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THE PARTICLE AND WAVE TRANSMISSION THROUGH A MEDIUM WITH A TWO-BAND DISPERSION LAW IN THE PRESENCE OF RANDOM POINT SCATTERERS

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1. Introduction

There is an extensive literature describing the scattering problem for the Schroedinger equation with a random potential (see e.g.[1],Chapter VII for the relevant review and references). This problem is, however, of a great interest for the Dirac equation as well. Reasons to study it are the following: first it is desirable to comprehend what complication or a new aspects may appear for a two-band spectrum, and second, there is a class of physical problems connected with the propagation of short random impulses through regular media which reduce to the Dirac equation (written in a moving coordinate system) with a random point interaction potential. In particular studying a nonlinear absorption of a stochastic acoustic signal by a superconductor, we arrive at an equation [2]

$$-iv(\sigma_z-\beta)\frac{d\psi}{dx} + \Delta\sigma_x\psi + \beta\,\varphi(x)\psi = E\psi\,, \qquad (1)$$

where $\varphi(x)$ is a random potential, v is the particle velocity in the laboratory coordinate system, s is the velocity of the reference frame connected with the acoustic wave (the sound velocity), $\beta = s/v$ and σ_x,σ_z are the Pauli matrices. Following [3] we would call (1) the tilted Dirac equation .We would moreover say that $\beta<1$ corresponds to the weakly tilted case and $\beta>1$ to the strongly tilted case,respectively.

In the one-dimensional disordered systems the scattering problem plays an important role. The corresponding scattering characteristics behave in a specific manner reflecting in such a way the well-known fact of state localization .For instance the transmission coefficient for the Schroedinger equation turns out to be exponentially small as a function of the length of the disordered segment. Moreover the scattering characteristics are directly related to some kinetic quantities of such systems. For instance the electrical conductivity of a disordered segment of length L is given by the well-known Landauer formula

$$G_L = \frac{e^2}{h}\,\frac{\langle T_1\rangle_F}{\langle R_1\rangle_F} \qquad (2)$$

while its thermal conductivity equals to

$$K_L = \frac{\pi^2 k_F T}{3h} \langle E^2 T_i \rangle . \qquad (3)$$

Here $\langle\ \rangle_{F,B}$ denotes the averaging over energy with the weight $-\frac{\partial n_{F,B}}{\partial E}$, where $n_{F,B}$ is the Fermi (Bose) function and T_L, R_L are the transmission and reflection coefficients of the segment respectively.(The derivation and discussion of those formulae can be found for instance in [1].) The absorption rate of the sound by a superconductor is determined , analogously to (2),(3) , by a formula [2]

$$Q = \frac{1}{4} N(E_F)\beta^2 \int_{-\infty}^{\infty} \frac{d\beta'}{\beta'^3} \int_{-\infty}^{\infty} dE\ R_L(E)(\varepsilon_+ - \varepsilon_-)[n_F(\varepsilon_+) - n_F(\varepsilon_-)] .$$

Here $N(E_F)$ is the density of states on the Fermi level and $\varepsilon_\pm$ are the dispersion laws for the unperturbed equation (1) (for $\beta\varphi(x)=0$).

The scattering and the corresponding spectral problem for the equation (1) with a Markov-type potential has been considered in Ref.3. The main results obtained in [3] are the following: in the weakly tilted case the mean transmission coefficient for the disordered barrier of the length L is exponentially small for large L. Hence all states of the infinite system are localized and its spectrum is pure point only. In other words in this case the structure of the solutions to the weakly tilted Dirac equation with a random potential qualitatively resembles the structure of solutions to the Schroedinger equation with a random potential (see for instance [1]). On the other hand, for the strongly tilted case the reflection is replaced by a partial transformation of waves between two scattering channels with a mean disbalance coefficient being exponentially small. For infinite system all the states are delocalized and the spectrum is absolutely continuous.

One of the most popular models of the random potential is a potential generated by point scatterers randomly distributed over the axis. This potential serves as a basis for many exactly solvable models of one-dimensional disordered systems. In a recent paper [4] for instance the probability density of the transmission

coefficients in the Schroedinger case was obtained explicitly for this potential in the high energy limit.

In the present paper we study the properties of the tilted Dirac equation (1) with a potential $\beta\varphi(x)$ generated by random point scatterers. As already mentioned this problem arises in particular when investigating the transmission of random sequences of extremely short impulses through a regular medium and is interesting from the mathematical as well as from the physical point of view[1]. This is partially connected with the fact that the point potentials which are for the Dirac equation more complicated that for the Schroedinger one remain still poorly understood. A particular attention is paid to the scattering problem. We compute the probability densities for the transmission and transformation coefficients in the high-energy limit in both the weakly and strongly tilted cases. It can be shown that the spectral properties of the problem on the whole line are fully analogous to those for the Markovian potential already discussed and we would not describe them here.

2. Statistical properties of the transmission coefficient in the weakly tilted case

In the weakly tilted case we write the solutions to the equation (1) on the right and on the left side of the disordered segment $(0, z_0)$ in the form

$$|\psi\rangle = \begin{cases} \psi_+(\quad)|u_+\rangle e^{ip_+z} + \psi_-(0)|u_-\rangle \quad {}^{ip_-z} , & z < 0 \\ \psi_+(z_0)|u_+\rangle e^{ip_+(z-z_0)} + \psi_-(z_0)|u_-\rangle e^{ip_-(z-z_0)} , & z_0 > 0 \end{cases} .$$

Then the transfer matrix T connecting these solutions

$$\begin{bmatrix} \psi_+(z_0) \\ \psi_-(z_0) \end{bmatrix} = T \begin{bmatrix} \psi_+(0) \\ \psi_-(0) \end{bmatrix}$$

1 For more extensive physical motivation see [2].

and fulfilling the current conservation law

$$|\psi_+(z)|^2 - |\psi_-(z)|^2 = \mathrm{const} \qquad (4)$$

looks as follows

$$T = \begin{bmatrix} \left(\frac{\gamma+1}{2}\right)^{1/2} \exp(i\varphi_\alpha) & \left(\frac{\gamma-1}{2}\right)^{1/2} \exp(i\varphi_\beta) \\ \left(\frac{\gamma-1}{2}\right)^{1/2} \exp i(\lambda-\varphi_\beta) & \left(\frac{\gamma-1}{2}\right)^{1/2} \exp i(\lambda-\varphi_\alpha) \end{bmatrix} . \qquad (5)$$

It is to be noted that in contrast to the Schroedinger and nontilted ($\beta=0$) Dirac equations the transfer matrix in the tilted case depends on four (rather then three) parameters. The point is that in the first two cases there exist a simple procedure that transforms a given solution into a linearly independent one. This transformation is given by a simple complex conjugation in the Schroedinger case, whereas for the non-tilted Dirac equation it is given by the charge conjugation (complex conjugation combined with a simultaneous transposition of the components). The existence of these operations reveals itself as additive constraints on the T-matrix elements, i.e., there are only three independent parameters left in these cases. There is, however, no such an operation for the tilted Dirac equation.

