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BETHE-ANSATZ SOLUTIONS OF A NON-STRING TYPE: NUMERICAL RESULTS

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The problem of diagonalising the hamiltonian for quantum integrable models in the context of the co-ordinate [1] or algebraic [2] Bethe ansatz is reduced to solving the system of the Bethe-ansatz equations. For the simplest case of the XXX Heisenberg antiferromagnet and its integrable generalisation [3] to arbitrary spin s the equations have the form

$$\left(\frac{\lambda_j+is}{\lambda_j-is}\right)^N = -\prod_{k=1}^{M}\frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k-i}, \quad j=1\ldots M. \tag{1}$$

Here, N is the number of sites of the spin ring, and the number M of complex parameters λ_j may be $0\ldots sN$. The energies E - eigenvalues of the hamiltonian

$$H=\sum_{n=1}^{N} h(\vec{S}_n \vec{S}_{n+1}), \quad h(x)=-\sum_{j=0}^{2s-1}\left(\sum_{k=j+1}^{2s} 1/k\right)\prod_{\substack{m=0\\ m\neq j}}^{2s}\frac{x-x_m}{x_j-x_m},$$

$$x_m = m(m+1)/2 - s(s+1), \quad \vec{S}_n^2 = s(s+1), \quad \vec{S}_{N+1} = \vec{S}_1, \text{ -}$$

momenta P, and spins S of the states are expressed through the solutions $\{\lambda_j\}_M$ to system (1)

$$E=-\sum_{j=1}^{M}\frac{s}{\lambda_j^2+s^2}, \quad P \overset{2\pi}{=} \frac{1}{i}\sum_{j=1}^{M}\ln\frac{\lambda_j+is}{\lambda_j-is}, \quad S=sN-M. \tag{2}$$

According to the "string" hypothesis [1,4,5], as $N\to\infty$, all the parameters λ should assemble into n-strings

$$\lambda = x + i[(n+1)/2 - m], \quad m=1\ldots n, \tag{3}$$

where a positive integer n specifies the length of the string, and a real x its centre position. The antiferromagnetic vacuum comprises a sea of $M=sN$ $2s$-strings [5]. The string hypothesis gives a rather accurate general qualitative classification of states, their total number being in agreement [6] with the assumption of completeness. However, the assertion about an exponential (in N) accuracy of strings proves wrong in a number of cases. Even for $s=1/2$, on the background of the sea of real roots (1-strings), non-string configurations - quartets and wide pairs - have been predicted [7]. For $s>1/2$, quartets are changed to multiplets and narrow pairs may appear [8]. Deformations of the sea strings become also possible. The numerical computations [9] have shown that the minimal deviations from formula (3) for the vacuum and two-hole states behave as $O(1/N)$ while the maximum is $O(1)$; however, these considerable string deformations weakly affect the energy. The object of the present letter is to find out multiplet-type solutions explicitly and to study finite-size corrections for them.

At large N, one can describe the sea of $2s$-strings with a density function. The Bethe-ansatz equations for the sea are rewritten as an integral equation for the density, which may be solved by the Fourier transformation. The study of the equations for complex-root configurations on the sea background shows that there are only three possibilities [8]: free narrow pairs $|\mathrm{Im}\,\lambda|<s-1/2$, wide pairs $|\mathrm{Im}\,\lambda|>s+1/2$, and multiplets

$$\lambda = x \pm i(y+s-m), \quad m=0\ldots 2s, \quad 0<y<1/2. \tag{4}$$

Real parameters x and y determine the positions of all the pairs of the multiplet, $2s-1$ narrow and two intermediate ones $||\mathrm{Im}\,\lambda|-s|<1/2$. As well as in strings, for each complex root of a multiplet λ (except lower members of its intermediate pairs) there is a successor λ' lying an imaginary unit below it. The deviations $\Delta x+i\Delta y = \lambda-\lambda'-i$ should be exponentially small [8]

$$\Delta x^2+\Delta y^2= \exp(-KN), \quad K = \ln\frac{\cosh(\pi x)+\cos(\pi y)}{\cosh(\pi x)-\cos(\pi y)} . \tag{5}$$

The sea contributions to the equations for the allowed configurations can be evaluated. Thereafter, the equations are reduced to the higher-level Bethe-ansatz form, where the N factors of equations (1) are cancelled. Contributions of the complex roots to the energy and momentum are exactly compensated by the backflow reaction of the sea. Thus, the energy and momentum are completely determined by physical excitations, holes in the sea. In the limit of an infinite size of the ring their positions may be arbitrary. However, at finite N they are discrete and correspond to half-integer values of the integral of the density for 2s-strings together with holes. This can also be written [9] as higher-level Bethe-ansatz equations.

