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CONTINUAL HEISENBERG MODELS DEFINED ON GRADED SU(3) AND SU(2,1) ALGEBRAS

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Pseudo-spin approach gets more and more popular in condense matter physics since it gives the quantization procedure a beautiful geometrical meaning. In one dimension there are classes of integrable models (with pseudo-spin ones between them) which are connected with each other through gauge transformations. Geometrical picture of the situation is very evident to make help to understand underlying symmetries. For example, the model of Bose gas with point-like attraction has as a counterpart the continual Heisenberg ferromagnet (with SU(2) symmetry), on the contrary "repulsive" Bose gas, the pseudo-spin model with SU(1,1) symmetry.

Recently Hubbard model and its extensions has attracted a great attention of researchers working in theory of HTSC(the high temperature superconductivity).These can be expressed in terms of Lie superalgebras (graded algebras), viz.,Spl(n/m). Therefore construction of integrable models of the Heisenberg type incorporating Grassmannian fields is of positive interest.

In what follows we present such models constructed on graded SU(3) and SU(2,1) algebras. A number of connections between them and quantum problem of superfluidity is given.

1. Bose gas with point-like repulsive interaction in Bogolubov's approximation is governed by the Hamiltonian

$$\hat{H} = \sum_{p\neq 0}\left(p^2 + \nu\frac{N}{V}\right)a_p^+ a_p + \nu\frac{N}{2V}\sum_{p\neq 0}\left(a_p^+ a_{-p}^+ + a_p a_{-p}\right) + \mathrm{const} \qquad (1)$$

with a_p and a_p^+ being conventional annihilation and creation operators generating Heisenberg-Weil group, ν the coupling constant and $m = \frac{1}{2}$. It can be rewritten in terms of SU(1,1) group generators (Solomon /1/)

$$\hat{H} = \mathrm{const} + \nu\sum_{p\neq 0}\oplus\frac{N}{V}\left[\hat{K}_1^{(p)} + \mu_p\left(\hat{K}_3^{(p)} - \tfrac{1}{2}\right)\right] \in su(1,1)\ , \qquad \mu_p = 1 + \frac{p^2 V}{N\nu} \qquad (2)$$

Since the system is supposed to be put in the volume $V = L^3$ and momentum p takes quasidiscrete values $p = \frac{2\pi}{L}(n_1, n_2, n_3)$ with n_α $(\alpha = \overline{1,3})$ being integers, Hamiltonian (2) describes a lattice model of non-interacting pseudo-spins $\vec{K}_{(n_\alpha)} = (K_1, K_2, K_3)_{(n_\alpha)}$. To the analogous model is reduced the Heisenberg antiferromagnet after fixation of the duble-sublattice Neel ground state associated with magnon condensation in the momentum space /2/.

As $\hat{H} = \mathrm{const} + \sum_p \oplus \hat{H}_p$, the eigenstate $|\Psi_n\rangle$ that corresponds to the energy E_n is factorized in a set of one-site states: $|\Psi_n\rangle = \prod_p \otimes |\Psi_n\rangle_p$. Hamiltonian (2), one can easily check, has no discrete spectrum bounded from below that is due to non-compactness of the dynamical symmetry group $\prod_p \otimes SU(1,1)_p$. Performing in each site of the momentum lattice the rotation R_p at the angle ϑ_p which is generated by the operator $\hat{K}_2^{(p)}$:

$$R = \prod_p \otimes R_p\ , \quad R_p = R(\vartheta_p) = e^{-i\hat{K}_2^{(p)}\vartheta_p}, \qquad (3)$$

we arrive at the Hamiltonian

$$\tilde{\hat{H}} = R\hat{H}R^{-1} = \mathrm{const} + \sum_p \oplus\left(E_p \tilde{K}_3^{(p)} - \mu_p\frac{\nu N}{2V}\right) \qquad (4)$$

with the angle ϑ_p being

$$\mathrm{cth}\,\vartheta_p = -\mu_p$$

and

$$E_p = \nu\frac{N}{V}\mathrm{csech}\,\vartheta_p \equiv \left(p^4 + 2p^2\frac{\nu N}{V}\right)^{\frac{1}{2}}. \qquad (5)$$

Thus, in each site p we turned pseudo-spins $\vec{K}^{(p)}$ so that they all are directed along the axis ξ_3 related to the component $\hat{K}_3^{(p)}$. In other words, the ground state of our model, in terms of pseudo-spin variables, has a structure with "ferromagnetic" ordering along ξ_3.