The point interaction potentials which can be naturally thought as a short-range limit of the potentials in (1) also conserve the current (the corresponding operator is self-adjoint ; see the Appendix). The T-matrix which describes the evolution of the vector $\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$ when passing through the scatterer is obviously parametrized in the same way. In order to distinguish the T-matrix corresponding to one point scatterer from the T-matrix describing the whole segment $(0,z_0)$ we would use an index 1 labelling all its elements.

The particle transmission through the disordered segment $(0,z_0)$ is described by the corresponding T-matrix, i.e. by its elements γ, φ_α, φ_β, and λ. The parameter γ is simply connected with the

transmission coefficient T

$$\gamma = 2/T - 1.$$

Therefore in order to find the probability density $w(\gamma|z_0)$ of $\gamma(z_0)$ it is enough to integrate the probability density

$$w(T|z) \equiv w(\gamma,\varphi_\alpha,\varphi_\beta,\lambda|z_0)$$

over the remaining variables $\varphi_\alpha,\varphi_\beta$, and λ:

$$w(\gamma|z) = \int w(T|z)\, d\varphi_\alpha\, d\varphi_\beta\, d\lambda .$$

Taking into account the current conservation law (4) it is natural to parametrize the vector $\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$ as follows

$$\psi_+ = \left[\frac{I-J}{2}\right] e^{i(\varkappa+\varphi)} \qquad \psi_- = \left[\frac{I+J}{2}\right] e^{i(\varkappa-\varphi)} .$$

Here $I(z) = |\psi_+(z)|^2 + |\psi_-(z)|^2$ is the wave intensity and $J(z) = |\psi_+(z)|^2 - |\psi_-(z)|^2 = \mathrm{const}.$ is the conserved current. The dynamics of the variables I,φ is separated and is described by the following equations (c.f. equation (7) in [4])

$$I(z_0) = \gamma I_0 + \sqrt{\gamma^2-1}\,\sqrt{I_0^2-J^2}\,\cos(\psi) \; ; \; \psi = 2\varphi_0+\varphi_\alpha-\varphi_\beta \qquad (6)$$

$$e^{2i\varphi(z_0)}\sqrt{I^2(z_0)-J^2} = e^{i(\varphi_\alpha+\varphi_\beta-\lambda)}\left\{(\gamma^2-1)^{1/2} I_0+(I_0^2-J^2)^{1/2}(\gamma\cos\psi+i\sin\psi)\right\},$$

where $I_0=I(0)$ and $\varphi_0= \varphi(0)$. It can be easily seen that the Jacobian $\frac{\mathcal{D}(I,\varphi)}{\mathcal{D}(I_0,\varphi_0)}$ equals to 1 and hence the phase volume $dId\varphi$ is conserved under the transformation (6).

Let us now introduce the probability density of the pair I,φ at point z under a fixed current J:

$$W_J(\Gamma|z)\, d\Gamma \; ; \; \Gamma = \{I,\varphi\},\; d\Gamma = dId\varphi .$$

Then W_J solves the integral equation

$$W_J(\Gamma|z) = \int \delta(\Gamma-\hat{T}\Gamma_0)\, w(T|z)\, W_J(\Gamma_0|0)\, dT\, d\Gamma_0 . \qquad (7)$$

Here $\hat{T}$ is an operator defined by the system (6) with $z = z_0$ and with the parameters $\gamma,\varphi_\alpha,\varphi_\beta,\lambda$ corresponding to the T-matrix of the segment $(0,z_0)$.

Let the disorder be caused by point impurities which are independently and uniformly distributed over the segment $(0,z_0)$ with a mean density n. Then the probability density $W_J(\Gamma|z)$ solves an equation which follows from (7) (see [1] and [4] for details)

$$\frac{\partial W}{\partial z} + \frac{p}{2}\frac{\partial W}{\partial \varphi} + n(W - \tilde{W}) = 0,$$

where

$$p = p_+(E) - p_-(E) \quad \text{and} \quad \tilde{W} = W(I_0(I,\varphi)|z). \qquad (8)$$

In the high-energy limit $p \longrightarrow \infty$ we can find the solution of this equation as a series in the powers of p^{-1}. The zero order approximation is φ-independent, i.e.,

$$W_J^{(0)} = W_J^{(0)}(I|z) \qquad (9)$$

and the non-trivial equation for it can be obtained from the condition of the existence of a 2π periodic solutions of first order approximation equations. It has the form

$$\frac{1}{n}\frac{\partial W_J^{(0)}(I|z)}{\partial z} = \frac{1}{2\pi}\int_0^{2\pi} W_J^{(0)}(I_0(I,\varphi)|z)\,d\varphi - W_J^{(0)}(I|z). \qquad (10)$$

On the other hand, using (6),(7) and an obvious condition $W_1^{(0)}(\gamma|0)=\delta(\gamma-1)$ it is simple to see that in the same approximation holds the following equation

$$w^{(0)}(\gamma|z) = W_1^{(0)}(\gamma|z).$$

Taking J=1 in (10) we can see that the sought probability density $w^{(0)}(\gamma|z)$ solves an integral equation

$$\frac{1}{n}\frac{\partial w^{(0)}(\gamma|z)}{\partial z} = \frac{1}{2\pi}\int_0^{2\pi} w^{(0)}\left[\gamma\gamma_1-((\gamma^2-1)(\gamma_1^2-1))^{1/2}\cos 2\varphi|z\right]d\varphi - w^{(0)}(\gamma|z). \qquad (11)$$