In the present letter the simplest multiplet-type solutions are considered, with one quartet ($s=1/2$) or sextet ($s=1$) at even N and the minimal number of holes, four, the total spin being $S=0$. For this case the higher-level Bethe-ansatz equations [8] are reduced to the form

$$\prod_{j=1}^{4}\frac{x-x_j+i(y+1/2)}{x-x_j+i(y-1/2)} = \frac{2y+1}{2y-1} , \tag{6}$$

$$\pi Q_j= N\,[\pi/4-\operatorname{atan}\exp(-\pi x_j)] +\operatorname{atan}\frac{x-x_j}{1/2+y} +\operatorname{atan}\frac{x-x_j}{1/2-y}$$

$$+ \int_0^\infty \frac{dp}{p}\sum_{k=1}^{4} \sin[(x_j-x_k)p] \tag{7}$$

$$* \frac{2\exp[(s-1/2)p]-\exp[-(s-1/2)p]-\exp[-(s+1/2)p]}{2\cosh(p/2)\ 2\sinh(sp)} , \quad j=1\ldots4,$$

where Q_j are (half-)integer numbers – according to $Q_{max} =N/4+1/2-(2s)^{-1}$,– which specify the hole positions; $|Q_j|\le Q_{max}$.

Equation (6) for the parameters of the multiplet can be solved exactly. After eliminating the denominator and taking

the imaginary part, one gets $y(4y^2-1)(x_1+x_2+x_3+x_4-4x)=0$. It follows then that for the multiplet solution (4)

$$x = \frac{1}{4}(x_1+x_2+x_3+x_4). \tag{8}$$

The real part, after formula (8) is substituted, gives a biquadratic equation for y. Its solution can be represented as

$$y = \{\frac{1}{6}[\frac{1}{2} -2A_2\pm(1+4A_2+28A_2^2-12A_4)^{1/2}]\}^{1/2}, \quad A_n=\frac{1}{4}\sum_{j=1}^{4}(x_j-x)^n. \tag{9}$$

Equations (7) have to be solved numerically. One iterates the hole co-ordinates, using formulae (8) and (9) at every step. The result allows one to compute the leading approximation in $N\to\infty$ for the energy and momentum of the state

$$E_\infty= \sum_{j=1}^{4}\frac{1}{2}\pi/\cosh(\pi x_j) - N \begin{cases}\sum_{n=1}^{s} (2n-1)^{-1}, & s = \text{integer};\\ \ln 2 + \sum_{n=1}^{s-1/2} (2n)^{-1}, & s = \text{half-integer};\end{cases} \tag{10}$$

$$P \frac{2\pi}{} \pi sN -2 \sum_{j=1}^{4} \operatorname{atan}\exp(-\pi x_j). \tag{11}$$

The difference between the primary values (2) derived from the solutions to equations (1), on the one hand, and the higher-level approximation (5)-(11), on the other, is due to finite-size corrections. As a consequence of equations (6) and (7), formula (11) for the momentum proves to be exact because its values are multiples of $2\pi/N$. Numerical data presented below demonstrate that the energy correction $E-E_\infty$ behaves as $O(1/N)$, that is in the same way as for the vacuum and simplest excitations [9,10,11].

The numerical computations are performed by the Newton method for the logarithms of equations (1) [9]. Since multiplets should have exponentially small deviations from formula (4), quantities of essentially different scales may be present in the problem. And because the computer precision is limited, one has to store for each complex root, besides its absolute position, the value of $\Delta x+i\Delta y$. Furthermore, to improve the linear system solved at every step of the iterations, the equations are modified as follows. To each equation for a member of a string-like chain the equations for all its successors are added. This eliminates singularities in internal deformations of the chains from the equations for their higher members.

Table 1. Quartet-type solutions ($s=1/2, S=0$): N dependence.

N	Q_j	$x_{j\infty}$	$x_\infty, y_\infty; \lambda_{max}$	K_∞, K	$P\frac{N}{2\pi}$, E, δ, δ_v
50	-12	-1.2528320	.4069619	.223	-3
	10	.65310112	.4310766	.305	-33.9652960767
	11	.86233372	.415359545		-.890181
	12	1.3652448	.929075272		-.824879465969
80	-19.5	-1.40391672	.4828662	.305	37
	17.5	.803357612	.3823154	.342	-55.0177558735
	18.5	1.012484691	.491795306		-.895606
	19.5	1.519539278	.878948890		-.824136394052
128	-31.5	-1.5559310	.4663726	.364	59
	28.5	.82630748	.3651575	.388	-88.2980643647
	29.5	.95286674	.473087495		-.893067
	31.5	1.64224731	.861554593		-.823697869135
200	-48.5	-1.23532167	.4508747	.300	92
	44.5	.805860239	.3947886	.313	-138.110673055
	46.5	.963748648	.456742522		-.865510
	48.5	1.26921169	.894007428		-.823428050508
300	-72.5	-1.18403224	.4631337	.289	139
	67.5	.82859734	.3948913	.296	-207.433923708
	70.5	1.00230524	.468814566		-.8599531
	72.5	1.20566462	.894719545		-.823254238795