The discrete spectrum bounded from below of such a system is given by the irreducible unitary representation of positive discrete series $D^+(j_p)$, where

$$j_p = \tfrac{1}{2}\left(1 + |\Delta_p|\right)\ , \quad \Delta_p = \hat{n}_p - \hat{n}_{-p}\ .$$

Define the representation in the p-th site:

$$(\hat{K}_3^{(p)})^2 - (\hat{K}_1^{(p)})^2 - (\hat{K}_2^{(p)})^2 = j_p(j_p - 1) = \tfrac{1}{4}(\Delta_p^2 - 1)$$

$$\hat{K}_3^{(p)}|n_p\rangle = \left(n_p + \frac{|\Delta_p|}{2} + \frac{1}{2}\right)|n_p\rangle.$$

As a result we have for the energy spectrum

$$E(\{n\}) = \sum_p\left[E_p\left(n_p + \frac{|\Delta_p|}{2} + \frac{1}{2}\right) - \mu_p\frac{\nu N}{2V}\right] + \mathrm{const} \qquad (6)$$

and the eigenstates are the corresponding pseudo-spin coherent states (generalized coherent states of the group SU(1,1) /3,4/).

Here we have to underline that the physical ground state (vacuum), excitations over which are stable and related to the discrete energy spectrum, does not coincide with the referent vacuum $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ (that corresponds to the Bose condensate at $\nu = 0 \,:\, a_p|0\rangle = 0$) for constructing GCS.

The physical ground state, the Bogolubov condensate one, gets setting $\Delta_p = n_p = 0$, $j_p = \frac{1}{2}$ so that

$$E(|0\rangle) = \frac{1}{2}\sum_p \left[E_p - (p^2 + \nu \frac{N}{V})\right] + const \qquad (7)$$

with eigenstate (the Bogolubov condensate)

$$|\Psi_B\rangle = \prod_p \operatorname{sech}\frac{\theta_p}{2} \exp\left\{\sum_p th\frac{\theta_p}{2} a_p^+ a_{-p}^+\right\}|0\rangle =$$
$$= \prod_p e^{i\theta_p \hat{K}_2^{(p)}} |\tfrac{1}{2}, \tfrac{1}{2}\rangle . \qquad (8)$$

This state consists of all the multiparticle states $|\{n_p\}\rangle$ with probabilities $w_{n_p} = s_{op}^{2n}(1 - s_{op}^2)$, $s_{op} = th\frac{\theta_p}{2}$.

Let us consider observables of the Bogolubov condensate (8). By use of the Hamiltonian (4) and momentum operator

$$\hat{P} = \sum_p p\, \hat{n}_p = \sum_p p\left(\hat{K}_3^{(p)} + \frac{1}{2}\Delta_p - \frac{1}{2}\right)$$

one can find expectation values for the pseudo-spin matrix operator $\hat{K}^{(p)} = 2\begin{pmatrix} \hat{K}_3^{(p)} & \hat{K}_-^{(p)} \\ -\hat{K}_+^{(p)} & -\hat{K}_3^{(p)} \end{pmatrix}$ in the p-th site as a function of x and t :

$$\langle|\hat{K}_i^{(p)}|\rangle = \langle\Psi_B| e^{-i(\hat{P}x + \hat{H}t)} \hat{K}_i^{(p)} e^{i(\hat{P}x + \hat{H}t)} |\Psi_B\rangle .$$

Finally

$$\langle|K_{\pm}^{(p)}(x,t)|\rangle = \frac{2\rho}{\omega_B} e^{\mp i(\omega_B t + px + \pi)} \qquad (9)$$

$$\langle|K_3^{(p)}(x,t)|\rangle = \frac{2\rho + p^2}{\omega_B} \qquad (10)$$

and

$$\langle\Psi_B|\hat{n}_p|\Psi_B\rangle = \frac{1}{2}\frac{2\rho + p^2 - \omega_B}{\omega_B} = \langle|K_3^{(p)}|\rangle - \frac{1}{2} .$$

where $\omega_B(p) = |p|\sqrt{p^2 + 4\rho}$ is the Bogolubov frequency. Here we proceed to the thermodynamical limit $\frac{N}{V} \to \rho_{V\to\infty}$.