Here all the arguments of the unknown function are greater than one and it is therefore natural (like in the Schroedinger case [4]) to seek the solution in a form of an integral over the cone functions [5]

$$w^{(0)}(\gamma|z) = \int_0^{\infty} \tilde{w}^{(0)}(t|z)\,P_{-1/2+it}(\gamma)\,dt.$$

Inserting this representation into (11) and using the addition theorem for the Legendre functions we find

$$\frac{1}{n}\frac{\partial \tilde{w}^{(0)}(t|z)}{\partial z} = \left[P_{-1/2+it}(\gamma_1) - 1\right]\tilde{w}^{(0)}(t|z)$$

and finally

$$w^{(0)}(\gamma|z) = \int_0^{\infty} P_{-1/2+it}(\gamma)\, t\, \mathrm{th}(\pi t)\, \exp\{nz\,[P_{-1/2+it}(\gamma_1) - 1]\}dt.$$

This implies, in particular, that the inverse localization length, which is defined as a (self-averaged for $z \longrightarrow \infty$) decrement of the transmission coefficient on a given realization

$$l^{-1} = -\lim_{z\to\infty} z^{-1}\ln T(z) = -\lim_{z\to\infty} z^{-1}\langle \ln T(z)\rangle$$

can be in the high energy limit expressed using the density of scatterers and the transmission coefficient for one of them only.

$$l^{-1} = -n \ln T_1.$$

This formula coincides with the corresponding formula in the Schroedinger case. Thus the difference between the Schroedinger and Dirac cases is determined completely by the form of the energy dependence of the parameter γ which describes the scattering on a single point scatterer.

3. The point scatterers for the Dirac equation.

The general form of the T-matrix of the point scatterer in a weakly tilted Dirac equation is given by the formula (5). This is in a full agreement with the existence of a four parameter family of self-adjoint extensions of the Dirac operator which is the natural candidate for the point interaction Hamiltonians [6].

In order to obtain a concrete dependence of the T-matrix elements on the energy we consider first the equation (1) with a square well potential

$$\beta\varphi(x) = \begin{cases} U_0, & |x| < a \\ 0, & |x| > a. \end{cases} \qquad (12)$$

Taking the limit $U_0 \to \infty$, $a \to 0$, $2aU_0 \to k_0 = \text{const}$ the matrix T (5) corresponding to the potential (12) becomes equal to

$$T = e^{-2i\alpha_-}\left\{ I + \left[1 - e^{\frac{2ik_0}{v(1-\beta^2)}} \right] \begin{bmatrix} sh^2\theta_0 & , sh\,\theta_0\, ch\,\theta_0 \\ -sh\,\theta_0\, ch\,\theta_0 & , -ch^2\theta_0 \end{bmatrix} \right\}, \tag{13}$$

where

$$\alpha_{\pm} = \frac{k_0}{2v(1\pm\beta)}\ , \ cth\ 2\theta_0 = E/E_0\ ,\ E_0 = \Delta\ (1-\beta^2)^{1/2}.$$

The single scatterer reflection coefficient $R_1 = 1 - |T_{11}|^2$ equals to

$$R_1 = \left[1 + \left(sh^2 2\theta_0\ \sin^2 \frac{k_0}{v(1-\beta^2)} \right)^{-1} \right]^{-1} \tag{14}$$

and we get for the localization length

$$l^{-1} = n\ \ln \left\{ 1 + sh^2 2\theta_0 \sin^2 \frac{k_0}{v(1-\beta^2)} \right\}. \tag{15}$$

Using a natural definition of the integral

$$\int_{-\varepsilon}^{\varepsilon} \delta(x)\ \psi(x)\ dx = \frac{1}{2} \left[\psi(+0) + \psi(-0) \right]$$

the point perturbation described by the T-matrix (13) can be written down using a term

$$k_0 \delta(x) P\psi\ , \tag{16}$$

where P is a diagonal matrix of a form

$$P = \begin{bmatrix} \frac{tg\alpha_-}{\alpha_-} & , \ 0 \\ 0 & , \ \frac{tg\alpha_+}{\alpha_+} \end{bmatrix} . \tag{17}$$

In the case of a weak scattering ($\alpha_{\pm} \ll 1$) $P \rightarrow I$ and

$$R = \frac{k_0}{v^2(1-\beta^2)^2}\ sh^2 2\theta_0 \ .$$

In the limit case $n \rightarrow \infty$, $k_0 \rightarrow 0$ and $nk_0^2 \rightarrow 2d$ we get from (15) an expression for the localization length which corresponds to a white noise potential and which was previously published in [2,3]

$$l^{-1} = \frac{2d}{v^2(E-E_0)^2}\ \frac{E_0^2}{(1-\beta^2)^2} . \tag{18}$$

From (13),(14) it follows that in the case

$$\frac{k_0}{v(1-\beta^2)} = n\pi$$

the point scatterer becomes reflectionless ($R = 0$). The corresponding T-matrix equals to $\exp(-2i\alpha_-)I$ while the potential (12) is for $\alpha_{\pm} \neq \frac{(2n+1)\pi}{2}$ described by (16),(17). Let us stress that this fact is a property of the tilted Dirac equation. In the untilted case, $\beta=0$, the reflectionless potential corresponds to a T-matrix which is equal to $\pm$ I while the potential itself equals either zero or infinity, respectively. The inverse localization length (15) also equals to zero. Consequently we can construct a random potential which is made from the reflectionless scatterers and for which even in a typical "localized" weakly tilted case all the states will be delocalized. The state structure is extremely simple: only the phase of the wavefunction changes with the coordinate, being a random uniformly distributed quantity from the interval (0.2π) at each point x. Let us note that the mechanism of the above described reflectionless potential is completely different from the mechanism of the so-called Bargmann potentials [8].

