D-1} dr,$$

estimating integral we get

$$\ddot{B} = 8E + \frac{4\pi}{n}\{2 - D(n-1)\}\frac{N^n}{V^{n-1}} = 8E + T.$$

The stationary point $8|E| = T$, $\ddot{B}(t) = 0$ is stable when $E < 0$, $T > 0$, and stable stationary packets exist, if

$$1 < n < 1 + \frac{2}{D} \qquad (D = 1,\ n = 2).$$

When $T < 0$ there are no stable stationary packets.

2. $U = -\alpha\Phi - \frac{1}{2}\Phi^2 + \frac{1}{3}\Phi^3$

Such choice of function U corresponds to the equation

$$i\psi_t + \Delta\psi + \alpha\psi + (|\psi|^2 - |\psi|^4)\psi = 0 \tag{7}$$

with the energy integral

$$E = \int \left\{ |\psi_r|^2 - \alpha|\psi|^2 - \frac{1}{2}|\psi|^4 + \frac{1}{3}|\psi|^6 \right\} d^D r.$$

The expression for $\ddot{B}$ assumes the form

$$\ddot{B} = 8F - 2(D-2)\int \Phi^2 d^D r + \frac{8}{3}(D-1)\int \Phi^3 d^D r \qquad (8)$$

or, evaluating integrals, we obtain

$$\ddot{B} \approx 8F - 2(D-2)\frac{N^2}{V} + \frac{8}{3}(D-1)\frac{N^3}{V^2}, \qquad (9)$$

where

$$F = \int \left\{ |\varphi_r|^2 - \frac{1}{2}\Phi^2 + \frac{1}{3}\Phi^3 \right\} d^D r .$$

Under the perturbation $V = V_0 + \delta V$, soliton-like solutions (SLS) will be stable if the ratio $\delta\ddot{B}/\delta V$ is negative. We consider (9) for dimensions D=1,2,3.

D=1, $\delta\ddot{B}/\delta V < 0$ -SLS are stable always

D=2, $\delta\ddot{B}/\delta V < 0$ -SLS are stable always

$$D=3,\ \delta\ddot{B} = -2\frac{N^2}{V_0^2}\left(\frac{16}{3}\frac{N}{V_0} - 1\right)\delta V . \qquad (10)$$

Here SLS would be stable if $N/V_0 = a^2 > 3/16$, where a^2 is weighed square amplitude of packet.

It is worth to use the mechanical analogy not only for modelling the wave packet time behaviour, as it was done above, but for evaluating its stationary amplitude as well /6a/.

Since solutions are given up to a phase θ_0, we put it zero, thereby passing to the packet rest frame

$$\varphi = y e^{i\theta_0}, \text{ where } \theta_0 = 0,\ \varphi = y . \qquad (11)$$

In the stationary case equation (7) with allowance for (11) looks like

$$y_{\xi\xi} + y^3 - y^5 + \alpha y = -\frac{D-1}{\xi} y_\xi . \qquad (12)$$

Multiplying (12) by y_ξ, we obtain

$$\frac{1}{2}\frac{d}{d\xi}\left[y'^2 + \frac{1}{2}y^4 - \frac{1}{3}y^6 + \alpha y^2 \right] = -\frac{D-1}{\xi} y_\xi^2 . \qquad (13)$$

Call

$$U_M = \alpha y^2 + \frac{1}{2}y^4 - \frac{1}{3}y^6 .$$

Equation (13) describes the motion of mechanical particle in an external field under the action of a friction force.

From the form of function U_M one can conclude, that particle-like solutions exist if $\alpha < 0$. The relief of function $U_M(y)$ for this case is represented in Fig.2.

