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FINITE-SIZE SCALING FOR THE MEAN SPHERICAL MODEL WITH INVERSE POWER LAW INTERACTION

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1. INTRODUCTION

Recently Singh and Pathria [1], and Shapiro and Rudnick [2] have developed systematic approaches to the derivation of critical finite-size scaling properties of the fully finite spherical model with nearest neighbour interactions. It seems desirable to extend the existing mathematical tools to deal with the case of an arbitrary ferromagnetic pair interaction potential with an inverse power law decay at large distances r of the form $r^{-d-\sigma}$, where d is the dimensionality of the system and $\sigma > 0$ is a parameter. The thermodynamic properties of the spherical model with such an interaction were studied by Joyce [3] (see also [4]) who found the critical indices to be σ-dependent in dimensionalities $d \in (\sigma, 2\sigma)$. It would be interesting to obtain the σ-dependence of the finite-size scaling functions, as well.

The first major step towards the development of systematic finite-size technique applicable to the mean spherical model with an arbitrary inverse power law ferromagnetic interaction was made in [5], where the finite size scaling form of the equation for the spherical field was derived. In the present work, a new analytical technique is suggested, which allows one to handle the finite-size corrections to the free energy density as well.

The main idea consists in the replacement of the two mathematical identities

$$\ln(1 + z) = \int_0^\infty dx(1 - e^{-zx}) \frac{e^{-x}}{x} \qquad (1.1)$$

and

$$(1 + z)^{-1} = \int_0^\infty dx\, e^{-zx} e^{-x} \qquad (1.2)$$

with $\mathrm{Re}\, z > -1$, used in the method of Singh and Pathria [1], by more general identities (see Appendix A)

$$\ln(1+z^{\alpha}) = \int_0^{\infty} dx(1-e^{-zx}) \frac{G_{\alpha}(x)}{x} \qquad (1.3)$$

and

$$(1+z^{\alpha})^{-1} = \int_0^{\infty} dx\, e^{-zx} F_{\alpha}(x), \qquad (1.4)$$

respectively, where

$$G_{\alpha}(x) = \alpha E_{\alpha}(-x^{\alpha}), \qquad (1.5a)$$

$$F_{\alpha}(x) = x^{\alpha-1} E_{\alpha,\alpha}(-x^{\alpha}), \quad \alpha > 0. \qquad (1.5b)$$

Here $E_{\alpha}(z) = E_{\alpha,1}(z)$ and $E_{\alpha,\beta}(z)$ is the entire function of the Mittag-Leffler type[6] defined by the power series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \quad (\alpha > 0). \qquad (1.6)$$

The use of identities (1.3) and (1.4) with $\alpha=\sigma/2$ and

$$z = y^{-2}(n_1^2 + \dots + n_d^2) \equiv y^{-2}|\vec{n}|^2, \qquad (1.7)$$

where

$$y = \left(\frac{\phi}{\rho_{\sigma}}\right)^{1/\sigma} \frac{N_0}{2\pi}, \quad 0 < \sigma \le 2, \qquad (1.8)$$

allows one to easily calculate d-dimensional Fourier transforms of the summands in expressions like

$$U_{d,\sigma}^{(N)}(\phi) = \ln(\phi/\rho_{\sigma}) + N_0^{-d} \sum_{\vec{n} \in S_{N,d}} \ln[1 + (|\vec{n}|y^{-1})^{\sigma}] \qquad (1.9)$$

and

$$W_{d,\sigma}^{(N)}(\phi) = \phi^{-1} N_0^{-d} \sum_{\vec{n} \in S_{N,d}} [1 + (|\vec{n}|y^{-1})^{\sigma}]^{-1}, \qquad (1.10)$$

where the summation is carried over the set (N_0 odd integer)

$$S_{N,d} = \{-\frac{N_0-1}{2}, \dots, 0, \dots, \frac{N_0-1}{2}\}^d. \qquad (1.11)$$

The further asymptotic analysis of sums (1.9) and (1.10), which enter into the expressions for the free energy density and the equation for the spherical field (ϕ is a linear function of the latter, see Eq.(2.9) below), respectively, may be accomplished with the aid of the Poisson summation formula[1] and the Ewald summation technique[2] in a complete analogy with the case of nearest neighbour interactions, which formally corresponds to $\sigma = 2$.

In the present paper it is found convenient to expound the new method in close parallel to the approach of Singh and Pathria[1]. This is expected to facilitate the extension of other available at $\sigma = 2$ results to the case of $0 < \sigma < 2$.

In Section 2, the notation used in the description of the model is introduced and basic expressions for the free energy per particle and the equation for the spherical field are given. The method of derivation of the asymptotic form of sums (1.9) and (1.10) when $N_0 \to \infty$ and $\phi \to 0$ so that ϕN_0^{σ} remains constant, is expounded in Section 3. Section 4 contains a derivation of a new finite-size scaling form of the equation for the spherical field. The main result on the finite-size scaling function for the free energy per particle is obtained in Section 5. Some mathematical aspects of the suggested technique and new consequences of the general results obtained here are mentioned in the discussion, Section 6. The proofs of identities (1.3) and (1.4), and some other necessary properties of the Mittag-Leffler type functions are given in Appendix A. Appendix B contains a brief re-derivation of a different representation of the equation for the spherical field obtained first in[5] by a Laplace transformation technique.

2. THE MODEL

We consider the ferromagnetic mean spherical model[7] in a fully finite hypercubic geometry with periodic boundary conditions. The model Hamiltonian is defined on a d-dimensional torus $\mathbf{T}_{N,d} = \{1, \dots, N_0\}^d$ of $N = N_0^d$ sites where N_0 is an odd integer, and has the form

$$\mathcal{H}_N(\{\sigma_r\}) = -\frac{1}{2}\sum_{\vec{r},\vec{r}'} J(|\vec{r}-\vec{r}'|)\sigma_{\vec{r}}\sigma_{\vec{r}'} - H\sum_{\vec{r}}\sigma_{\vec{r}}, \qquad (2.1)$$

where $\vec{r},\vec{r}' \in \mathbf{T}_{N,d}$, $\sigma \in \mathbf{R}^1$ is the spin variable at site $\vec{r}$, $J(|\ell|)$ is the pair interaction potential, $H \in \mathbf{R}^1$ is an external magnetic field.