The structure of the self-adjoint extensions for the Dirac equation was previously investigated in the work [6]. There were in particular distinguished two one parameter families which are in some sense analogous to the point scatterers of the type $\delta(x)$ and $\delta'(x)$ in the Schroedinger equation. In the case of a weakly tilted Dirac equation the T-matrices corresponding to these scatterers have (in the basis ψ_+,ψ_-) a form

$$T = I + \frac{i\alpha}{2}\ e^{\pm 2\theta_0} \begin{bmatrix} 1 & , \ \pm 1 \\ \mp 1 & , \ -1 \end{bmatrix} . \tag{19}$$

Here the upper and lower signs correspond to the scatterer of the type $\delta(x)$ and $\delta'(x)$, respectively. Their Schroedinger analogues are (in the same basis) the following:

$$T = I + \frac{i}{2}(k_0/k)^{\pm 1}\begin{bmatrix} 1 , & \pm 1 \\ \pm 1, & -1 \end{bmatrix}. \qquad (20)$$

Comparing now (19) with (20) and taking into account (13) we can see that in both cases the perturbations of the type $\delta(x)$ become infinitely strong for $E \to E_0$ $(k \to 0)$ while the perturbations of the type $\delta'(x)$ tend to zero. In the high energy limit, however, the behavior of these perturbations in the Schroedinger and tilted Dirac equations becomes quite different. In the Schroedinger case the perturbation of the δ type becomes infinitely weak while the δ' type becomes infinitely strong. On the other hand in the tilted Dirac equation both the perturbations, δ and δ',tend to a finite limit as $E \to \infty$ which equals

$$T_\infty = I + \frac{i}{2} v k_0 \begin{bmatrix} 1 , & \pm 1 \\ \mp 1, & -1 \end{bmatrix}.$$

One can introduce the δ-like potential directly also to the tilted Dirac equation. The wavefunctions become now discontinuous in the points $x=x_j$ where the scatterers are localized and the corresponding integral is defined as follows:

$$\int_{x_j-0}^{x_j+0} \delta(x-x_j)|\psi(x)\rangle dx = \frac{1}{2}\left[|\psi(x_j+0)\rangle+|\psi(x_j-0)\rangle\right].$$

The equation (1) which describes the system with scatterers of a δ type has a form

$$-iv(\sigma_z-\beta)\frac{d\psi}{dx} + \Delta\sigma_x \psi - v\sum_j k_0\, \delta(x-x_j)\, P_\beta\, \psi = E\,\psi , \qquad (21)$$

where projector P_β is given by

$$P_\beta = \begin{bmatrix} \frac{1-\beta}{2} , & \frac{(1-\beta^2)^{1/2}}{2} \\ \frac{(1-\beta^2)^{1/2}}{2} , & \frac{1+\beta}{2} \end{bmatrix}.$$

For the reflection coefficient corresponding to one point scatterer of δ type we find

$$R_1 = \frac{k_0^2}{4v^2}\,\frac{E+E_0}{E-E_0}\left[\frac{E+E_0}{2}\right]^2.$$

Hence the inverse localization length is equal for $k_0 << V$ to

$$l^{-1} = \frac{n k_0^2}{4V^2(E^2-E_0^2)}$$

and is in the $n \to \infty$, $k_0 \to 0$, $nk_0^2 \to 2d$ limit considerably different from the expression (18). This means that in the Dirac case the point potential in (21) analogous to the δ function in the Schroedinger equation in the sense that it gives the same T-matrix in the basis ψ_+,ψ_- and the point potential (16) obtained as a short range limit of square well potential (12) and generating the white noise potential correspond to two different families of self-adjoint extensions. In the Schroedinger case, however, these families coincide.

4.Statistical properties of the transformation coefficient in the strongly tilted case.

In the strongly tilted case both the free solutions propagate in one direction only. The solutions of the equation (1) on the left and on the right side of the disordered segment $(0,z_0)$ respectively are connected by the transmission matrix T

$$\begin{bmatrix} \psi_+(0) \\ \psi_-(0) \end{bmatrix} = T \begin{bmatrix} \psi_+(z_0) \\ \psi_-(z_0) \end{bmatrix}$$

which has a form

$$T = \begin{bmatrix} \left(\frac{1+\gamma}{2}\right)^{1/2}\exp(i\varphi_\alpha) & \left(\frac{1-\gamma}{2}\right)^{1/2}\exp(i\varphi_\beta) \\ -\left(\frac{1-\gamma}{2}\right)^{1/2}\exp(i(\lambda-\varphi_\beta)) & \left(\frac{1+\gamma}{2}\right)^{1/2}\exp(i(\lambda-\varphi_\alpha)) \end{bmatrix}. \qquad (22)$$

In such a way a wave of the first type which is incident on the right end of the disordered segment$(\psi_-(z_0) = 0)$ is partially transformed into the wave of the second type. The difference of the squared moduli of the free solutions on the right is proportional to the quantity γ. The transformation is absent for $\gamma=1$ while for $\gamma=-1$ the solutions of the first type are completely transformed into the solutions of the second type. It is therefore natural to call γ the disbalance coefficient of these transformation.

Analogously as in the chapter 2 we introduce a probability density $w(\gamma,\varphi_\alpha,\varphi_\beta,\lambda|z)$ which being integrated over the variables φ_α, φ_β, λ gives the probability density of the disbalance coefficient. Since in the weakly tilted case the conserved quantity is given by the intensity $|\psi_+(z)|^2+|\psi_-(z)|^2$ = I= const. and not by the current $J(z) = |\psi_+(z)|^2-|\psi_-(z)|^2$, it is natural to parametrize the vector (ψ_+,ψ_-) as

$$\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} \left(\frac{I_0+J}{2}\right)^{1/2} e^{i(\varkappa+\varphi)} \\ \left(\frac{I_0-J}{2}\right)^{1/2} e^{i(\varkappa-\varphi)} \end{bmatrix} .$$

The dynamics of the pair (J,φ) is separated and is determined by

$$J(z) = \gamma J_0 + \left[(1-\gamma^2)(I^2-J_0^2)\right]^{1/2} \cos\psi, \quad \psi = 2\varphi_0+\varphi_\alpha-\varphi_\beta \tag{23}$$

$$\left[I^2-J(z)^2\right]^{1/2} e^{2i\varphi(z)} = e^{i(\varphi_\alpha+\varphi_\beta-\lambda)}\left\{-J_0\left[1-\varkappa^2\right]^{1/2}+\left[I^2-J_0^2\right](\gamma\cos\psi+i\sin\psi)\right\},$$

The Jacobian $\frac{\mathcal{D}(J,\varphi)}{\mathcal{D}(J_0,\varphi_0)}$ is again equal to one and the phase space measure is therefore conserved.

