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INHOMOGENEOUS ISING MODEL ON THE BETHE LATTICE

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1. INTRODUCTION

So far, much of interest has been directed towards the study of low-dimensional systems. Many of them, in spite of the restricted applicability, correctly describe a number of phenomena associated with higher dimensions [1]. They are usually solvable exactly. However, when the system becomes inhomogeneous, the space variation of the model parameters prevents to express the free energy explicitly except in some special examples.

In the theory of the one-dimensional inhomogeneous Ising model, there have been examined the cases of:

i) nonconstant interactions and a uniform external field at zero temperature [2].

ii) the fixed nearest-neighbour interactions and a site-dependent field [3-6].

iii) variable both interactions and a field [7].

The partition function and its derivatives for some special models were obtained in ref. [6] only at a specific temperature. Otherwise, the results were expressed in terms of continued fractions [5], nonlinear recurrence relations [7]. More encouraging results were obtained in the so-called inverse problem initiated by Percus [3]. It consists in finding the external potential needed to evoke a given magnetization profile. The original solution of Percus [3,4] for constant nearest-neigbour interactions has been generalized recently in ref. [7] to variable nearest-neighbour bonds.

In this paper, we report a new approach to inhomogeneous Ising models. In contrast to the above methods, it works from the beginning to end with the quantities having a clear physical meaning. It turns out that the method is a very convenient means for formal solving the inverse problem for a large class of models.

An introductory outline of the theory for a one-dimensional Ising chain is presented in section 2. Within the framework of the proposed formulation it becomes apparent how to solve the inverse problem (section 3). A natural and straightforward extension of the method to the Bethe lattice with arbitrary coordination number q is given with a discussion of its applicability in section 4.

2. FORMULATION FOR A ONE-DIMENSIONAL INHOMOGENEOUS ISING MODEL

We are concerned with an Ising chain of N spins. Its Hamiltonian is

$$\mathcal{H} = -\sum_{n=1}^{N-1} J_n s_n s_{n+1} - \sum_{n=1}^{N} H_n s_n , \qquad (2.1)$$

where $s_n (= \pm 1)$ denotes the usual Ising spin variable at site n, J_n a nonconstant interaction that couples s_n to s_{n+1} and H_n a site-dependent external field. The equilibrium statistical properties of the system are determined by the partition function

$$Z_N(K_1,\dots K_{N-1};h_1,\dots h_N) = \sum_{\{s\}_N} \exp\Big(\sum_{n=1}^{N-1} K_n s_n s_{n+1} + \sum_{n=1}^{N} h_n s_n\Big) , \qquad (2.2)$$

where $K_n = \beta J_n$, $h_n = \beta H_n$, β is the reciprocal temperature and the summation runs over all possible configurations of N spins. A second thermodynamic quantity which we shall concentrate on is the spin expectation value,

$$\langle s_i \rangle = \frac{S_i(K_1,\dots K_{N-1};h_1,\dots h_N)}{Z_N(K_1,\dots K_{N-1};h_1,\dots h_N)} , \qquad (2.3a)$$

$$S_i(K_1,\dots K_{N-1};h_1,\dots h_N) = \sum_{\{s\}_N} s_i \exp\Big(\sum_{n=1}^{N-1} K_n s_n s_{n+1} + \sum_{n=1}^{N} h_n s_n\Big) . \qquad (2.3b)$$

In the present method, we shall eliminate consecutively spins from the chain and investigate the thermodynamic quantities of interest of the remaining fragments. We start with the spin at site 1. Using the identity

$$\exp(K_1 s_1 s_2) = \cosh K_1 + s_1 s_2 \sinh K_1 \quad \text{for } s_1, s_2 = \pm 1 ,$$

and taking in (2.2) the sum over spin variable s_1, we easily find

$$Z_N(K_1,\dots K_{N-1};h_1,\dots h_N) = 2\cosh K_1 \cosh h_1 \{1 + \tanh h_1 \tanh K_1 \langle s_2 \rangle_1 \} Z_{N-1}(K_2,\dots K_{N-1};h_2,\dots h_N) . \qquad (2.4)$$

Here,

$$Z_{N-1}(K_2,\dots K_{N-1};h_2,\dots h_N) = \sum_{\{s\}_{N-1}} \exp\Big(\sum_{n=2}^{N-1} K_n s_n s_{n+1} + \sum_{n=2}^{N} h_n s_n\Big) , \qquad (2.5a)$$

$$\langle s_2\rangle_1 = \frac{\sum_{\{s\}_{N-1}} s_2 \exp\left(\sum_{n=2}^{N-1} K_n s_n s_{n+1} + \sum_{n=2}^{N} h_n s_n\right)}{\sum_{\{s\}_{N-1}} \exp\left(\sum_{n=2}^{N-1} K_n s_n s_{n+1} + \sum_{n=2}^{N} h_n s_n\right)}, \tag{2.5b}$$

represent the thermodynamic quantities of the chain without site 1. Following the same procedure in the case of $S_1(K_1,\dots K_{N-1}; h_1,\dots h_N)$, we arrive at

$$S_1(K_1,\dots K_{N-1};h_1,\dots h_N) = 2\cosh K_1 \cosh h_1 \{\tanh h_1 + \tanh K_1 \langle s_2\rangle_1\} Z_{N-1}(K_2,\dots K_{N-1};h_2,\dots h_N), \tag{2.6}$$

and so

$$\langle s_1\rangle = \frac{\tanh h_1 + \tanh K_1 \langle s_2\rangle_1}{1+\tanh h_1 \tanh K_1 \langle s_2\rangle_1}. \tag{2.7}$$