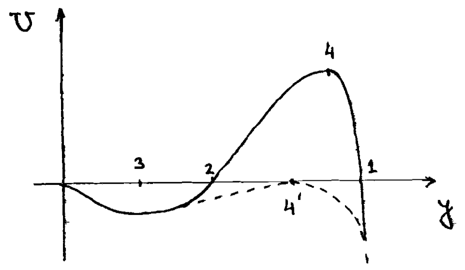

Fig.2

Let put $U_M(y) = 0$. By solving equation

$$\alpha y^2 + \frac{1}{2}y^4 - \frac{1}{3}y^6 = 0 . \qquad (14)$$

we obtain the intersection points of $U_M(y)$ with the axis OY

$$y_{1,2} = \frac{3}{4}\left(1 \pm \sqrt{1 - \frac{16}{3}|\alpha|}\right),\quad y_0 = 0 . \qquad (14a)$$

From the radical in (14a) one can see, that at $|\alpha| = 3/16$ the point y'_a is just a contact point of the function U_M with the OY axis. When $|\alpha| < \frac{3}{16}$, curve crosses the axis OY in points 1 and 2. When $|\alpha| > \frac{3}{16}$, there are no crossing points. Therefore the conditions of existence of particle-like solutions (PLS) are $\alpha < 0$, $0 < |\alpha| < \frac{3}{16}$.

The points of extremum are

$$y_{3,4} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4|\alpha|}\right),\quad y_0 = 0, \qquad (14b)$$

whence $|\alpha| < 1/4$.

From formulae (14a) for limit values of α we obtain an estimation for the amplitude:

when $|\alpha| \ll \frac{3}{16}$, $y_2 \approx 2|\alpha|$ (y_1 is out of our interest)

$$|\alpha| \approx \frac{3}{16},\ y_2 \approx \frac{3}{4} . \qquad (14c)$$

Combining conditions in (14c) gives $0 < a^2 < 3/4$ when D=1. In the above we have got the conditions of SLS stability: $a^2 > 3/16$ (see(10)). We call $\langle a^2 \rangle = 3/16$ the point separating regions of "stability" and "unstability" of SLS, the critical amplitude value. Substituting it in Eq.(14a) with allowance for conditiond of PLS existence, we obtain the stability region for parameter α :

$$0.08 < |\alpha| < 0.1875 .$$

The threshold value of α one can also evaluate using the computer data of /4/,/5/, see Fig.3. Recall, that the weighed mean-square value of amplitude $\langle a_{cr} \rangle = 3/16 = 0.433$ then the proper value of amplitude for estimations one can put $\sqrt{2}\langle a_{cr}\rangle$, giving $\alpha_{cr} = 0.03$ (see Fig.3) what is in good agreement (for such a rough estimate) with result /4/ ($\alpha \approx 0.025$) also given by the Q-theorem, $\frac{dN}{d\alpha} = 0$ (see /6/).

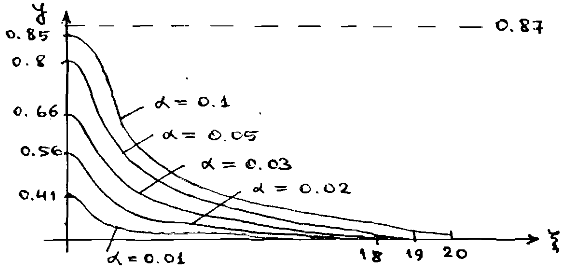

Fig.3. Stationary nodeless solutions to Eq.(12) for few values of α. In order to pass from our model (7) to the model of the Ref. /4/, /5/, one should make a substitution $\gamma^2 = \alpha\psi^2$, $\xi^2 = x^2/\alpha$.