Let us introduce further the probability density of the pair (J,φ) at a point $y=z_0-z$ which corresponds to a fixed "intensity" I

$$W_I(\Gamma|y)\ d\Gamma\ ,\ \Gamma = \{J,\varphi\},\ d\Gamma = dJ\ d\varphi\ .$$

Then W solves an integral equation

$$W_I(\Gamma|y) = \int\delta(\Gamma-T\Gamma_0)w(T|y)W_I(\Gamma_0|0)\ dT\ d\Gamma_0\ , \tag{24}$$

where T is an operator given by the system (23).

If the disorder is caused by independent uniformly distributed scatterers localized on the segment $(0,z_0)$ with a density n then the probability density $W_I(\Gamma|y)$ solves an equation

$$\frac{\partial W}{\partial y} + \frac{p}{2}\frac{\partial W}{\partial\varphi} + n(W - \tilde{W}) = 0,$$

where (analogously to (8))

$$p = p_- - p_+\ ,\ \tilde{W}_I = W_I(J_0(J,\varphi)|y).$$

The equations and solutions of the zero-order approximation in the parameter 1/p are in the high-energy limit obtained from (9) and (10) changing J $\longleftrightarrow$ I and z $\longleftrightarrow$ y respectively and taking $J_0(J,\varphi)$ from (23). Moreover we get from (23),(24) and from the condition

$$W_I^{(0)}(J|0) = \delta(J-1)$$

that in the zero-order approximation it holds

$$w^{(0)}(\gamma|y) = W_1^{(0)}(\gamma|y).$$

Therefore the integral equation for $w^{(0)}(\gamma|z)$ acquires a form

$$\frac{\partial w^{(0)}(\gamma|y)}{\partial y} = \frac{n}{2\pi}\int_0^{2\pi} w^{(0)}\left[\gamma\gamma_1-\left[(1-\gamma^2)(1-\gamma_1^2)\right]^{1/2}\cos2\varphi\,\middle|\,y\right]d\varphi - n\, w^{(0)}(\gamma|z). \tag{25}$$

All the arguments of the unknown function are less than one. It is therefore natural to seek the solution as a series in the Legendre polynomials:

$$w^{(0)}(\gamma|y) = \sum_{k=0}^{\infty} w_k^{(0)}(z)P_k(\gamma).$$

Inserting this series into (25) and using the addition formula for the Legendre polynomials we find

$$\frac{1}{n}\frac{\partial w_k^{(0)}}{\partial z} = [P_k(\gamma_1)-1]w_k^{(0)}$$

and finally

$$w^{(0)}(\gamma|z) = \sum_{k=0}^{\infty}\frac{2k+2}{2}P_k(\gamma)\ e^{nz(P_k(\gamma_1)-1)}. \tag{26}$$

From this expression it follows that the mean value of the disbalance coefficient is exponentially small for large z.

$$\langle\gamma\rangle = e^{-2z/l},\ l = \frac{2}{n[1-P_1(\gamma_1)]}.$$

It is therefore natural to call l the mixing length. In the weak scattering case, $\gamma_1=1-2R_1$, $R_1 \ll 1$ the mixing length equals to

$$l = (nR_1)^{-1}$$

and the probability density (26) is transformed into

$$w^{(0)}(\gamma|z) = \sum^{\infty} \frac{2k+1}{2} P_k(\gamma)\, e^{-k(k+1)z/l},$$

Approximating the point potential by a square well potential with a shrinking support we find an explicit form of the T-matrix

$$T = e^{-2i\alpha_-}\left\{I + \left[\exp\left(\frac{2ik_0}{v(\beta^2-1)}\right) - 1\right]\begin{bmatrix}\cos^2\theta_0 & ,\sin\theta_0\cos\theta_0 \\ \sin\theta_0\cos\theta_0, & \sin^2\theta_0\end{bmatrix}\right\},$$

where $\alpha_\pm = \frac{k_0}{2v(\beta\pm1)}$ and $\mathrm{ctg}2\theta_0 = -\frac{E}{\Delta(\beta^2-1)^{1/2}}$.

For the transformation coefficient we get

$$R_1 = |T_{12}|^2 = \sin^2 2\theta_0 \sin^2 \frac{k_0}{v(\beta^2-1)} \qquad (27)$$

which yields in the weak transformation limit an expression for the mixing length

$$l^{-1} = \frac{nk_0^2}{2}\frac{\sin^2 2\theta_0}{2}$$

coinciding with those obtained originally in [2,3]. For

$$\frac{k_0}{v(\beta^2-1)} = n\pi$$

the transformationless transition $R_1 = 0$ (27) through the particular point scatterer occurs.

Let us note at the conclusion that for a sound signal which is the sequence of extremely short (point) pulses just as for the white noise signal [2] the rate of nonlinear absorption by the superconductor is essentially higher that that of a periodic signal of a finite length and is practically independent on the signal root mean square amplitude.

Appendix

Here we make comments on some mathematical questions concerning the point interaction in the tilted Dirac equation. Namely we construct the self-adjoint operators (Hamiltonians) which correspond to the four-parameter family of T matrices (5) and show that they represent short-range limits of the operators corresponding to (1) with local potentials φ.

A. Construction of the Hamiltonians

Let us start with a free tilted Dirac operator which is defined on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})\oplus\mathbb{C}^2$ by the differential expression

$$H = -iv(\sigma_z-\beta)\frac{d}{dx} + \Delta\sigma_x.$$

It is simple to prove that H with $D(H) = \{ f \in \mathcal{H}\ ;\ f \in AC(\mathbb{R})\oplus\mathbb{C}^2, Hf \in \mathcal{H} \}$ is a self adjoint operator for all $\beta \in \mathbb{R}$. In order to construct the Hamiltonian leading to the T-matrices (5) we follow the standard procedure which is used when dealing with potentials supported on sets with a measure zero [7]. The first step is to remove the interaction point by defining a restricted operator H_0

$$H_0 = H \mid C_0^\infty(\mathbb{R}\backslash\{0\}) \oplus \mathbb{C}^2.$$

The operator H_0 is symmetric but it is not self-adjoint. A direct inspection shows that it has for $\beta \neq \pm1$ deficiency indices equal to (2,2). The second step is to construct all self-adjoint extension of H_0 which become then the candidates for the possible Hamiltonian of the system. Because the deficiency indices of H_0 equal (2,2) we have for $\beta \neq \pm1$ a four parameter family of such Hamiltonians. It is not difficult to show that there is a one-to-one correspondence between these Hamiltonians and the transfer matrices (5). In order to illustrate it we restrict ourselves to the one parameter subfamily (19) only. As already mentioned this family is in the case of the δ-type scatterers described formally by the equation (21). Hence in order to find the corresponding family of Hamiltonians it is enough to find a self-adjoint realization of the heuristic expression

$$h = -iv(\sigma_z-\beta)\frac{d}{dx} + \Delta\sigma_x - v\lambda\,\delta(x)\,P_\beta.$$