$-\pi^2/12=-.822467033424$

The results of computing multiplet-type states with different N but about the same hole positions are presented in tables 1 and 2. The index ∞ relates to the higher-level Bethe-ansatz approximation (5)-(10). On the other hand, the real and imaginary parts of the highest multiplet member λ_{max} ($\mathrm{Im} \simeq y_\infty+s$), the coefficient $K=-\ln(\Delta x^2+\Delta y^2)_{max}/N$ charactering its deviation from the successor, the momentum and energy of the state (2) are computed through the solutions

to equations (1). The quantity $\delta=(E-E_\infty)N$ controls the accuracy of the approximation (10); the δ_v values correspond to the vacuum solutions.

For $s=1$ the information about the sea-string deformations is presented in addition: Δ_{max} and Δ_{mean}, the maximum and average Δy over all the string-like chains. It should be noted that at the points x_j the sign of the deformations alters, like in two-hole states [9]. This is reflected on the average and maximum values.

Table 2. Sextet-type solutions ($s=1, S=0$): N dependence.

N	Q_j	$x_{j\infty}$	$x_\infty, y_\infty; \lambda_{max}$	K_∞, K	$P\frac{N}{2\pi}$, E, δ, δ_v	$\Delta_{max}, \Delta_{mean}$
30	-7.5	-1.2662	.4428	.120	13	-.0528279
	5.5	.58747	.45935	.296	-29.2420778022	-.0124799
	6.5	.87213	.461383788		-.273	
	7.5	1.57761	1.46389221		-1.24359396024	
50	-12.5	-1.4273916	.527411	.189	23	-.0557867
	10.5	.7508006	.417459	.236	-49.5404404411	-.0129662
	11.5	1.0358171	.548039600		-.21263	
	12.5	1.7504168	1.42389289		-1.24008319675	
80	-20	-1.581445	.555043	.183	37	-.0571100
	17	.740698	.413279	.198	-79.5904903094	-.00991790
	19	1.176765	.578650910		-.18748	
	20	1.884152	1.42377551		-1.23829584077	
128	-31	-1.1257839	.468014	.234	59	:0769850
	28	.7739528	.414146	.253	-127.435220911	-.00124705
	30	1.0066041	.478827931		-.24932	
	31	1.2172837	1.41526422		-1.23718650132	
150	-36.5	-1.17646748	.472369	.233	69	.0771949
	32.5	.74751712	.413384	.246	-149.452818318	-.00126993
	35.5	1.05462039	.483363361		-.237879	
	36.5	1.26380619	1.41500005		-1.23690487880	

$-\pi^2/8=-1.23370055014$

Another projection, different multiplet-type states at the same N, is presented in tables 3 and 4. One can observe how the parameters of the multiplets vary with shifting the holes.

The following general conclusion can be made from the computations: As well as strings, multiplets are perfectly reliable configurations for sufficiently large N. They may degenerate into strings only when y approaches 1/2 or 0.

Table 3. Quartet-type solutions (s=1/2,N=300,S=0): Q dependence.

Q_j	$x_{j\infty}$	$x_\infty, y_\infty; \lambda_{max}$	K_∞, K	$P\frac{N}{2\pi}$, E, δ
-73.5	-1.36482157	.5879376	.3808	145
71.5	1.09433226	.2906191	.3799	-207.696353473
72.5	1.21614273	.595746011		-.86834
73.5	1.40609720	.789259871		
-73.5	-1.36364014	.1723006	.5446	118
62.5	.65594717	.4011303	.5862	-206.817631923
63.5	.68345220	.168892334		-.85231
64.5	.71344296	.896267176		
-73.5	-1.36381005	-.0647769	.3519	64
44.5	.35668617	.4431056	.3905	-205.206089162
45.5	.36811928	-.071645209		-.83273
46.5	.37989702	.938997428		
-73.5	-1.36480479	-.2441374	.1673	-17
17.5	.12222059	.4651560	.1973	-203.556189286
18.5	.12939748	-.252869538		-.807541
19.5	.13663694	.961900751		
-73.5	-1.36962506	-.5362784	.0286	121
-36.5	-.26748256	.4873218	.0558	-204.409235838
-35.5	-.25843997	-.547265385		-.787864
-34.5	-.24956619	.985490076		

Table 4. Sextet-type solutions (s=1,N=150,S=0): Q dependence.