The partition function of the Gaussian model with Hamiltonian (2.1), in the canonical Gibbs ensemble with temperature $T = \beta^{-1}$ and spherical field s is defined as

$$Z_N(K, L, s) = \int_{R^N} d\sigma_1 \dots d\sigma_N \exp\{-\beta \mathcal{H}_N(\{\sigma_{\vec{r}}\}) - s \sum_{\vec{r}} \sigma_{\vec{r}}^2\}, \qquad (2.2)$$

where

$$K = \beta \hat{J}(0), \quad L = \beta H. \qquad (2.3)$$

The exact evaluation of the multidimensional integral in (2.3) is readily achieved by using the Fourier transformation for diagonalization of the quadratic form in Hamiltonian (2.1). The fourier transform $\hat{J}(\vec{q})$ of the interaction potential,

$$\hat{J}(\vec{q}) = \sum_{\vec{\ell} \in S_{N,d}} J(|\vec{\ell}|) e^{-i\vec{\ell}\cdot\vec{q}}, \quad \vec{q} = 2\pi\vec{n}/N_0, \quad \vec{n} \in S_{N,d}, \qquad (2.4)$$

defined for convenience on the d-dimensional torus (1.11), is assumed to have the long-wavelength asymptotic form

$$\hat{J}(\vec{q}) \simeq \hat{J}(\vec{0})(1 - \rho_\sigma |\vec{q}|^\sigma), \quad |\vec{q}| \to 0, \quad \sigma > 0, \qquad (2.5)$$

which corresponds to the inverse power law behavior

$$J(|\vec{r} - \vec{r}'|) \sim |\vec{r} - \vec{r}'|^{-d-\sigma}, \quad |\vec{r} - \vec{r}'| \to \infty, \quad \sigma > 0, \qquad (2.6)$$

at large separations $\vec{r} - \vec{r}'$. The asymptotic form (2.5) determines the leading finite-size corrections to the thermodynamic properties.

The thermodynamic potential per particle for the Gaussian model,

$$a_N(K, L, s) = -\frac{1}{\beta N} \ln Z_N(K, L, s), \qquad (2.7)$$

is given with sufficient accuracy by its long-wavelength approximation

$$\beta a_N(K, L, s) \simeq \frac{1}{2} \ln \frac{\rho_\sigma K}{2\pi} - \frac{L^2}{2K\phi} + \frac{1}{2} U_{d,\sigma}^{(N)}(\phi), \qquad (2.8)$$

where $U_{d,\sigma}^{(N)}(\phi)$ is the d-fold sum defined by Eq.(1.9), and

$$\phi = 2s/K - 1 \qquad (2.9)$$

is a parameter related to the spherical field s.

The free energy per particle for the mean spherical model, $f_N(K, L)$ is defined through the Legendre transformation

$$\beta f_N(K, L) = \sup_s [\beta a_N(K, L, s) - s]. \qquad (2.10)$$

The supremum in the right-hand side of Eq.(2.10) is attained at a point $s = s_N(K, L)$ which obeys the equation

$$\beta \frac{\partial}{\partial s} a_N(K, L, s) = 1, \qquad (2.11)$$

or explicitly,

$$W_{d,\sigma}^{(N)}(\phi) \simeq K[1 - (\frac{L}{K\phi})^2], \qquad (2.12)$$

where the d-fold sum $W_{d,\sigma}^{(N)}(\phi)$ is defined by Eq.(1.10).

3. GENERAL ASYMPTOTIC ANALYSIS

We need asymptotic expressions for sums (1.9) and (1.10) when $N_0 \to \infty$, $\phi \to 0$, so that y (see Eq.(1.8)) remains finite. The technique suggested here is based on identities (1.3) and (1.4), the application of which to the summands in (1.9) and (1.10), respectively, allows one to factorize the d-fold summation. Thus we obtain the representations

$$U_{d,\sigma}^{(N)}(\phi) = \ln \tilde{\phi} + \int_0^\infty dx \{1 - [Q_{N_0}(x\tilde{\phi}^{-2/\sigma})]^d\} \qquad (3.1)$$

and

$$R_{d,\sigma}^{(N)}(\phi) = \rho_\sigma^{-1} \tilde{\phi}^{-1} \int_0^\infty dx [Q_{N_0}(x\tilde{\phi}^{-2/\sigma})]^d F_{\sigma/2}(x), \qquad (3.2)$$

where $\tilde{\phi} = \phi/\rho_\sigma$ and

$$Q_{N_0}(a) = \frac{1}{N_0} \sum_{n=-(N_0-1)/2}^{(N_0-1)/2} e^{-a(2\pi n/N_0)^2}. \qquad (3.3)$$

The asymptotic analysis of (3.3) when $N_0 \to \infty$ follows standard procedures. First we define a periodic function of $n \in Z^1$ with period N_0 by setting

$$g^{(p)}\left(\frac{2\pi n}{N_0}; a\right) = \exp\left[-a\left(\frac{2\pi n}{N_0}\right)^2\right], \quad n \in \left[-\frac{N_0}{2}, \frac{N_0}{2}\right], \tag{3.4}$$

and

$$g^{(p)}\left(\frac{2\pi}{N_0}(n + kN_0); a\right) = g^{(p)}\left(\frac{2\pi n}{N_0}; a\right), \quad k \in \mathbf{Z}^1. \tag{3.5}$$

Then we have the Fourier series expansion

$$g^{(p)}\left(\frac{2\pi n}{N_0}; a\right) = \sum_{k=-\infty}^{\infty} e^{2\pi i k n/N_0}\, \hat{g}(k; a), \tag{3.6}$$

where

$$\hat{g}(k; a) = \frac{1}{N_0}\int_{-N_0/2}^{N_0/2} dp\, e^{-2\pi i k p/N_0}\, g^{(p)}\left(\frac{2\pi p}{N_0}; a\right) =$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-ik\theta - a\theta^2} = (4\pi a)^{-1/2}\, e^{-k^2/4a}\, \mathrm{Re}\,\Phi\left(\pi a^{1/2} + \frac{1}{2} ika^{-1/2}\right). \tag{3.7}$$

Thus, by inserting (3.6) into (3.3) we obtain

$$Q_{N_0}(a) = \sum_{\ell=-\infty}^{\infty} \hat{g}(\ell N_0; a), \tag{3.8}$$

where

$$\hat{g}(0; a) = (4\pi a)^{-1/2}\, \Phi(\pi a^{1/2}) \tag{3.9}$$

and, when $\ell \neq 0$,

$$\hat{g}(\ell N_0; a) = (4\pi a)^{-1/2}\, e^{-\ell^2 N_0^2/4a}\, \mathrm{Re}\,\Phi\left(\pi a^{1/2} + \frac{1}{2} i\ell N_0 a^{-1/2}\right) \simeq$$
$$\simeq (4\pi a)^{-1/2}\, e^{-\ell^2 N_0^2/4a}, \tag{3.10}$$

since for all $a > 0$,

$$\left|\pi a^{1/2} + \frac{1}{2} i\ell N_0 a^{-1/2}\right|^2 \geq \pi N_0 |\ell| \to \infty, \quad N_0 \to \infty, \quad \ell \neq 0, \tag{3.11}$$

and the error function $\mathbf{\Phi(z)}$ tends to unity exponentially fast as $|z| \to \infty$ in the considered sector of the complex z-plane.