We note then the natural way to investigate Hamiltonian (1) is to use the pseudo-spin coherent states (PSCS of the group SU(1,1)). The system of such states has as a geometric image the homogeneous space SU(1,1)/U(1), i.e. a two-sheet hyperboloid.

2. The same image possesses a classical pseudo-spin Heisenberg model which in continual approximation is given by the Hamiltonian function /5-7/:

$$H = \frac{1}{2}\int\left\{(S_x^{(1)})^2 + (S_x^{(2)})^2 - (S_x^{(3)})^2\right\}dx \qquad (11)$$

with the vector $\vec{S}$ lying on the hyperboloid

$$(\vec{S})^2 = (S^{(3)})^2 - (S^{(1)})^2 - (S^{(2)})^2 = 1 . \qquad (12)$$

In terms of the pseudospherical projection

$$S^+ = S^{(1)} + iS^{(2)} = \frac{2\xi}{1-|\xi|^2} , \quad S^{(3)} = \frac{1+|\xi|^2}{1-|\xi|^2} \qquad (13)$$

we have

$$H = \frac{1}{2}\int\frac{|\nabla\xi|^2 dx}{(1-|\xi|^2)^2} . \qquad (14)$$

Rewriting $\vec{S}$ as the matrix
$$S = \begin{pmatrix} S^{(3)} & iS^{(-)} \\ iS^{(+)} & -S^{(3)} \end{pmatrix}$$
we get the equation of motion

$$S_t = \frac{1}{2i}[S, \Delta S] \qquad (15)$$

and for pseudostereographic projection (13)

$$i\dot{\xi} + \Delta\xi + 2\bar{\xi}(\nabla\xi)^2(1-|\xi|^2)^{-1} = 0 .$$

There is a quasiclassical description of the Bose gas (also Ginzburg-Landau phenomenology for superfluidity)

$$i\Psi_t + \Delta\Psi - 2\left(|\Psi|^2 - \rho\right)\Psi = 0 \qquad (16)$$

$$H = \int \left\{ |\Psi_x|^2 - \left(|\Psi|^2 - \rho\right)^2 \right\} dx \qquad (16a)$$

with the Bose condensate $|\Psi|^2 = \rho$. The model has no Bogolubov condensate properties barring the spectrum of excitations for which $\omega = \omega_B$.

The condensation considered, strictly speaking, occurs in three-space dimensions at finite temperatures. In what follows to realize the connections mentioned above we shall consider one-dimensional models at zero temperature, for which many of such connections are exact. In this case the number of bosons becomes infinite.

3. Here we consider the AKNS-ZS problem

$$\Phi_x = U\Phi \ , \ \Phi_t = V\Phi \qquad \text{pair } (U,V) \in SU(1,1) \quad (17)$$

then both models (15) and (16) are two versions (the "plane" one and the "curve" or σ-model one) of the same linear problem formulated on the su(1,1) algebra(17) /5/.The particular system depends on the choice of the gauge of the pair(U,V):the plane version is NSE (16) and the curve one is the pseudo-spin model (11] They are related by the gauge coupling: $|\Psi|^2 = \frac{1}{2}\mathrm{Tr}(S_x)^2$, $S = g^{-1}\sigma_3 g$ with $g(x,t;\lambda_0)$ being Jost solution to the first eq. of (17)for the plane gauge with spectral parameter $\lambda = \lambda_0 > \sqrt{\rho}$.

By "dressing" the NSE quasiclassical Bose condensate with solitonless $g(x,t;\ \lambda_0)$ we get

$$S = \frac{1}{k}\begin{pmatrix} \sqrt{k^2+4\rho} & 2\sqrt{\rho}\, e^{i\alpha} \\ -2\sqrt{\rho}\, e^{-i\alpha} & -\sqrt{k^2+4\rho} \end{pmatrix} \qquad (18)$$

which is an exact pseudo-spin wave solution $S^{\pm} = S_0^{\pm} e^{\mp i(kx-\omega t)}$, $\partial_t S^{(3)} = 0$ to (15) /6,7/ with $\alpha = kx - \omega t$, where