Q_j	$x_{j\infty}$	$x_\infty, y_\infty; \lambda_{max}$	K_∞, K	$P\frac{N}{2\pi}$, E, δ	Δ_{max}, Δ_{mean}
-37.5	-1.784453	.705753	.325	73	-.0584750
35.5	1.100561	.231061	.295	-149.846266281	-.00895129
36.5	1.387729	.729061176		-.13287	
37.5	2.119174	1.248480914			
-37.5	-1.7701893	.3728061	1.006	70	.0708430
34.5	.9243322	.1932501	1.117	-149.648265432	-.00595921
35.5	1.0610943	.368102324		-.14313	
36.5	1.2759871	1.154708082			
-37.5	-1.76501546	.1721583	1.247	64	.0840352
32.5	.73894290	.2803942	1.474	-149.256159171	-.00412634
33.5	.81152492	.160400989		-.16679	
34.5	.90318069	1.250735093			
-37.5	-1.76233077	-.0067414	1.020	52	.0909902
28.5	.53843698	.3441881	1.205	-148.490623758	-.00257904
29.5	.57687237	-.023845271		-.20709	
30.5	.62005585	1.320130172			
-37.5	-1.76195649	-.1492694	.669	34	.0941040
22.5	.36444698	.3832854	.784	-147.427735359	-.00145736
23.5	.38779568	-.170197903		-.27493	
24.5	.41263609	1.362607767			
-37.5	-1.76478369	-.3183200	.330	-2	.0958628
10.5	.14892474	.4188086	.399	-145.842065398	-.000275055
11.5	.16372558	-.343302354		-.45789	
12.5	.17885336	1.401552230			
-37.5	-1.77047569	-.4448982	.177	-38	.0963235
-1.5	-.01623679	.4391816	.231	-145.283023731	.000500191
-.5	-.00303683	-.472627654		-.68493	
.5	.01015657	1.424216927			

Moreover, the deviations from the multiplet structure (4) are in fact exponentially small: K behaves like O(1) and agrees reasonably with the predicted values (5). At the same time, deformations of the sea strings (at s=1) are more considerable. They may probably be diminished, in the average, only owing to changes of their signs at the hole positions.

The higher-level Bethe ansatz (6)-(11) provides a rather good approximation. One sees from tables 1 and 2 that the quantity δ at large N approaches a constant (fluctuations are due to some drift of the holes). Hence, the finite-size energy correction is O(1/N). The leading asymptotics coefficient for the vacuum (previous numerical results:[10],s=1/2; [11],s=1; [9],s up to 9/2) well agrees with the value of the central charge in the conformal field theory [12]

$$\delta_v = (E-E_{\infty v})N \xrightarrow[N\to\infty]{} -\frac{1}{12}\pi^2 c, \quad c=3s/(s+1). \qquad (12)$$

For the excited states, δ differs from formula (12) and depends on the hole positions (tables 3 and 4). The comparison with the anomalous dimensions of the scaling operators [13] is, however, difficult because the considered states are too high-excited. The low-lying two-hole excitations [9] would be more appropriate, but there are problems for them either, due to the presence of logarithmic corrections [8,9,14] besides O(1/N). These corrections may furthermore depend on the sea-string deformations, to describe which there are still no efficient exact methods.

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Численный анализ неструнных решений уравнений анзатца Бете

Для интегрируемого XXX-антиферромагнетика Гейзенберга спина s=1/2 и 1 найдены численные решения уравнений анзатца Бете (sN до 150), содержащие неструнные конфигурации - мультиплеты. Результаты сравниваются с приближением вторичного анзатца Бете. Отличие точных энергий четырехдырочных мультиплетных состояний полного спина 0 от приближенных составляет O(1/N); коэффициент не такой, как для вакуума, и зависит от положений дырок. Как и ожидалось, мультиплеты имеют экспоненциальную точность, в то время как струны моря деформируются значительно сильнее.

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Bethe-Ansatz Solutions of a Non-String Type: Numerical Results

For the integrable XXX antiferromagnetic ring of N spins s=1 or s=1/2, the numerical solutions to the Bethe-ansatz equations are found, which involve non-string configurations, multiplets. The results up to sN=150 are compared with higher-level Bethe-ansatz predictions. The difference between the predicted and finite-N energies of the spin-zero states with a multiplet and four holes comprises O(1/N). The coefficient is not the same as for the vacuum and depends on positions of the holes. As has been expected, the multiplets are of an exponential accuracy in N, while sea strings are deformed much stronger.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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