By inserting the asymptotic form (3.10) into (3.8), one obtains

$$Q_{N_0}(a) \simeq (4\pi a)^{-1/2}\, \Phi(\pi a^{1/2}) + (4\pi a)^{-1/2} \sum_{\ell=-\infty}^{\infty}{}' e^{-\ell^2 N_0^2/4a}, \tag{3.12}$$

where the prime in the sum denotes that the term with zero summation index has been omitted.

Next, at raising (3.12) to the power d, one makes the approximation

$$[Q_{N_0}(a)]^d \simeq (4\pi a)^{-d/2}\left\{[\Phi(\pi a^{1/2})]^d + \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' e^{-|\vec{\ell}|^2 N_0^2/4a}\right\}, \tag{3.13}$$

which follows if in all terms of the form

$$[\Phi(\pi a^{1/2})]^m\, e^{-|\vec{\ell}|^2 N_0^2/4a} \tag{3.14}$$

with $1 \leq m \leq d-1$ and $|\vec{\ell}| \neq 0$ one replaces the error function $\Phi(\pi a^{1/2})$ by unity. This approximation is legitimate since the exponential in (3.14) effectively cuts off the contribution from small values of a.

The use of the asymptotic expression (3.13) in equations (3.1) and (3.2) completes the separation of the leading finite-size effects from the bulk contribution:

$$U^{(N)}_{d,\sigma}(\phi) = U_{d,\sigma}(\phi) + \delta U^{(N)}_{d,\sigma}(\phi), \tag{3.15}$$

$$W^{(N)}_{d,\sigma}(\phi) = W_{d,\sigma}(\phi) + \delta W^{(N)}_{d,\sigma}(\phi). \tag{3.16}$$

Here the corresponding bulk terms are

$$U_{d,\sigma}(\phi) = \ln\tilde{\phi} + \int_0^{\infty} dx \left\{1 - (4\pi x)^{-(d/2)}\tilde{\phi}^{-d/\sigma}\left[\Phi(\pi x^{1/2}\tilde{\phi}^{-(1/\sigma)})\right]^d\right\}\frac{G_{\sigma/2}(x)}{x}, \tag{3.17}$$

$$W_{d,\sigma}(\phi) =$$
$$= (4\pi x)^{-(d/2)}\rho_\sigma^{-1}\,\tilde{\phi}^{(d/\sigma)-1}\int_0^{\infty} dx\, x^{-(d/2)}\left[\Phi(\pi x^{1/2}\tilde{\phi}^{-(1/\sigma)})\right]^d F_{\sigma/2}(x), \tag{3.18}$$

and the leading finite-size corrections are given by

$$\delta U^{(N)}_{d,\sigma}(\phi) \simeq$$
$$\simeq -(4\pi)^{-(d/2)}\tilde{\phi}^{-d/\sigma}\sum_{\vec{\ell} \in \mathbf{Z}^d}{}' \int_0^{\infty} dx\, x^{-(d/2)} e^{-\pi^2 y^2 |\vec{\ell}|^2 x^{-1}}\,\frac{G_{\sigma/2}(x)}{x}, \tag{3.19}$$

$$\delta W^{(N)}_{d,\sigma}(\phi) \simeq$$

$$\simeq (4\pi)^{-(d/2)} \rho_\sigma^{-1} \tilde{\phi}^{d/\sigma - 1} \sum_{\vec{\ell} \in \mathbf{Z}^d} \int_0^\infty dx\, x^{-d/2} e^{-\pi^2 y^2 |\vec{\ell}|^2 x^{-1}} F_{\sigma/2}(x) . \qquad (3.20)$$

4. ASYMPTOTIC FORM OF THE EQUATION FOR THE SPHERICAL FIELD

The asymptotic form of Eq.(2.12) as $N_0 \to \infty$, $\phi \to 0$, so that ϕN_0^σ = const, has been studied in /5/ by using a Laplace transformation technique equivalent to the use of identity (1.4). It has been shown there that when $\sigma < d < 2\sigma$, the solution $\phi = \phi_N(K, L)$ of this equation, in the finite-size scaling critical region defined by the finite values of the scaled variables

$$x_1 = \rho_\sigma K_c [1 - K/K_c] N_0^{d-\sigma},$$
$$x_2 = (\rho_\sigma K_c)^{-1/2} L N_0^{(d+\sigma)/2}, \qquad (4.1)$$

has the asymptotic form

$$\phi_N(K, L) \simeq \rho_\sigma N_0^{-\sigma} g(x_1, x_2), \qquad (4.2)$$

where $g = g(x_1, x_2)$ is the solution of the equation (see Appendix B):

$$g^{-1} - g \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' (2\pi |\vec{\ell}|)^{-\sigma} [(2\pi |\vec{\ell}|)^\sigma + g]^{-1} = -\tilde{x}_1 - (x_2/g)^2, \qquad (4.3)$$

with

$$\tilde{x}_1 = x_1 + \frac{C_{d,\sigma}}{(2\pi)^\sigma \Gamma(\sigma/2)} . \qquad (4.4)$$

Here a new representation of the equation for the spherical field (2.12) is derived which is a direct extension of the equation due to Singh and Pathria /1/. To this end we make use of the integral representation, see Appendix A,

$$F_{\sigma/2}(x) = (4\pi)^{-1/2} x^{-3/2} \int_0^\infty dt\, t^\sigma E_{\sigma,\sigma}(-t^\sigma) e^{-t^2/4x} \qquad (4.5)$$

and transform expression (3.20) to

$$\delta W^{(N)}_{d,\sigma}(\phi) =$$

$$= \rho_\sigma^{-1} \pi^{-\frac{d+1}{2}} \left(\frac{2\pi}{N_0}\right)^{d-\sigma} \Gamma\left(\frac{d+1}{2}\right) \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' (2\pi|\vec{\ell}|)^{-d+\sigma} w_{d,\sigma}(2\pi|\vec{\ell}|y), \qquad (4.6)$$

where

$$w_{d,\sigma}(z) = \int_0^\infty dr\, r^\sigma (1 + r^2)^{-(d+1)/2} E_{\sigma,\sigma}(-r^\sigma z^\sigma) . \qquad (4.7)$$