$$\omega = k^2\sqrt{1 + S_0^+ S_0^-} = k\sqrt{k^2+4\rho} \ . \qquad (19)$$

Solution (18) describes pseudo-spin wave with vector $\vec{S}$ precessing on the hyperboloid (12) with Bogolubov frequency (19) that propagates along the x-axis with the momentum k. Comparing matrix (18) with one defined by (9) and (10) in the small k=p limit ($k^2 \ll 4\rho$) we see both expressions coincide:

$$2\langle|\hat{K}_3^{(p)}|\rangle = S^{(3)}_{(p)} = \frac{2\sqrt{\rho}}{p}, \quad 2\langle|\hat{K}_{\pm}^{(p)}|\rangle = S^{\pm}_{(p)} = \frac{2\sqrt{\rho}}{p} e^{\mp i(px - 2p\sqrt{\rho}t)} \qquad (20)$$

Thus, there is a correspondence between observables of the Bogolubov (superfluid) condensate $\langle\Psi_B|K_i^{(p)}(x,t)|\Psi_B\rangle$ and components of the vector of pseudo-spin $\vec{S}_{(p)}$ in the plane wave solution of the SU(1.1) Heisenberg model.These components can be regarded as the classical analogs of the partial waves of the Bogolubov condensate with the momentum p. Thus on the classical (or quasiclassical) level condensate solution $|\Psi|^2 = \rho$ to the NSE (classical version of the Bose gas model) corresponds to the Bose condensate on the quantum level, moreover,in the NSE version there is no analogs of the Bogolubov (i.e. superfluid) condensate. The (u-v) Bogolubov transformation,which is the Solomon's rotation in the algebra space, renders the Bose condensate into the Bogolubov one on the quantum level. On the classical level gauge transformation with a certain k of the Bose condensate (in NSE version) renders it into the partial wave with momentum k of the Bogolubov condensate. In this sense gauge function $g(x,t;\ \lambda_0\ )$ plays in solitonless sector a role of the Bogolubov transformation that connects two representations with different vacua. Thus we have connections on the algebraic basis (dynamical symmetry) and the geometric one (pseudo-spin CS).

geometric

algeb. rotat.	Bogolubov cond.	——	PS wave	gauge transfor.
	Bose condensate	——	NSE cond.	
	quantum level		quasiclassical level	

4. Let us make a minimal extension of the problem (17) via including Grassmannian variables (graded algebras su(2,1) or su(3)).

I) Noncompact version. We have the five parameter superalgebra ospu(1,1/1) :

$$E_j = \frac{1}{2}\begin{pmatrix} \tau_j & 0 \\ 0 & 0 \end{pmatrix}, \quad E_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\varkappa \\ 0 & 0 & 1 \\ 1 & \varkappa & 0 \end{pmatrix}, \quad E_5 = \frac{i}{2}\begin{pmatrix} 0 & 0 & \varkappa \\ 0 & 0 & 1 \\ 1 & -\varkappa & 0 \end{pmatrix}, \quad \varkappa = \pm 1$$
$$j = \overline{1,3}$$

with the commutation relations

$$\left[E_\alpha, E_\beta\right\} = E_\alpha E_\beta - (-1)^{P(\alpha)P(\beta)} E_\beta E_\alpha = i C_{\alpha\beta\gamma} E_\gamma .$$

For the curve (σ -model) version we have /9/

$$U = -i\lambda S \quad , \quad S = g^{-1} E_3 g$$

$$V = 2i\lambda^2 S + \lambda\left(2[S, S_x] + 3S^2 S_x S\right).$$

Since local U(1) transformations $g \to e^{iE_3\beta(x,t)} g$ preserve the matrix S, it belongs to the coset space : $S \in OSPU(1,1/1)/U(1,1)$ and may be parametrized as follows

$$S = \sum_{j=1}^{5} S_j E_j = \left(\begin{array}{cc|c} S^{(3)} & S^- & \bar{c} \\ -S^+ & -S^{(3)} & c \\ \hline c & \bar{c} & 0 \end{array}\right), \tag{21}$$

where $c = \frac{1}{2}(S_4 + iS_5)$, $S^{\pm} = -\frac{i}{2}(S_1 \pm iS_2)$, $p(c) = 1$, $p(S_i) = 0_{\,i=\overline{1,3}}$.