Theorem 1: The self-adjoint realization of the operator h_λ is for $\beta \neq \pm 1$ given by

$$H_\lambda = -iv(\sigma_z-\beta)\frac{d}{dx} + \Delta\sigma_x$$

$$D(H_\lambda) = \{ f = (f_1,f_2) \in \mathcal{H},\ f_i \in AC(\mathbb{R}\setminus\{0\}),\ i=1,2 \text{ and}$$

$$z_1 f_1(0_-) + \bar{z}_1 f_1(0_+) = -\frac{\lambda}{2}(1-\beta^2)^{1/2}[f_2(0_-) + f_2(0_+)]$$

$$z_2 f_2(0_-) + \bar{z}_2 f_2(0_+) = -\frac{\lambda}{2}(1-\beta^2)^{1/2}[f_1(0_-) + f_1(0_+)]$$

with

$$z_1 = \left[\frac{\lambda}{2} - iv\right](1-\beta),\quad z_2 = \left[\frac{\lambda}{2} + iv\right](1+\beta) \quad \}.$$

Proof: Let us first prove that the operator H_λ defined by the indicated boundary conditions at the origin represents actually a self adjoint extension of H_0. We know from the general theory [7,8] that any self adjoint extension of H_0 is determined by two independent boundary conditions at 0. It is therefore enough to show that the used boundary conditions nullify the boundary form

$$b(f,g) = (f, H_0^* g) - (g, H_0^* f)$$

(here (,) denotes the scalar product in $\mathcal{H}$). A simple calculation leads to

$$b(f,f) = iv(1-\beta)\left[|f_1(0_+)|^2 - |f_1(0_-)|^2\right] + iv(1+\beta)\left[|f_2(0_-)|^2 - |f_2(0_+)|^2\right]$$

and the rest is verified by inserting the boundary conditions directly into b(,). The operator H_λ is hence self-adjoint. In order to show that it represents the realization of the heuristic operator h_λ we use the formula

$$\int_{x_j-0}^{x_j+0} \delta(x-x_j)|\psi(x)\rangle dx = \frac{1}{2}\left(|\psi(x_j+0)\rangle + |\psi(x_j-0)\rangle \right)$$

and insert it into the equation

$$-iv(\sigma_z-\beta)\frac{d\psi}{dx} + \Delta\sigma_x\psi - v\lambda\,\delta(x)\,P_\beta\psi = E\,\psi.$$

An integration by parts leads then to the boundary conditions which determine $D(H_\lambda)$.

B. The short-range approximation

We show now that the point interaction Hamiltonians which we obtained applying the von Neumann's theory represent a short-range limit of Hamiltonians with local potential. We will, however, not discuss here the general case because the corresponding proof is very long and technical. For this reason we restrict ourselves to a particular one-parameter family of Hamiltonians, which corresponds to the class of heuristic operators

$$h_\alpha = -i(\sigma_z-\beta)\frac{d}{dx} + \Delta\sigma_x + \alpha\begin{pmatrix} 1; & -1 \\ -1; & 1 \end{pmatrix}\delta(x).$$

We show that h_α are short-range limits of Hamiltonians of the type

$$H_\varepsilon = -i(\sigma_z-\beta)\frac{d}{dx} + \Delta\sigma_x + \frac{1}{\varepsilon}\begin{pmatrix} 1; & -1 \\ -1; & 1 \end{pmatrix} V(x/\varepsilon)$$

with V being a smooth short-range potential

$$V \in C_0^\infty(\mathbb{R}).$$

Let us now pass to the precise mathematical formulations of the above statements. We introduce a one parameter family of self-adjoint operators H_α which represents a self-adjoint realization of the heuristic operators h_α :

$$H_\alpha = -i(\sigma_z-\beta)\frac{d}{dx} + \Delta\sigma_x$$

$$D(H_\alpha) = \{ f=(f_1,f_2) \in \mathcal{H},\ f_i \in AC(\mathbb{R}\setminus\{0\});\ i=1,2 \text{ and}$$

$$z_1 f_1(0_-) + \bar{z}_1 f_1(0_+) = \frac{\alpha}{2}(f_2(0_-) + f_2(0_+))$$

$$z_2 f_2(0_-) + \bar{z}_2 f_2(0_+) = \frac{\alpha}{2}(f_1(0_-) + f_1(0_+))$$

with

$$z_1 = \frac{\alpha}{2} + i(\beta-1) \ ; \quad z_2 = \frac{\alpha}{2} + i(\beta+1) \quad \}$$

(In order to show that H_α actually represents the self-adjoint realization of h_α one has to follow step by step the proof of the Theorem 1.)

Theorem 2:

$$H_\alpha = \underset{\varepsilon \to 0}{\text{N R lim}}\ H_\varepsilon$$

with α given by

$$\alpha = - \frac{1}{2(1+\beta)} \left(u,\ (1+T)^{-1} v \right)_{L^2(\mathbb{R})}, \qquad \text{(A.1)}$$

where $u(x) = |V(x)|^{1/2}$, $v(x) = |V(x)|^{1/2}\text{sgn}(V(x))$ and T is an Hilbert-Schmidt operator on $L^2(\mathbb{R})$ with a kernel

$$T(x,y) = \frac{i\beta}{2(1-\beta^2)}\, v(x)\ \text{sgn}(x-y)\ u(y)$$

(N.R.lim means the limit in the norm-resolvent topology and $(\ ;\ )_{L^2(\mathbb{R})}$ denotes the scalar product in $L^2(\mathbb{R})$.

The proof of this Theorem is rather long and we split it therefore into several lemmas. Our first step will be to transform the operators H_α and H_ε into a more convenient form.