At $\sigma = 2$, by making use of the fact that

$$E_{2,2}(-x) = \frac{\sin x^{1/2}}{x^{1/2}}, \quad x \geq 0, \qquad (4.8)$$

and by integration by parts in Eq. (4.7) with account of the integral representation of the modified Bessel function

$$K_\nu(yz) = (2z)^\nu \pi^{-1/2} y^{-\nu} \Gamma(\nu + \tfrac{1}{2}) \int_0^\infty dt (z^2 + t^2)^{-\nu - 1/2} \cos(ty), \qquad (4.9)$$

$$(y > 0, \; |\arg z| < \tfrac{\pi}{2}),$$

one verifies that expression (4.6) reduces (up to a slight difference in notation) to the well-known form /1/

$$\delta W^{(N)}_{d,2}(\phi) \simeq$$

$$\simeq (2\rho_2)^{-1} \pi^{-d/2} \left(\frac{\pi y}{N_0}\right)^{d-2} \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' (\pi y |\vec{\ell}|)^{-(d-2)/2} K_{(d-2)/2}(2\pi|\vec{\ell}|y) . \qquad (4.10)$$

In the general case of $0 < \sigma < 2$, the equation for the spherical field (2.12) in the finite-size scaling critical region (4.1) now follows from (4.6) and the known expression for the bulk term

$$W_{d,\sigma}(\phi) \simeq K_c - \rho_\sigma^{-1} D_{d,\sigma} \tilde{\phi}^{d/\sigma - 1} . \qquad (4.11)$$

It reads:

$$2^{d-\sigma}\pi^{(d-1)/2-\sigma}\Gamma\left(\frac{d+1}{2}\right)\sum_{\vec{\ell}\in \mathbf{Z}^d}{}' (2\pi|\vec{\ell}|)^{-d+\sigma}\, w_{d,\sigma}(2\pi|\vec{\ell}|y) - \tag{4.12}$$

$$- D_{d,\sigma}(2\pi y)^{d-\sigma} = - x_1 - x_2^2(2\pi y)^{-2\sigma} .$$

In the limit $y \to 0^+$ one may use the approximation

$$\sum_{\vec{\ell}\in \mathbf{Z}^d} (2\pi|\vec{\ell}|)^{-d+\sigma} w_{d,\sigma}(2\pi|\vec{\ell}|y) \simeq \tag{4.13}$$

$$\simeq \frac{y^{-\sigma}}{(2\pi)^d}\,\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\int_0^\infty dr\, r^{\sigma-1} w_{d,\sigma}(r) \simeq \frac{\pi^{1/2}\, y^{-\sigma}}{2^d \pi^{d/2}\,\Gamma(\frac{d+1}{2})},$$

where integral (A.16) has been used. Hence

$$\delta W^{(N)}_{d,\sigma}(\phi) \simeq \rho_\sigma^{-1} N_0^{-d+\sigma}(2\pi y)^{-\sigma}, \quad y \to 0^+, \tag{4.14}$$

and equation (4.12) reduces to the asymptotic form

$$x_2^2(2\pi y)^{-2\sigma} + (2\pi y)^{-\sigma} \simeq -x_1 ,$$

which has the (positive) solution

$$(2\pi y)^\sigma \simeq \frac{1}{2}|x_1|^{-1}\left[(1 + 4|x_1|x_2^2)^{1/2} + 1\right] \tag{4.15}$$

when $x_1 \to -\infty$.

An approximation in the limit $y \to +\infty$ is most readily obtained from the initial expression (3.20). By substituting there the asymptotic form, see (A.14),

$$F_{\sigma/2}(x) \simeq \frac{\sigma}{2\pi}\sin\left(\frac{\sigma\pi}{2}\right)\Gamma\left(\frac{\sigma}{2}\right)x^{-\sigma/2-1}, \quad x \to +\infty, \tag{4.16}$$

and integrating, one obtains for $0 < \sigma < 2$,

$$\delta W^{(N)}_{d,\sigma}(\phi) \simeq \rho_\sigma^{-1} M_{d,\sigma} N_0^{-d+\sigma}\, 2^\sigma\, (2\pi y)^{-2\sigma}, \quad y \to +\infty, \tag{4.17}$$

where

$$M_{d,\sigma} = \pi^{-d/2}\frac{\sigma}{2\pi}\sin\left(\frac{\sigma\pi}{2}\right)\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{d+\sigma}{2}\right)\sum_{\vec{\ell}\in \mathbf{Z}^d}{}' |\vec{\ell}|^{-d-\sigma} . \tag{4.18}$$

It should be noted that this result cannot be continued smoothly to the case $\sigma = 2$, since then $F_{\sigma/2}(x)$ falls of exponentially fast,

$$F_1(x) = e^{-x} , \tag{4.19}$$

and, correspondingly, from (4.10) one obtains in the limit $y \gg 1$:

$$\delta W^{(N)}_{d,2}(\phi) \simeq \rho_2^{-1} N_0^{-d+2}(4\pi)^{-1}\, d\, y^{(d-3)/2} e^{-2\pi y}, \quad y \to \infty . \tag{4.20}$$

In any case $\delta W^{(N)}_{d,\sigma}(\phi)$ does not contribute to the leading asymptotic form of the eqation for the spherical field, which in the limit under consideration is

$$D_{d,\sigma}(2\pi y)^{d-\sigma} \simeq x_1 . \tag{4.21}$$

Hence

$$(2\pi y)^d \simeq D_{d,\sigma}^{-\sigma/(d-\sigma)}\, x_1^{\sigma/(d-\sigma)}, \qquad x_1 \to +\infty . \tag{4.22}$$

5. ASYMPTOTIC FORM OF THE FREE ENERGY PER PARTICLE

It is convenient to transform the bulk term (3.17) with the aid of the identity (see Appendix A, Eq.(A.18))

$$\ln\tilde{\phi} = -\int_0^\infty dx\, x^{-1}\left[G_{\sigma/2}(x\tilde{\phi}^{2/\sigma}) - G_{\sigma/2}(x)\right] \tag{5.1}$$

to the form

$$U_{d,\sigma}(\phi) = A_{d,\sigma} - B_{d,\sigma}(\tilde{\phi}), \tag{5.2}$$

where

$$A_{d,\sigma} = \int_0^\infty dt\, t^{-1}\left\{1 - (4\pi t)^{-d/2}\left[\Phi(\pi t^{1/2})\right]^d\right\}G_{\sigma/2}(t) \tag{5.3}$$

and

$$B_{d,\sigma}(\tilde{\phi}) = (4\pi)^{-d/2}\int_0^\infty dt\, t^{-d/2-1}\left[\Phi(\pi t^{1/2})\right]^d\left[G_{\sigma/2}(x\tilde{\phi}^{-2/\sigma}) - G_{\sigma/2}(x)\right]. \tag{5.4}$$