It follows from the definition that

$$S^3 = S \tag{22}$$

which is a generalization of the σ-model condition $S^2 = I$ in the theory of the SU(1,1) Heisenberg model /5,7/ to the superalgebra su(1,1/1).

The equations of motion are

$$iS_t = 2\left[S, S_{xx}\right] + 3\left(S^2 S_x, S\right)_x \tag{23}$$

and

$$H = \frac{1}{4}\int dx \, \mathrm{Str}\left(S_x^2 + 3\left[(S^2)_x\right]^2\right) \tag{24}$$

or

$$H = \frac{1}{2}\int dx \left\{ (S_x^{(3)})^2 - S_x^+ S_x^- - 2\varkappa \bar{c}_x c_x - 3\bar{c}_x c_x \bar{c} c - 6\varkappa\left(\varkappa S^{(3)}\bar{c} - iS^- c\right)_x \left(\varkappa S^{(3)} c + iS^+ \bar{c}\right)_x \right\}.$$

Symmetry transformations generated by Grassmann generators E_4 and E_5

$$R = \exp i(\vartheta_1 E_4 + \vartheta_2 E_5) \equiv \exp i(\vartheta q_2 - \bar{\vartheta} q_1), \quad \vartheta = \frac{1}{2}(\vartheta_1 + i\vartheta_2), \quad q_1 = -\varkappa(E_4 + iE_5), \quad q_2 = E_4 - iE_5$$

allow to dress the Bogolubov condensate (13) $S = \tilde{S}$:

$$S = \tilde{S}(1 + \bar{\vartheta}\vartheta), \quad c = \frac{i}{\varkappa}\left(-\tilde{S}^{(3)}\vartheta + \tilde{S}^- \bar{\vartheta}\right).$$

The plane NSE version is /8/

$$i\varphi_t + \varphi_{xx} - 2\left(|\varphi|^2 - \rho + 2\varkappa\bar{\psi}\psi\right)\varphi + 4i\psi\psi_x = 0 \tag{25a}$$

$$i\psi_t + 2\psi_{xx} - \left(|\varphi|^2 - \rho\right)\psi + i\varkappa\left(2\varphi\bar{\psi}_x + \varphi_x\bar{\psi}\right) = 0 \tag{25b}$$

with φ being the "boson" field and ψ the "fermion" (Grassmannian) one. Hamiltonian of the system (21) and the integral of the number of particles are

$$H = \int dx \left\{ |\varphi_x|^2 + (|\varphi|^2 - \rho + 2\bar{\Psi}\Psi)^2 + 8\bar{\Psi}_x \Psi_x - 4i(\bar{\varphi}\Psi\Psi_x + \varphi\bar{\Psi}\bar{\Psi}_x) \right\} \qquad (26a)$$

$$N = \int dx \left\{ (|\varphi|^2 - \rho) + 2\bar{\Psi}\Psi \right\} . \qquad (26b)$$

The latter is associated with the invariance of (22) under the global U(1) group : $\varphi \to \varphi' = e^{2i\alpha}\varphi$, $\Psi \to \Psi' = e^{i\alpha}\Psi$.

Gauge coupling :

$$-\frac{1}{8} \mathrm{Str}\left\{ (S_x)^2 + 3\left[(S^2)_x\right]^2 \right\} = |\varphi|^2 + 2\varkappa\bar{\Psi}\Psi . \qquad (27)$$

For two dimensional Grassmann algebra

$$\varphi = \varphi_1 + \varphi_2 \bar{\vartheta}\vartheta , \qquad \Psi = \Psi_1 \vartheta + \Psi_2 \bar{\vartheta}$$

we have (φ_i and Ψ_i are c-functions)

$$i\dot{\varphi}_1 + \varphi_{1xx} - 2(|\varphi_1|^2 - \rho)\varphi_1 = 0 \qquad (NSE) \qquad (28a)$$

$$i\dot{\varphi}_2 + \varphi_{2xx} - 4|\varphi_1|^2\varphi_2 - 2\varphi_1^2\bar{\varphi}_2 + 2\rho\varphi_2 - 4(|\Psi_1|^2 - |\Psi_2|^2)\varphi_1 \qquad (28b)$$
$$+ 4i(\Psi_2\Psi_{1x} - \Psi_1\Psi_{2x}) = 0$$