Lemma 1: The operators H_α and H_ε are unitary equivalent to

$$\tilde{H}_\alpha = -i(\sigma_x - \beta)\frac{d}{dx} - \Delta\sigma_z$$

$$D(\tilde{H}_\alpha) = \{\ f=(f_1,f_2) \in \mathcal{H},\ f_i \in AC(\mathbb{R}\backslash\{0\});\ i=1,2 \text{ and}$$
$$f_1(0_-) - f_1(0_+) = \beta\,(f_2(0_-) - f_2(0_+))$$
$$f_2(0_-) - \bar{z}_2 f_2(0_+) = \frac{i\alpha}{2}\,(f_1(0_-) + f_1(0_+))\quad \}$$

and

$$\tilde{H}_\varepsilon = -i(\sigma_y - \beta)\frac{d}{dx} - \Delta\sigma_z + \frac{2}{\varepsilon}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V(x/\varepsilon),$$

respectively. The unitary mapping is in both cases given by a constant matrix

$$U = (1/2)^{1/2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Proof: Direct verification

From this Lemma it follows that in order to prove the Theorem 2 it

is enough to prove that

$$\tilde{H}_\alpha = \underset{\varepsilon \to 0}{\text{N R lim}}\ \tilde{H}_\varepsilon$$

with α given by the expression (A.1). From now we will work only with the operators $\tilde{H}_\alpha$, $\tilde{H}_\varepsilon$, ... We however drop the tilde in all expressions.

Our next step will be to pass from the operators H_α, H_ε to the corresponding resolvents and use the Krein's formula [7]. For this purpose the following Lemma turns to be very helpful.

Lemma 2: The family H_α ; $\alpha \in \mathbb{R} \cup \{\infty\}$ exhausts all self-adjoint extensions of the operator

$$H_0 = -i(\sigma_x - \beta)\frac{d}{dx} - \Delta\sigma_z$$

$$D(H\ ) = \{\ f=(f\ ,f\ ) \in \mathcal{H},\ f\ \in AC(\mathbb{R});\ i=1,2 \text{ and } f\ (0) = 0\ \}.$$

Proof: The operator H_0 has deficiency indices equal to (1,1). This can be easily seen when we evaluate the corresponding adjoint operator H_0^* which is determined by

$$D(H_0^*) = \{\ f=(f_1,f_2) \in \mathcal{H},\ f_i \in AC(\mathbb{R}\backslash\{0\});\ i=1,2 \text{ and}$$
$$f_1(0_-) - f_1(0_+) = \beta\,(f_2(0_-) - f_2(0_+))\ \}.$$

Hence all the functions from $D(H_0^*)$ are already constrained by one boundary condition at 0, which yields that the deficiency indices of H_0 must be equal to (1,1). (note that $D(H_0^*) \neq D(H_0)$ and the indices are therefore not zero) Consequently H_0 has exactly one parameter family of self-adjoint extensions. This family must, however, coincide with H_α because $D(H_\alpha) \subset D(H_0^*)$ for all $\alpha \in \mathbb{R}$.

Knowing that H_α represent self-adjoint extension of H_0 we can apply the Krein's formula and obtain the corresponding resolvent. In order to avoid complicated expressions we restrict ourselves to the case $\beta < 1$ and we calculate the resolvent only for the spectral parameter equal to zero.

Lemma 3:

$$H_\alpha^{-1} = H^{-1} + \frac{\alpha}{2(1-\beta^2)(1-\beta-\alpha\sqrt{1-\beta^2})} |f_1 \rangle\langle f_2|$$

where $|f_1 \rangle\langle f_2|$ is a rank-one operator

$$|f_1 \rangle\langle f_2| : f \longrightarrow f_1 (f_2;f)$$

with

$$f_1(x) = \begin{bmatrix} \beta\mathrm{sgn}(x) - \sqrt{\beta^2-1} \\ \mathrm{sgn}(x) \end{bmatrix} \exp\left[- \frac{\Delta|x|}{(1-\beta^2)^{1/2}}\right]$$

$$f_2(x) = \begin{bmatrix} \beta\mathrm{sgn}(x) + \sqrt{\beta^2-1} \\ \mathrm{sgn}(x) \end{bmatrix} \exp\left[- \frac{\Delta|x|}{(1-\beta^2)^{1/2}}\right]$$

H^{-1} is an integral operator with the kernel

$H^{-1}(x,y) =$

$$= \frac{i}{2(1-\beta^2)} \begin{bmatrix} \beta\mathrm{sgn}(x-y) - \sqrt{\beta^2-1} \;; & \mathrm{sgn}(x-y) \\ \mathrm{sgn}(x-y) & ;\ \beta\mathrm{sgn}(x-y) + \sqrt{\beta^2-1} \end{bmatrix} \exp\left[- \frac{\Delta|x-y|}{(1-\beta^2)^{1/2}}\right]$$

Proof: Direct evaluation shows that for all $\varphi \in \mathscr{H}$

$$H_\alpha^{-1}\varphi \in D(H_\alpha)$$

and

$$H_\alpha H_\alpha^{-1}\varphi = \varphi .$$

Proof of the Theorem 2: We suppose that $\beta < 1$ and that $V \geq 0$. In order to indicate explicitly the dependence of the resolvent on the parameter Δ we will write $H^{-1}(\Delta)$, $H_\alpha^{-1}(\Delta)$ and $H_\varepsilon^{-1}(\Delta)$ instead of H^{-1}, H_α^{-1} and H_ε^{-1}. Introducing the scaling group

$$U_\varepsilon : f(x) \longrightarrow \varepsilon^{-1/2} f(x/\varepsilon)$$

we find for $H_\varepsilon^{-1}(\Delta)$

$$H_\varepsilon^{-1}(\Delta) = H^{-1}(\Delta) - \frac{1}{\varepsilon} H^{-1}(\Delta) U_\varepsilon v \mathbb{P}(1 + v\mathbb{P}H^{-1}(\varepsilon\Delta)\mathbb{P}v)^{-1}\mathbb{P}vU_\varepsilon^{-1}H^{-1}(\Delta) =$$

$$= H^{-1}(\Delta) - A_\varepsilon(1 + B_\varepsilon)^{-1}C_\varepsilon ,$$

where $v(x) = V(x)^{1/2}$, $\mathbb{P} = \begin{bmatrix} 1 & , & 0 \\ 0 & , & 0 \end{bmatrix}$ and A_ε, B_ε, C_ε are