By using identity (A.11) one directly verifies that

$$\frac{d}{d\phi}U_{d,\sigma}(\phi) = W_{d,\sigma}(\phi), \tag{5.5}$$

and, therefore, from Eq.(4.11) one finds

$$U_{d,\sigma}(\tilde\phi) \simeq U_{d,\sigma}(0) + \rho_\sigma K_c \tilde\phi - \frac{\sigma}{d} D_{d,\sigma} \tilde\phi^{d/\sigma}. \tag{5.6}$$

The finite-size term (3.19) may be transformed with the use of the integral representation, see (A.7),

$$G_{\sigma/2}(x) = (4\pi x)^{-1/2} \int_0^\infty G_\sigma(u)\, e^{-u^2/4x} du, \tag{5.7}$$

to the form

$$\delta U^{(N)}_{d,\sigma}(\tilde\phi) =$$

$$= -N_0^{-d} \sigma 2^d \pi^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right) \sum_{\vec\ell \in Z^d}{}' (2\pi|\vec\ell|)^{-d} u_{d,\sigma}(2\pi|\vec\ell| y), \tag{5.8}$$

where

$$u_{d,\sigma}(z) = \int_0^\infty d\tau (1+\tau^2)^{-(d+1)/2} E_\sigma(-\tau^\sigma z^\sigma). \tag{5.9}$$

One may notice that at $\sigma = 2$,

$$E_2(-\tau^2 z^2) = \cos(\tau z), \tag{5.10}$$

and, with the aid of the integral representation of the modified Bessel function (4.9), expression (5.8) reduces to the form

$$\delta U^{(N)}_{d,2}(\tilde\phi) = -2\pi^{-d/2}\left(\frac{\pi y}{N_0}\right)^d \sum_{\vec\ell \in Z^d}{}' (\pi y \vec\ell)^{-d/2} K_{d/2}(2\pi y \vec\ell), \tag{5.11}$$

known (in a slightly different notation) for the mean spherical model with nearest neighbour interactions [1].

Thus, by collecting the results (2.8), (3.5), (5.6) and (5.11), one obtains for the thermodynamic potential per particle of the Gaussian model with spherical field s given by (see Eqs.(1.8) and (2.9))

$$s = \frac{1}{2} K\left[1 + \rho_\sigma \left(\frac{2\pi y}{N_0}\right)^\sigma\right], \quad y\text{-fixed}, \tag{5.12}$$

the following asymptotic expression

$$\beta a_N(K,L,s) \simeq \frac{1}{2} U_{d,\sigma}(0) + \frac{1}{2}\ln\frac{\rho_\sigma K}{2\pi} - \frac{L^2}{2\rho_\sigma K}\left(\frac{N_0}{2\pi y}\right)^\sigma + \frac{\rho_\sigma K}{2}\left(\frac{2\pi y}{N_0}\right)^\sigma -$$

$$- N_0^{-d}\left(\frac{\sigma}{2}\right)\left[d^{-1} D_{d,\sigma}(2\pi y)^d + 2^d \pi^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right) \sum_{\vec\ell \in Z^d}{}' (2\pi|\vec\ell|)^{-d} u_{d,\sigma}(2\pi|\vec\ell| y)\right]. \tag{5.13}$$

Finally, in the finite-size scaling critical region (4.1) the free energy per particle of the mean spherical model, defined by Eq.(2.10), takes the form

$$\beta f_N(K,L) \simeq \frac{1}{2} U_{d,\sigma}(0) + \frac{1}{2}\ln\frac{\rho_\sigma K}{2\pi} - \frac{1}{2}K + N_0^{-d} Y_{d,\sigma}(x_1,x_2), \tag{5.14}$$

where the finite-size scaling function $Y_{d,\sigma}(x_1,x_2)$ is given by

$$Y_{d,\sigma}(x_1,x_2) = -\frac{x_2^2}{(2\pi y)^\sigma} + \left(\frac{1}{2} - \frac{\sigma}{2d}\right) D_{d,\sigma}(2\pi y)^d -$$

$$-(4\pi)^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right)\left[\sigma \sum_{\vec\ell \in Z^d}{}' (2\pi\vec\ell)^{-2} u_{d,\sigma}(2\pi \vec\ell y) +\right.$$

$$\left. + y^\sigma \sum_{\vec\ell \in Z^d}{}' (2\pi \vec\ell)^{-d-\sigma} w_{d,\sigma}(2\pi\vec\ell y)\right]. \tag{5.15}$$

where $y = y(x_1,x_2)$ is the solution of Eq.(4.12). This equation generalizes the corresponding results of Singh and Pathria [1] to the case of arbitrary $0 < \sigma \le 2$.

With the use of Eq.(4.12) one can write (5.15) in an alternative form

$$Y_{d,\sigma}(x_1,x_2) = \frac{1}{2} g(x_1,x_2) - \frac{1}{2}\frac{x_2^2}{g(x_1,x_2)} - \frac{\sigma}{2d} D_{d,\sigma}\, g^{d/\sigma}(x_1,x_2) -$$

$$-\sigma(4\pi)^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right) \sum_{\vec\ell \in Z^d}{}' (2\pi \vec\ell)^{-d} u_{d,\sigma}(\vec\ell\, g^{1/\sigma}(x_1,x_2)). \tag{5.16}$$

Let us consider now the asymptotic forms of $Y_{d,\sigma}(x_1,x_2)$ as $x_1 \to \pm\infty$. In the limit $y \to 0^+$ $(x_1 \to -\infty)$, we use the approximation

$$\sum_{\vec{\ell}\in \mathbf{Z}^d}{}' (2\pi|\vec{\ell}|)^{-d}\, u_{d,\sigma}(2\pi|\vec{\ell}|y) \simeq \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{2\pi y}^{\infty} dr\, r^{-1} u_{d,\sigma}(r) \simeq \tag{5.17}$$

$$\simeq (4\pi)^{-d/2}\frac{\pi^{1/2}}{\Gamma(\frac{d+1}{2})}\ln\frac{1}{y} + \text{const},$$

where it has been taken into account that

$$\int_{2\pi y}^{\infty} dr\, r^{-1} E_\sigma(-r^\sigma\tau^\sigma) = -\ln(2\pi y\tau)E_\sigma(-(2\pi y\tau)^\sigma) + \text{const} \simeq \tag{5.18}$$

$$\simeq -\ln(2\pi y\tau) + \text{const}.$$

Therefore,

$$\delta U^{(N)}_{d,\sigma}(\phi) \simeq -N_0^{-d}\sigma(\ln\frac{1}{y} + \text{const}), \quad y \to 0^+, \tag{5.19}$$

and the leading asymptotic form of $Y_{d,\sigma}(x_1,x_2)$ when $x_1 \to -\infty$, x_2 finite, becomes independent of σ:

$$Y_{d,\sigma}(x_1,x_2) \simeq -\frac{1}{2}(1+4|x_1|x_2^2)^{1/2} + \frac{1}{2}\ln\left[\frac{(1+4|x_1|x_2^2)^{1/2}+1}{2|x_1|}\right]. \tag{5.20}$$