$$i\dot{\Psi}_j + 2\Psi_{jxx} - (|\varphi_1|^2 - \rho)\Psi_j - \left[2\varphi_1(\tau_1)_{j\ell}\bar{\Psi}_{\ell x} + \varphi_{1x}(\tau_1)_{j\ell}\bar{\Psi}_\ell\right] = 0 . \qquad (28c)$$

$$\varkappa = 1$$

where $\tau_{1,3}$ are the generators of the SU(1,1) group

$$\tau_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 .$$

Eq. (24a) is just the NSE and can be solved via the conventional linear problem

$$U = -i\lambda\tau_3 + i\begin{pmatrix} 0 & \bar{\varphi}_1 \\ -\varphi_1 & 0 \end{pmatrix} \qquad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \qquad (29)$$
$$V = -2\lambda U + i\begin{pmatrix} |\varphi_1|^2 - \rho & -i\bar{\varphi}_{1x} \\ -i\varphi_{1x} & \rho - |\varphi_1|^2 \end{pmatrix}$$

Eliminating λ from (17),(19) one gets for the vector $\Phi = (\bar{\varphi}_1, \varphi_2)^{tr}$ just the equations (28c). Thus, the Jost solutions in the frame-work of the NSE not only render the Bose condensate $|\psi|^2 = \rho$ into the pseudo-spin one, but also dress the Bose condensate with Grassmannian fields inside the superNSE to give again the Bogolubov condensate partial waves :

$$S^{(3)} = \frac{1+|\xi|^2}{1-|\xi|^2} , \quad S^+ = \frac{2\xi}{1-|\xi|^2} , \quad \xi = \Psi_2/\Psi_1 , \text{ where}$$

$\Phi = \begin{pmatrix} \bar{\Psi}_1 & \bar{\Psi}_2 \\ \Psi_2 & \Psi_1 \end{pmatrix}$ and Ψ_i are "superpartners".

There are the relations:

$$\text{NSE} \longrightarrow \text{Bose cond.} \xRightarrow[\text{Jost solutions}]{\text{solitonless}} \text{PS waves of HM}$$

$$\text{sNSE} \longrightarrow \text{Bose cond.} \Longrightarrow \text{dressed Bose cond.} \longrightarrow \text{PSW of HM}$$

Note that the pseudo-spin model (11) can be obtained from the conventional isotropic Heisenberg antiferromagnet

$$\hat{H} = \sum_i J \hat{\vec{S}}_i \hat{\vec{S}}_{i+1}$$

in representation of $\hat{S}^\pm = i\hat{K}^\pm$, $\hat{S}^{(3)} = \hat{K}^{(3)}$ via averaging over the pseudo-spin coherent states in continual limit:

$$H_{cl} = \langle \Psi | \hat{H} | \Psi \rangle ,$$

where $|\Psi\rangle$ is a pseudo-spin CS. Underline the <u>noncompact</u> symmetry of the system so obtained. Interesting to note as well that one soliton excitation of the k_0 partial wave has the spectrum, in the condensate limit $k_0 << 2\sqrt{\rho}$ /6,7/:

$$E_s = \sqrt{\rho}\,\sin P_s/2 \qquad , \qquad P_s \in [0, 2\pi]$$

which up to a numerical factor coincides with the exact formula of Des Cloizeaux and Pearson for the dispersion of antiferromagnons , the localized excitations of the Heisenberg antiferromagnet /10/. For small p_s, $E(p) = 2\sqrt{\rho}\, p$ is just the one-magnon spectrum (hole-like spectrum).

II) Compact version SU(2/1). There are two models we discuss here.

1) Eight-parameter subalgebra with one scalar (boson) complex field φ and a Grassmannian (fermion) field ψ . Generators are

$$\vec{T} = \frac{1}{2}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\vec{\sigma}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 0 \end{array}\right), \quad T_4 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\frac{1}{2} I} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right), \quad T_5 = \frac{1}{2}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{matrix}1\\0\end{matrix} \\ \hline 1 & 0 & 0 \end{array}\right)$$

$$T_6 = \frac{i}{2}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{matrix}-1\\0\end{matrix} \\ \hline 1 & 0 & 0 \end{array}\right), \quad T_7 = \frac{1}{2}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{matrix}0\\1\end{matrix} \\ \hline 0 & 1 & 0 \end{array}\right), \quad T_8 = \frac{i}{2}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{matrix}0\\-1\end{matrix} \\ \hline 0 & 1 & 0 \end{array}\right). \quad \mathrm{Str}\, T_i = 0 .$$