3. Model with the saturable nonlinearity

Let us choose the function

$$U = \ln(1+\Phi) - (1-m^2)\Phi, \qquad \text{with } \Phi = |\Psi|^2 . \quad (15)$$

It corresponds to the equation with the saturated nonlinearity

$$i\Psi_t + \Delta\Psi + \Psi \frac{|\Psi|^2}{1+|\Psi|^2} = 0 \quad (16)$$

with the energy integral

$$E = \int \left\{ |\Psi_r|^2 + \ln(1+\Phi) - \Phi \right\} d^D r . \quad (17)$$

Proceeding from function Ψ to φ

$$\Psi = \varphi(r,t)\, e^{i\omega t} \quad (18)$$

instead of Eq.(16) gives

$$i\varphi_t + \Delta\varphi + \omega\varphi + \varphi\frac{|\varphi|^2}{1+|\varphi|^2} = 0 . \quad (19)$$

One can write the free energy in the form

$$F = E - \omega N = \int \left\{ |\Psi_r|^2 + \ln(1+\Phi) - (1+\omega)\Phi \right\} d^D r . \quad (20)$$

Examine the conditions of PLS existence for equation (19). To do it more visual, as above, we use the mechanical analogy.

Let $\varphi_t = 0$. Then the equation (19) is

$$\varphi_{xx} + \omega\varphi + \varphi\frac{\varphi^2}{1+\varphi^2} = -\frac{D-1}{x}\varphi_x \quad (21)$$

or

$$\frac{d}{dx}\left\{ \varphi_x^2 + \omega\varphi^2 + \varphi^2 - \ln(1+\varphi^2) \right\} = -2\frac{D-1}{x}\varphi_x^2 . \quad (22)$$

We introduce following notations

$$U_M(\xi) = \frac{\omega}{2}\xi^2 + \frac{1}{2}\left(\xi^2 - \ln(1+\xi^2)\right) \quad (23a)$$

$$E_M = \frac{1}{2}\dot{\xi}^2 + U_M \quad (23b)$$

$$F_{fr} = -\frac{D-1}{t}\dot{\xi}^2 . \quad (23c)$$

Equation (21) is just the equation of motion of a mechanical particle in the external potential U_M under the action of the friction force F_{fr}.

The condition for particle-like solutions to exist, as one can see from (23a), is $\omega < 0$. The function $U_M = U_M(\xi)$ is plotted in Fig.4. From formula (23a) we find the point

$$\xi_0^2 = \frac{|\omega|}{1-|\omega|} \quad (24)$$

corresponding to the function U_M minimum.

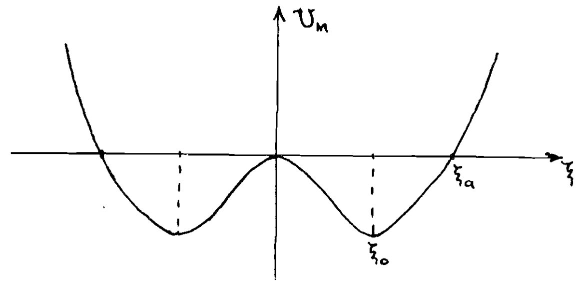

Fig.4

Whence the limitation for parameter ω appears $|\omega| < 1$, and hence conditions of PLS existence is $\omega < 0$, $|\omega| < 1$.

Now we investigate the stability of the PLS to equation (16). The magnitude $\ddot{B}$ is now of the form

$$\ddot{B} = 8F + 4\int \left\{ D\frac{\Phi}{1+\Phi} + 2(1+\omega)\Phi - (D+2)\ln(1+\Phi) \right\} d^D r \quad (25)$$

with free energy

$$F = \int \left\{ |\Psi_r|^2 - (1+\omega)\Phi + \ln(1+\Phi) \right\} d^D r . \qquad (26)$$

We estimate in (25) the integrals as follows: $N = \tilde{\Phi} V$, $\tilde{\Phi} = N/V$, where $\tilde{\Phi}$ is volume average (weighed) value of Φ . Then we write (25) in the form

$$\ddot{B} = 8F + 4V \left\{ D \frac{N/V}{1+N/V} + 2(1+\omega)\frac{N}{V} - (D+2)\ln\left(1+\frac{N}{V}\right) \right\} . \qquad (27)$$

Introduce the perturbation $V = V_0 + \delta V$. Then expression (27), with allowance for $\delta F = 0$, assumes the form

$$\delta\ddot{B} = 4 \left\{ \frac{D(N/V)^2}{(1+N/V)^2} - (D+2)\ln(1+N/V) + \frac{(D+2)N/V}{1+N/V} \right\} \delta V . \qquad (28)$$

For PLS of the equation (16) to be stable, the expression in curly brackets must be negative, i.e. $\delta\ddot{B}/\delta V < 0$.