Hilbert-Schmidt operators with kernels

$$A_\varepsilon(x,y) = H^{-1}(\Delta,x,\varepsilon y)\mathbb{P}v(y)$$

$$B_\varepsilon(x,y) = v(x)\mathbb{P}H^{-1}(\varepsilon\Delta,x,y)\mathbb{P}v(y)$$

$$C_\varepsilon(x,y) = v(x)\mathbb{P}H^{-1}(\Delta,\varepsilon x,y)$$

($H^{-1}(\Delta,x,y)$ denotes the kernel of $H^{-1}(\Delta)$, c.f. Lemma 3.).
Taking now the $\varepsilon \longrightarrow 0$ limit we find that the operators $A_\varepsilon, B_\varepsilon, C_\varepsilon$ converge in norm to A,B,C, where A and C are rank-one operators with kernels

$$A(x,y) = \frac{i}{2(1-\beta^2)} \begin{bmatrix} \beta\mathrm{sgn}(x) - \sqrt{\beta^2-1} & ;0 \\ \mathrm{sgn}(x) & ;0 \end{bmatrix} \exp\left[\frac{\Delta|x|}{(1-\beta^2)^{1/2}}\right] \mathbb{P}\, v(y)$$

$$C(x,y) = \frac{iv(x)}{2(1-\beta^2)} \mathbb{P} \begin{bmatrix} \beta\mathrm{sgn}(x) - \sqrt{\beta^2-1} ; \mathrm{sgn}(x\ ) \\ 0 \quad ;\ 0 \end{bmatrix} \exp\left[- \frac{\Delta|x|}{(1-\beta^2)^{1/2}}\right]$$

and the operator B is described by

$$B(x,y) = \frac{i}{2(1-\beta^2)} \begin{bmatrix} \beta v(x)\mathrm{sgn}(x-y)v(y) - v(x)v(y)\sqrt{\beta^2-1}; & 0 \\ 0 & ;\ 0 \end{bmatrix} .$$

We obtained in such a way that the uniform limit of H_ε^{-1} exists and is equal to

$$\text{n-lim}_{\varepsilon \to 0} H_\varepsilon^{-1} = H^{-1}(\Delta) - A\ (1 + B\ )^{-1}C\ . \qquad \text{(A.2)}$$

The last step in the proof is to invert the operator (1+B). This can be, however, easily done if we split the operator B into two parts

$$B = B_1 + T$$

with B_1 being a rank-one operator

$$B_1 = \frac{1}{2(1-\beta^2)^{1/2}} \begin{bmatrix} v(x)v(y) & , & 0 \\ 0 & , & 0 \end{bmatrix} .$$

We get then

$$(1+B)^{-1} = [1 + (1+T)^{-1}B_1]^{-1}(1+T)^{-1}$$

(Note that a rank-one operator can be inverted explicitly.) Inserting this formula into (A.2) and comparing the result with the formula for H_α^{-1} we obtain the assertion of the Theorem.

The coupling constant α is a quite irregular function of the potential V. In order to illustrate this peculiarity we multiply the potential V by a constant λ. In such a way α becomes a function of λ

$$\alpha(\lambda)=-\frac{\lambda}{2(1+\beta)}(u,(1+\lambda T)^{-1}v)_{L^2(\mathbb{R})} .$$

The function $\alpha(\lambda)$ is, however, not smooth. It has singularities which are localized at points λ_k for which $-1/\lambda_k$ is an eigenvalue of the operator T. On the other hand T is Hilbert-Schmidt and the behaviour of its eigenvalues is very well known [9]. Applying results from [9] we get the following assertion:

Let $N(\Lambda)$ be the number of singularities of the function $\alpha(\lambda)$ which are localized in an interval $[0, \Lambda]$:

$$N(\lambda) = \# \{ \lambda \in [0,\Lambda];\ \alpha^{-1}(\lambda) = 0 \} .$$

Then

$$\lim_{\Lambda \to \infty} \frac{N(\Lambda)}{\Lambda} = \frac{\beta}{2\pi(1-\beta^2)} \int_{\mathbb{R}} V(x)\, dx .$$

To illustrate the behaviour of $\alpha(\lambda)$ in more detail we choose $V(x)$ in a square well form

$$V(x) = \begin{cases} 0 & ;\ x \notin [0,1] \\ V_0 & ;\ x \in [0,1] \end{cases}$$

and decompose $\alpha(\lambda)$ with respect to λ

$$\alpha(\lambda) = -\frac{1}{2(1+\beta)} [\ \lambda(u,v) - \lambda^2(u,Tv) + \dots + (-1)^n\lambda^{n+1}(u,T^n v) + \dots \quad (A.3)$$

For $(u,T^n v)$ we get

$$(u,T^{2n+1}v) = 0 \text{ for all } n$$

$$(u,T^{2n}v) = (-1)^n V_0^{2n+1}\left[\frac{\beta}{2(1-\beta^2)}\right]^{2n} \int_0^1\int_0^1 \dots \int_0^1 \operatorname{sgn}(x_1-x_2)\operatorname{sgn}(x_2-x_3)\cdot \dots \operatorname{sgn}(x_{2n}-x_{2n+1})\, dx_1 dx_2 \dots dx_{2n+1} .$$

Noting that

$$\int_0^1\int_0^1 \dots \int_0^1 \operatorname{sgn}(x_1-x_2)\operatorname{sgn}(x_2-x_3)\cdot \dots \operatorname{sgn}(x_{2n}-x_{2n+1})\, dx_1 dx_2 \dots dx_{2n+1} =$$

$$= \frac{2^{2(n+1)}(2^{2(n+1)}-1)}{(2n+2)!} B_{2n+2} ,$$

where B_{2n} are the Bernoulli numbers and inserting this result into (A.3) we get finally

$$\alpha(\lambda) = \frac{1-\beta}{\beta}\ \mathrm{tg}\left[\frac{\lambda V_0 \beta}{2(1-\beta^2)}\right]$$

which is in a good correspondence with the result obtained for the short-range approximation of the transmission matrix (13).

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