In the limit $y\to\infty$ $(x_1 \to +\infty)$ the term

$$\delta U^{(N)}_{d,\sigma}(\phi) \simeq -N_0^{-d} M_{d,\sigma}(\pi y)^{-\sigma}, \quad y\to\infty, \tag{5.21}$$

does not contribute to the leading asymptotic form of $Y_{d,\sigma}(x_1,x_2)$ which becomes

$$Y_{d,\sigma}(x_1,x_2) \simeq \frac{d-\sigma}{2d} D_{d,\sigma}^{-\sigma/(d-\sigma)} x_1^{d/(d-\sigma)}, \quad x_1 \to +\infty. \tag{5.22}$$

We note again that the asymptotic expression (5.21) cannot be continued smoothly to the case $\sigma = 2$, when it becomes exponentially small:

$$\delta U^{(N)}_{d,2}(\phi) \simeq -N_0^{-d}\, dy^{(d-1)/2} e^{-2\pi y}. \tag{5.23}$$

However, the asymptotic form (5.22) reduces at $\sigma = 2$ to the known expression [1].

6. DISCUSSION

In the present paper the σ-dependent scaling function $Y_{d,\sigma}(x_1,x_2)$ for the free energy per particle of the mean spherical model with an interaction potential falling with distance r as $r^{-d-\sigma}$ when $r\to\infty$, has been found. A convenient representation (5.16) of $Y_{d,\sigma}(x_1,x_2)$ has been obtained, which involves integral transforms, see Eq.(5.9), of the simple square-integrable over $(0,\infty)$ function

$$v_d(\tau) = (1+\tau^2)^{-(d+1)/2}, \quad \tau\in(0,\infty),$$

with the Mittag-Leffler kernel $E_\sigma(-\tau^\sigma z^\sigma)$. Such transforms are a particular case of more general transformations with Mittag-Leffler type kernels

$$\tau^{\beta-1}E_{a,\beta}(e^{i\phi}\tau^a x^a), \quad x>0, \quad \frac{1}{2}a\pi \le \phi \le 2\pi - \frac{1}{2}a\pi,$$

in the class of square-integrable over $(0,\infty)$ functions, the mathematical theory of which has been developed [9]. The suggested new analytical technique may be successfully used to generalize a number of results on the spherical model with different geometry and boundary conditions [1].

Here we point out that some new information about the contribution of the long-distance asymptotics of the interaction potential to the formation of the critical bulk singularities of the mean spherical model can be derived from our results.

When $t = (T-T_c)/T_c \to 0$, the singular part, $c_N^{(s)}(K,0)$, of the zero-field specific heat per particle is given by

$$c_N^{(s)}(K,0) \simeq -\rho_\sigma^2 K_c^2 N_0^{-2\sigma+d}\frac{\partial^2}{\partial x_1^2}Y_{d,\sigma}(x_1,0). \tag{6.1}$$

The differentiation of the scaling function (5.16) with respect to x_1, by taking into account Eq.(4.12), yields

$$\frac{\partial}{\partial x_1}Y_{d,\sigma}(x_1,x_2) = \frac{1}{2}g(x_1,x_2). \tag{6.2}$$

Therefore

$$c_N^{(s)}(K,0) \simeq -\frac{1}{2}\rho_\sigma^2 K_c^2 N_0^{-2\sigma+d}\frac{\partial}{\partial x_1}g(x_1,0). \tag{6.3}$$

In the limit $x_1 \to -\infty$ one may use Eq.(4.15) to obtain from (6.3):

$$c_N^{(s)}(K,0) \simeq -\frac{1}{2}\rho_\sigma^2 K_c^2 N_0^{-2\sigma+d} |x_1|^{-2}, \quad x_1 \to -\infty. \qquad (6.4)$$

Hence, the singular part of the specific heat just below the critical point behaves as

$$c_N^{(s)}(K,0) \simeq -\frac{1}{2} N_0^{-d} |t|^{-2}, \quad t \to 0^-, \qquad (6.5)$$

independently of the interaction potential parameter σ.

When $x_1 \to +\infty$, the use of Eqs.(4.22) and (6.3) yields

$$c_N^{(s)}(K,0) \simeq -\frac{\sigma}{2(d-\sigma)} \cdot N_0^{-2\sigma+d} (\rho_\sigma K_c)^2 D_{d,\sigma}^{-\sigma/(d-\sigma)} x_1^{(2\sigma-d)/(d-\sigma)}, \qquad (6.6)$$

i.e., just above the critical point one obtains in the leading order

$$c_N^{(s)}(K,0) \simeq -\frac{\sigma}{2(d-\sigma)} (\rho_\sigma K_c)^{d/(d-\sigma)} D_{d,\sigma}^{-\sigma/(d-\sigma)} t^{(2\sigma-d)/(d-\sigma)}, \qquad (6.7)$$

$t \to 0^+$.

The known value of the critical exponent α_s for the singular part of the specific heat follows from Eq.(6.7):

$$\alpha_s = -\frac{2\sigma-d}{d-\sigma}, \quad \sigma < d < 2\sigma. \qquad (6.8)$$

Thus we see that the low-temperature branch of the singular part of the bulk specific heat, $c_\infty^{(s)}(K,0)$, is asymptotically built out of a (vanishing in the thermodynamic limit) function, see Eq.(6.5), which does not depend on the decay parameter σ of the interaction potential.

An analogous situation is observed in the case of the magnetic susceptibility $\chi_N(K,L)$. By differentiation of the magnetization per particle,

$$m_N(K,L) = -\frac{\partial}{\partial L}\beta f_N(K,L) = \frac{H}{\rho_\sigma \hat{J}(\vec{0})} \tilde{\phi}_N^{-1}, \qquad (6.9)$$

with allowance for the dependence of $\tilde{\phi}_N$ on H through the equation of state, see Eqs.(4.2), (4.3), one obtains

$$T\chi_N(K,L) = \frac{N_0^\sigma}{\rho_\sigma K g(x_1,x_2)}\left[1 - \frac{x_2}{g(x_1,x_2)} \frac{\partial g(x_1,x_2)}{\partial x_2}\right]. \qquad (6.10)$$

Therefore, for the zero-field susceptibility in the limit $x_1 \to -\infty$, one finds, by using (4.15), the following leading-order expression

$$\chi_N(K,0) \simeq \frac{N_0^\sigma}{\rho_0 \hat{J}(\vec{0})} |x_1|, \quad x_1 \to -\infty. \qquad (6.11)$$

Hence

$$\chi_N(K,0) \simeq \beta_c |t| N_0^d, \quad t \to 0^-, \qquad (6.12)$$

i.e., the low-temperature branch of the bulk zero-field susceptibility per particle is again asymptotically built out of a (diverging in the thermodynamic limit) function that does not depend on the decay parameter σ of the interaction potential.