When D=1, the condition of stability for (28) is

$$-3\ln(1+x) + \frac{x}{x+1}\left(3 + \frac{x}{x+1}\right) < 0 \qquad (29)$$

where $x = N/V$.

Let $x \ll 1$, expanding (29) in series of X from (28) we have

$$\frac{\delta\ddot{B}}{\delta V} = -\frac{1}{6}x^2 . \qquad (30)$$

When $x \gg 1$, from (29) one gets

$$3 - 3\ln x < 0 .$$

Thus, when D=1, PLS of (16) are stable.

Let D=2. When $x \ll 1$, from (28) we have

$$\frac{\delta\ddot{B}}{\delta V} = -\frac{4}{3}x^2 .$$

When $x \gg 1$, the coefficient of δV is equal to

$$6 - 4\ln x < 0$$

therefore, in this case also PLS are stable.

When D=3 for small $x \ll 1$, from (28) it follows

$$\frac{\delta\ddot{B}}{\delta V} = \frac{1}{2}x^2 ,$$

i.e. PLS are unstable, and when $x \gg 1$,

$$8 - 5\ln x < 0$$

they become stable. In the point N/V = 0.3 the magnitude of $\delta\ddot{B}$ changes the sign, i.e. there is a threshold for stability. Since $a^2 = |\Psi|^2 = \tilde{\Phi}$ = N/V from (24) we have the estimation for the threshold

$$|\omega| > \frac{(1/2)(N/V)}{1+(1/2)(N/V)} \approx 0.23 .$$

III. Solitons in the condensate

Self-localization of Schrödinger wave packets in drop statement has been considered.

We apply the similar method to investigate localized solutions in condensate, i.e. under nontrivial boundary conditions. As examples we take the models Ψ^3 and $\Psi^3 - \Psi^5$ in condensate statement. Integrals of particle number and energy should be given now by

$$N = \int \left\{ 1 - |\Psi|^2 \right\} dx \qquad (31)$$

$$E = \int \left\{ |\Psi_x|^2 + U \right\} dx \qquad (32)$$

for being finite and

$$|\Psi|^2 \to 1 , \quad x \to \pm\infty \qquad (33)$$

$$U = \begin{cases} (1-|\Psi|^2)^2 , & (\Psi^3\text{- model}) \\ (|\Psi|^2-1)^2(|\Psi|^2-A) & (\Psi^3\text{-}\Psi^5\text{- model}) \end{cases} \qquad (34)$$

according the boundary conditions fixed.

Both the models possess solutions that describe density cavity moving through the condensate "bubbles".

Perfoming as above we introduce a functional

$$B = \int (1-|\Psi|^2) r^2 dr = N\langle r^2 \rangle \qquad (35)$$

proportional the middle square of density cavity dimension.

Differentiating twice with respect to time we get

$$\ddot{B} = 2N\langle \dot{r}^2 \rangle + 2N\langle \ddot{r}\, r \rangle \qquad (36)$$

or for

$$\ddot{A} \equiv \ddot{B} - 2N\langle \dot{r}^2 \rangle = -8F + \begin{cases} 4\int \left\{ 1 - |\Psi|^2 \right\}^2 dx \\ -4(2+A)\int \left\{ 1 - |\Psi|^2 \right\}^2 dx \end{cases} , \qquad (37)$$

where

$$F = E - \begin{cases} N\left(1-\frac{v^2}{4}\right) \\ \left(1-A-\frac{v^2}{4}\right)N \end{cases} , \quad v \text{ - is the bubble velocity.} \tag{38}$$

The upper line as above corresponds to the ψ^3-model, and the lower to the $\psi^3-\psi^5$-model.