The <u>pláne</u> (sNSE) version is

$$\begin{aligned} & i\dot{\varphi} + \varphi_{xx} + 2\varphi\left(|\varphi|^2 + \bar{\psi}\psi\right) = 0 \\ & i\dot{\psi} + \psi_{xx} + 2\psi|\varphi|^2 = 0 \end{aligned} \qquad (30)$$

$$U = i\left(\begin{array}{cc|c} 0 & \bar{\varphi} & \bar{\psi} \\ \varphi & \lambda & 0 \\ \hline \psi & 0 & \lambda \end{array}\right), \quad V = i\left(\begin{array}{cc|c} -(|\varphi|^2 + \bar{\psi}\psi) & -\lambda\bar{\varphi} - i\bar{\varphi}_x & -\lambda\bar{\psi} - i\bar{\psi}_x \\ -\lambda\varphi + i\varphi_x & |\varphi|^2 - \lambda^2 & \varphi\bar{\psi} \\ \hline -\lambda\psi + i\psi_x & \psi\bar{\varphi} & \psi\bar{\psi} - \lambda^2 \end{array}\right) \qquad (31)$$

(compare to (24)).

The <u>curve</u> (σ-model) version is

$$S = g^{-1}\Sigma g , \quad \Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = i\lambda S , \; V = -i\lambda^2 S - \lambda[S, S_x]$$

$$S_t = i\left[S, S_{xx}\right], \qquad S^2 = S . \qquad (32)$$

Coupling: $\frac{1}{2}\mathrm{Str}\,(S_x)^2 = |\varphi|^2 + \bar{\psi}\psi$ (compare with (27)).

On two-dimensional Grassmann algebra

$$S_j = S_j^0 + S_j^{(1)}\bar{\vartheta}\vartheta , \quad c_1 = f_1\vartheta , \quad c_2 = f_2\vartheta ;$$

where

$$S = 2\sum_{j=1}^{4} S_j T_j + 2\sum_{j=5}^{8} c_j T_j$$

and

$$c_1 = c_5 + i c_6 , \quad c_2 = c_7 + i c_8$$

the conditions $S^2 = S$ read

$$S_4^0 = \frac{1}{2} , \quad S_4^{(1)} = \frac{1}{2}\left(|f_1|^2 + |f_2|^2\right) \qquad (33a)$$

$$(S_1^0)^2 + (S_2^0)^2 + (S_3^0)^2 = \frac{1}{4} . \qquad (33b)$$

One can easily see among these the usual σ-model condition occurring in the SU(2) Heisenberg ferromagnet for the magnetization vector in the classical limit. Matrix S is invariant under local transformations generated by T_3, T_4, T_7, and T_8 from SU(2/1) that implies the superNSE (30) to be invariant under the same but now global transformations.*)

2) Subalgebra with two Grassmannian fields

$$\vec{T} = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline \begin{matrix}0\\0\end{matrix} & \multicolumn{2}{c}{\vec{\sigma}} \end{array}\right), \quad T_4 = \frac{1}{2}\left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline \begin{matrix}0\\0\end{matrix} & \multicolumn{2}{c}{I} \end{array}\right), \quad T_5 = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline \begin{matrix}1\\0\end{matrix} & \multicolumn{2}{c}{0} \end{array}\right)$$

$$T_6 = \frac{i}{2}\left(\begin{array}{c|cc} 0 & 0 & -1 \\ \hline \begin{matrix}0\\1\end{matrix} & \multicolumn{2}{c}{0} \end{array}\right), \quad T_7 = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline \begin{matrix}1\\0\end{matrix} & \multicolumn{2}{c}{0} \end{array}\right), \quad T_8 = \frac{i}{2}\left(\begin{array}{c|cc} 0 & -1 & 0 \\ \hline \begin{matrix}1\\0\end{matrix} & \multicolumn{2}{c}{0} \end{array}\right)$$

with the plane (sNSE) version pure Grassmannian

$$\begin{aligned} & i\dot{\psi}_1 + \psi_{1xx} + 2\psi_1\bar{\psi}_2\psi_2 = 0 \\ & i\dot{\psi}_2 + \psi_{2xx} + 2\psi_2\bar{\psi}_1\psi_1 = 0 \end{aligned} \qquad (34)$$