In the limit $x_1 \to \infty$ from (4.22) it follows that

$$\chi_N(K,0) \simeq N_0^\sigma (\rho_\sigma \hat{J}(\vec{0}))^{-1} D_{d,\sigma}^{\sigma/(d-\sigma)} x_1^{-\sigma/(d-\sigma)}, \qquad (6.13)$$

which implies that the singularity at the critical point from above is characterized by the σ-dependent critical exponent $\gamma = \sigma/(d-\sigma)$:

$$\chi_N(K,0) \simeq \beta_c D_{d,\sigma}^{\sigma/(d-\sigma)} (\rho_\sigma K_0)^{-d/(d-\sigma)} t^{-\sigma/(d-\sigma)}, \quad t \to 0^+. \qquad (6.14)$$

It is interesting to note that the low-temperature asymptotic expressions (6.5) and (6.12) hold true even in the extreme case of the infinitely coordinated Husimi - Temperley mean spherical model with $N = N_0^d$ spins.

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APPENDIX A

The Mittag-Leffler type functions are entire functions of finite order of growth, defined by the power series[6]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0. \tag{A.1}$$

In particular, the function $E_\alpha(z) = E_{\alpha,1}(z)$ has been introduced by Mittag-Leffler. A rather complete study of these functions can be found in the book [8] (see also [6]).

Here we are interested in the properties of functions (A.1) when $0 < \alpha < 2$ and $\beta \geq 0$.

To derive identity (1.3), one may start with the known integral

$$\int_0^\infty dt e^{-t} E_\alpha(t^\alpha z) = \frac{1}{1-z}, \tag{A.2}$$

which converges in the complex z-plane to the left of the line $\mathrm{Re}\, z^{1/\alpha} = 1$, $|\arg z| \leq \frac{1}{2}\alpha\pi$. By setting here $z = -p^{-\alpha}$, $p > 0$, and $t = xp$, one obtains the Laplace transformation[9]

$$\int_0^\infty dx\, e^{-px} E_\alpha(-x^\alpha) = \frac{p^{\alpha-1}}{1+p^\alpha}, \quad \mathrm{Re}\, p > 0. \tag{A.3}$$

Equation (1.3) now follows by integration of (A.3) over p from zero to z.

The identity (1.4) may be derived from a more general integral

$$\int_0^\infty dt e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^\alpha y) = \frac{1}{1-y}, \tag{A.4}$$

which is readily obtained by means of term by term integration with the use of series (A.1). By setting in (A.4) $y = -z^{-\alpha}$, $z > 0$, and $t = xz$, one obtains the Laplace transformation[10]

$$\int_0^\infty dx\, e^{-zx} x^{\beta-1} E_{\alpha,\beta}(-x^\alpha) = \frac{z^{\alpha-\beta}}{1+z^\alpha}. \tag{A.5}$$

Hence $\beta = \alpha$ yields Eq.(1.4).

Particular cases (1.1) and (1.2) follow from general identities (1.3) and (1.4), respectively, considering that

$$E_1(z) = E_{1,1}(z) = e^{-z}. \tag{A.6}$$

The integral representation (5.7) is equivalent to

$$E_\alpha(-t^\alpha) = (\pi t)^{-1/2} \int_0^\infty du E_{2\alpha}(-u^{2\alpha}) e^{-u^2/4t}, \tag{A.7}$$

which may be obtained by means of term by term integration of the series representing the integrand.

In order to derive the integral representation (4.5) we write down

$$(4\pi x)^{-1/2} \int_0^\infty dt\, t^\sigma E_{\sigma,\sigma}(-t^\sigma) e^{-t^2/4x} =$$

$$= (4\pi)^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{\sigma(k+1)} x^{\frac{1}{2}\sigma(k+1)}}{\Gamma(\sigma(k+1))} \Gamma(\tfrac{1}{2}\sigma(k+1) + \tfrac{1}{2}) = \tag{A.8}$$

$$= x^{\sigma/2} \sum_{k=0}^{\infty} (-1)^k \frac{x^{\frac{1}{2}\sigma k}}{\Gamma(\frac{1}{2}\sigma(k+1))} = x^{\sigma/2} E_{\sigma/2,\sigma/2}(-x^{\sigma/2}).$$

The differential relation (5.5) follows from the identities

$$\frac{d}{dz}[z^\alpha E_{\alpha,\alpha+1}(-z^\alpha)] = z^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha k}}{\Gamma(\alpha k+\alpha)} = z^{\alpha-1} E_{\alpha,\alpha}(-z^\alpha) \tag{A.9}$$

and

$$z^\alpha E_{\alpha,\alpha+1}(-z^\alpha) = -\sum_{k=1}^{\infty} (-1)^k \frac{z^{\alpha k}}{\Gamma(\alpha k+1)} = 1 - E_\alpha(-z^\alpha). \tag{A.10}$$

Hence

$$-\frac{d}{dz} E_\alpha(-z^\alpha) = z^{\alpha-1} E_{\alpha,\alpha}(-z^\alpha). \tag{A.11}$$

In the derivation of asymptotic expansions (4.16) and (5.21) we have used the leading asymptotic behavior of $E_\alpha(-x^\alpha)$ and $E_{\alpha,\alpha}(-x^\alpha)$ when $x\to\infty$ which follows from the lemma below.

Lemma [8]. Let $0<\alpha<2$, β be an arbitrary complex number and γ be a real number obeying the condition

$$\frac{1}{2}\alpha\pi<\gamma<\min\{\pi,\alpha\pi\}.$$

Then for any integer $p\geq 1$ the following asymptotic expressions hold when $|z|\to\infty$:

1. At $|\arg z|\leq\gamma$.

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}}-\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+\mathcal{O}(|z|^{-p-1}). \qquad (A.12)$$

2. At $\gamma\leq|\arg z|\leq\pi$.

$$E_{\alpha,\beta}(z)=-\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+\mathcal{O}(|z|^{-p-1}). \qquad (A.13)$$

Notice that since $\Gamma(0)=\infty$, from (A.13) it follows that

$$E_{\alpha,\alpha}(-x^\alpha)\simeq-\frac{x^{-2\alpha}}{\Gamma(-\alpha)},\quad x\to\infty,\quad \alpha\neq 1. \qquad (A.14)$$

By integration of Eq.(A.11) one gets

$$\int_0^t dz\, z^{1-\alpha}E_{\alpha,\alpha}(-z^\alpha)=1-E_\alpha(-t^\alpha). \qquad (A.15)$$