Just from these expressions (37),(38) one can see the difference between the models: the signs of the second terms are opposite.

By use of the known solutions

$$\psi = \varkappa\, \mathrm{th}\, \varkappa\xi + i\frac{v}{2}, \quad \varkappa^2 = 1-\frac{v^2}{4}, \quad \xi = x - vt \tag{39}$$

in the first case and

$$\psi = \frac{\sqrt{2}\, \mathrm{ch}(a\xi - i\mu)}{\left[\frac{2-A}{\sqrt{A^2+v^2}} + \mathrm{ch}\, 2a\xi\right]^{1/2}}; \quad \cos 2\mu = \frac{2A+v^2}{2\sqrt{A^2+v^2}}, \quad \varkappa^2 = 1-A-\frac{v^2}{4}, \quad \xi = \varkappa(x-vt) \tag{40}$$

in the second case /7/ we get $\ddot{A}=0$.

A rough*) analysis of expressions (37) shows kinks (39) to be stable of the ψ^3-model and the total region $v \in [0,2)$ in accordance with the results of /8/, and bubbles (40) ($A>0$) unstable which for $v=0$ corresponds to the result of /9/.

VI. Resume

We saw in the above that the usage of functionals leads to a good agreement between semi-quantitative results and numerical ones. It is also important that only two conservation laws are sufficient to derive the functionals under consideration that allows one to generalize the techique described involving other soliton-like solution bearing equations. In particular in the case of Klein-Gordon equation the charge density $q = \frac{i}{2}(\bar{\psi}_t \psi - \bar{\psi}\psi_t)$ comes instead of ρ .

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*) More precise analysis is supposed to be published elsewhere.

References.

1. V.I.Talanov. Letters in ZhETF, 1965, v.2, 5, p.218-222.(in Russian)
2. V.E.Zakharov. ZhETF, 1972, v.62, 5, p.1745-1759. (in Russian).
3. Kh.O.Abdulloev, A.V.Makhankov. The Holstein-Primakoff transformation for magnetic systems and self-localization of spin waves. JINR, P17-88-165, Dubna 1988. (in Russian).
4. V.G.Kartavenko. Soliton-like solutions in nuclear hydrodynamics. JINR, P4-83-411, Dubna 1983. N.Ph., 1984, v.40, p.377-388.(Russian)
5. D.Anderson. Stability of the time-dependent particle-like solutions in non-linear field theories. J.Math.Phys. 1971, 12, p.945.

6a. Yu.P.Rybakov. Regular solutions conditional stability of nonlinear field theory. Gravitation and elementary particle theory problems. Atomizdat, M., 1979, p.194-202. (in Russian).

b. V.G.Makhankov. Dynamics of classical solitons. Phys.Rep. 1978, 35, p.1-128.

c. N.G.Vakhitov, A.A.Kolokolov. Stationary solutions of wave equation with the saturable nonlinearity. Izv.VUZov. Radiophysics, 1973, 16, p.1020-1028. (in Russian).

7. I.V.Barashenkov, V.G.Makhankov. Soliton-like excitations in a one-dimensional nuclear matter. JINR E2-84-173, Dubna 1984.
8. V.E.Zakharov, A.B.Shabat. ZhETF, 1973, 64, p.1627. (in Russian).
9. I.V.Barashenkov, V.G.Makhankov. Soliton-like "bubbles" in a system of integrating bosons. Phys.Lett.A, 1988, v.128, p.52-56.
10. R.T.Glassey. On the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations. J.Math.Phys. 1977, v.18, 9, p.1794-1797.

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