$$U = i\left(\begin{array}{c|cc} 2\lambda & \bar{\psi}_1 & \bar{\psi}_2 \\ \hline \psi_1 & \lambda & 0 \\ \psi_2 & 0 & \lambda \end{array}\right), \quad V = i\left(\begin{array}{c|cc} 2\lambda^2 - \bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 & \lambda\bar{\psi}_1 - i\bar{\psi}_{1x} & \lambda\bar{\psi}_2 - i\bar{\psi}_{2x} \\ \hline \lambda\psi_1 + i\psi_{1x} & \lambda^2 + \psi_1\bar{\psi}_1 & \psi_1\bar{\psi}_2 \\ \lambda\psi_2 + i\psi_{2x} & \psi_2\bar{\psi}_1 & \lambda^2 + \psi_2\bar{\psi}_2 \end{array}\right). \qquad (35)$$

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*Details of this point are supposed to be published elsewhere.

The curve version

$$\Sigma = 2T_4 \,, \quad S = g^{-1}\Sigma g \in SU(1/2)/SU(2)\otimes U(1)$$

$$\mathcal{V} = i\lambda S \,, \quad V = i\lambda^2 S + 2\lambda S S_x - 3\lambda S_x \,,$$

$$S_t = \frac{1}{i}[S, S_{xx}] \,, \quad S^2 = 3S - 2I \qquad (36)$$

and the coupling

$$\frac{1}{2}\,\mathrm{Str}\,(S_x)^2 = \bar{\Psi}_1\Psi_1 + \bar{\Psi}_2\Psi_2$$

5. In the first part of the paper we have found intimate relations between Bogolubov condensate, noncompactness of the dynamical symmetry group (here SU(1,1) one and pseudo-spin ordering) superfluidity on the quantum level from one hand and quasiclassical repulsive NSE, pseudo-spin Heisenberg model (and related to it Heisenberg antiferromagnet) defined on the hyperboloid (noncompact symmetry SU(1,1) again) on the classical level from another one. One may then assume that noncompact classical pseudo-magnet associated so close with Bogolubov condensate is associated naturally with superfluidity too.

Compact dynamical symmetry, Fermi systems and Heisenberg ferromagnets (SU(2) and spin ordering) have as their classical descendants "attractive" NSE as a plane version and Landau-Lifshitz equation as the curve one.

Now we can speculate, for example, as follows : For the system defined on a graded algebra (superalgebra) one, in the spirit of Klauder /3/, Makhankov et al./11/ and Wiegman /12/, can derive corresponding classical system via averaging the former over proper generalized CS and then symmetry analysis of the boson sector (in curve version) should be carried out. Then compactness of this sector implies pure fermions or ferromagnetism initially to be (anyway at T=0). Superconductivity in such models is associated with fermion coupling (for Cooper pairs see, e.g. /13/). Noncompactness of the boson sector can imply boson (hole) superfluidity or (may be along with) antiferromagnetism. Influence of fermions (Grassmannian fields) needs to be investigated.

One of such models is the Hubbard model with strong electron correlations $U \to \infty$ /12/. Our models obtained above are distinguished due to the fact that they are integrable in one dimension. Their relation to classical descendants of the Hubbard model are now under study.

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Континуальная модель Гейзенберга, определенная на градуированных алгебрах SU(3) и SU(2,1)

Обсуждается связь между различными /на первый взгляд/ структурами такими, как Бозе и Боголюбовский конденсаты, ферромагнетик Гейзенберга и антиферромагнитные модели и их классические аналоги, модели Бозе-газа в квазиклассическом пределе и нелинейное уравнение Шредингера, включающее грассмановы поля. Обсуждаются также связи с теорией высокотемпературной сверхпроводимости.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Makhankov V.G., Pashaev O.K. E17-88-517

Continual Heisenberg Models Defined on Graded SU(3) and SU(2,1) Algebras

In this report we discuss the connections occurring between such different (at first sight) structures as Bose and Bogolubov condensates, Heisenberg magnet and antiferromagnet models and their classical descendants, Bose-gas models in quasiclassical limit and nonlinear Schrödinger equations involving Grassmannian fields. Speculations are given on related topics of high temperature superconductivity.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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