Passing here to the limit $t\to\infty$, with account of (A.13) one finds

$$\int_0^\infty dz\, z^{1-\alpha}E_{\alpha,\alpha}(-z^\alpha)=1. \qquad (A.16)$$

As a direct consequence of (A.16) one obtains for any $t>0$

$$\int_0^\infty dx\, x^{1-\alpha}E_{\alpha,\alpha}(-x^\alpha t)=t^{-1}. \qquad (A.17)$$

The integration of Eq.(A.17) over t from $\phi>0$ to one yields the identity

$$-\ln\phi=\int_0^\infty dx\,x^{-1}\int_\phi^1 dt\,x^\alpha E_{\alpha,\alpha}(-x^\alpha t)=$$
$$=\alpha\int_0^\infty dx\,x^{-1}[E_\alpha(-x^\alpha\phi)-E_\alpha(-x^\alpha)], \qquad (A.18)$$

where use has been made of the relationship

$$\alpha\frac{d}{dt}E_\alpha^\alpha(-x^\alpha t)=\sum_{k=1}^\infty\frac{(-1)^k t^{k-1}x^{\alpha k}}{\Gamma(\alpha k)}=-x^\alpha E_{\alpha,\alpha}(-x^\alpha t) \qquad (A.19)$$

APPENDIX B

For the sake of completeness, a short derivation of equations (4.2) and (4.3), obtained first in [5], is given here.

One starts by noticing that with the aid of the d-dimensional version of the Jacobi identity (see, e.g. [1]),

$$\sum_{\ell\in\mathbf{Z}^d}e^{-a|\ell|^2}=\left(\frac{\pi}{a}\right)^{d/2}\sum_{\vec{\ell}\in\mathbf{Z}^d}e^{-\pi^2|\vec{\ell}|^2a^{-1}}, \qquad (B.1)$$

the finite-size term (3.20) may be cast in the form

$$\delta W_{d,\sigma}^{(N)}(\phi)=$$

$$=\rho_\sigma^{-1}\tilde{\phi}^{-1}N_0^{-d}\{1+\int_0^\infty dx[\sum_{\vec{\ell}\in\mathbf{Z}^d}{}'e^{-x|\vec{\ell}|^2y^{-2}}-\left(\frac{\pi y^2}{x}\right)^{d/2}]F_{\sigma/2}(x)\}. \qquad (B.2)$$

The integral in the right-hand side of Eq.(B.2) may be identically written as a sum of two terms, $I_1(y)+I_2(y)$, where

$$I_1(y)=\int_0^\infty dx[\sum_{\vec{\ell}\in\mathbf{Z}^d}{}'e^{-x|\vec{\ell}|^2y^{-2}}-\left(\frac{\pi y^2}{x}\right)^{d/2}][F_{\sigma/2}(x)-\frac{x^{\sigma/2-1}}{\Gamma(\sigma/2)}],$$

$$I_2(y) = \frac{1}{\Gamma(\sigma/2)} \int_0^\infty dx\, x^{\sigma/2-1} \Big[\sum_{\vec{\ell} \in \mathbf{Z}^d}{}' e^{-x|\vec{\ell}|^2 y^{-2}} - \Big(\frac{\pi y^2}{x}\Big)^{d/2} \Big]. \qquad (B.3)$$

Now, by making use of the identity (1.4), we see that

$$\sum_{\vec{\ell} \in \mathbf{Z}^d}{}' \int_0^\infty dx\, e^{-x|\vec{\ell}|^2 y^{-2}} \Big[F_{\sigma/2}(x) - \frac{x^{\sigma/2-1}}{\Gamma(\sigma/2)} \Big] =$$
$$= y^\sigma \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' |\vec{\ell}|^{-\sigma} [(|\vec{\ell}| y^{-1})^\sigma + 1]^{-1} \qquad (B.4)$$

and, taking into account the small argument asymptotic behavior of $F_{\sigma/2}(x)$, we may set

$$\pi^{d/2} y^d \int_0^\infty dx\, x^{-d/2} \Big[F_{\sigma/2}(x) - \frac{x^{\sigma/2-1}}{\Gamma(\sigma/2)} \Big] = -(2\pi y)^d D_{d,\sigma}, \qquad (B.5)$$

whereby the constant $D_{d,\sigma}$ is defined.

Next, $I_2(y)$ may be written as

$$I_2(y) = \frac{1}{\Gamma(\sigma/2)} \lim_{\delta \to 0} \Big\{ \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' \int_\delta^\infty dx\, x^{\sigma/2-1} e^{-x|\vec{\ell}|^2 y^{-2}} -$$
$$- \pi^{d/2} y^d \int_\delta^\infty dx\, x^{-(d-\sigma)/2-1} \Big\} = \frac{y^\sigma}{\Gamma(\sigma/2)} C_{d,\sigma}, \qquad (B.6)$$

where $C_{d,\sigma}$ is the Madelung type constant[5]

$$C_{d,\sigma} = \lim_{\delta \to 0} \Big\{ \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' \Gamma\Big(\frac{\sigma}{2}, \delta|\vec{\ell}|^2\Big) |\vec{\ell}|^{-\sigma} - \int_{\mathbf{R}^d} d^d r\, \Gamma\Big(\frac{\sigma}{2}, \delta|\vec{r}|^2\Big) |\vec{r}|^{-\sigma} \Big\}. \qquad (B.7)$$

Collecting the results (B.2) to (B.7), we get[5]

$$W^{(N)}_{d,\sigma}(\phi) \simeq W_{d,\sigma}(\phi) + \rho_\sigma^{-1} N_0^{d-\sigma} \Big\{ (\tilde{\phi} N_0^\sigma)^{-1} + \frac{C_{d,\sigma}}{(2\pi)^\sigma \Gamma(\sigma/2)} +$$
$$+ D_{d,\sigma} (\tilde{\phi} N_0^\sigma)^{d/\sigma - 1} - \tilde{\phi} N_0^\sigma \sum_{\vec{\ell} \in \mathbf{Z}^d}{}' (2\pi|\vec{\ell}|)^{-\sigma} [(2\pi|\vec{\ell}|)^\sigma + \tilde{\phi} N_0^\sigma]^{-1} \Big\}. \qquad (B.8)$$

Now Eqs.(4.2), (4.3) follow by substitution of (B.8) and (4.1) into (2.12) and by taking into account Eq.(4.11). The finite-size temperature shift is identified as[5]

$$\epsilon_N = -(\rho_\sigma K_c)^{-1} N_0^{-d+\sigma} \frac{C_{d,\sigma}}{(2\pi)^\sigma \Gamma(\sigma/2)}. \qquad (B.9)$$

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