We proceed further in this manner for spins 2,...,N-1,N and readily get

$$\begin{aligned} Z_{N-1}(K_2,\dots K_{N-1};h_2,\dots h_N) &= 2\cosh K_2 \cosh h_2 \{1 + \tanh h_2 \tanh K_2 \langle s_3\rangle_2\} Z_{N-2}(K_3,\dots K_{N-1};h_3,\dots h_N),\\ &\dots\dots\dots\\ Z_2(K_{N-1};h_{N-1},h_N) &= 2\cosh K_{N-1}\cosh h_{N-1} \{1+\tanh h_{N-1}\cdot \tanh K_{N-1} \langle s_N\rangle_{N-1}\} Z_1(h_N),\\ Z_1(h_N) &= 2\cosh h_N, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \langle s_2\rangle_1 &= \frac{\tanh h_2 + \tanh K_2 \langle s_3\rangle_2}{1+\tanh h_2 \tanh K_2 \langle s_3\rangle_2},\\ &\dots\dots\dots\\ \langle s_{N-1}\rangle_{N-2} &= \frac{\tanh h_{N-1} + \tanh K_{N-1}\langle s_N\rangle_{N-1}}{1+\tanh h_{N-1}\tanh K_{N-1}\langle s_N\rangle_{N-1}},\\ \langle s_N\rangle_{N-1} &= \tanh h_N. \end{aligned} \tag{2.9}$$

Note that the elimination of the n th spin in $\langle s_{n+1}\rangle_n$ means the simultaneous elimination of all spins with indices lower than n.

To simplify the formalism we introduce the auxiliary quantities $\{a_n\}_{n=1}^N$ as follows

$$\tanh a_n = \tanh K_n \langle s_{n+1}\rangle_n \qquad (K_N \equiv 0). \tag{2.10}$$

Then using

$$\tanh(x+y) = (\tanh x + \tanh y)/(1+\tanh x \tanh y)$$

we rewrite (2.4,8), (2.7,9) in a more convenient form

$$Z_N(K_1,\dots K_{N-1};h_1,\dots h_N) = 2^N\left(\prod_{n=1}^N \cosh K_n \cosh h_n\right)\cdot \prod_{n=1}^N (1+\tanh h_n \tanh a_n), \tag{2.11}$$

$$\begin{aligned} \langle s_1\rangle &= \tanh(h_1 + a_1),\\ \langle s_2\rangle_1 &= \tanh(h_2+a_2),\\ &\dots\dots\dots\\ \langle s_{N-1}\rangle_{N-2} &= \tanh(h_{N-1}+a_{N-1}),\\ \langle s_N\rangle_{N-1} &= \tanh(h_N + a_N), \end{aligned} \tag{2.12}$$

where the quantities $\{a_n\}_{n=1}^N$ satisfy the recursion relations

$$\begin{aligned} \tanh a_1 &= \tanh K_1 \tanh(h_2+a_2),\\ \tanh a_2 &= \tanh K_2 \tanh(h_3+a_3),\\ &\dots\dots\dots\\ \tanh a_{N-1} &= \tanh K_{N-1} \tanh(h_N + a_N),\\ a_N &= 0. \end{aligned} \tag{2.13}$$

Analogously, performing the successive elimination of spins starting from site N, ending at site 1, and defining $\tanh b_n = \tanh K_{n-1} \langle s_{n-1}\rangle_n$ (n = 0,1,...,N; $K_{-1} \equiv 0$), one easily derives

$$Z_N(K_1,\dots K_{N-1};h_1,\dots h_N) = 2^N\left(\prod_{n=1}^N \cosh K_n \cosh h_n\right)\cdot \prod_{n=1}^N (1+\tanh h_n \tanh b_n), \tag{2.14}$$

$$
\begin{aligned}
&\langle s_N \rangle = \tanh(h_N + b_N)\,,\\
&\langle s_{N-1} \rangle_N = \tanh(h_{N-1} + b_{N-1})\,,\\
&\ldots\ldots\ldots\ldots\\
&\langle s_2 \rangle_3 = \tanh(h_2 + b_2)\,,\\
&\langle s_1 \rangle_2 = \tanh(h_1 + b_1)\,,
\end{aligned}
\tag{2.15}
$$

where the auxiliary quantities $\{b_n\}_{n=1}^{N}$ are given by

$$
\begin{aligned}
&\tanh b_N = \tanh K_{N-1} \tanh(h_{N-1} + b_{N-1})\,,\\
&\tanh b_{N-1} = \tanh K_{N-2} \tanh(h_{N-2} + b_{N-2})\,,\\
&\ldots\ldots\ldots\ldots\\
&\tanh b_2 = \tanh K_1 \tanh(h_1 + b_1)\,,\\
&b_1 = 0\,.
\end{aligned}
\tag{2.16}
$$

3. THE INVERSE PROBLEM IN ONE DIMENSION

The recursion schemes (2.13) and (2.16) determine unambiguously the respective sequences $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$, whose knowledge should enable us to compute the partition function directly from (2.11) or (2.14). However, the variables $\{a_n\}_{n=1}^{N}$, $\{b_n\}_{n=1}^{N}$ are highly nonlinear in the model parameters $\{K_n\}_{n=1}^{N-1}$, $\{h_n\}_{n=1}^{N}$ and so the exact solution can be found only in some special cases.

On the other hand, the procedure for solving the inverse problem is straightforward. To express a specific field, h_i for example, as a function of the magnetization profile and the nonconstant couplings, we will eliminate the spin i th from the system and investigate the consequent modification of naturally chosen quantities $Z(K_1,\ldots,K_{N-1};\, h_1,\ldots,h_N)$, $S_i(K_1,\ldots,K_{N-1};\, h_1,\ldots,h_N)$, $S_{i+1}(K_1,\ldots,K_{N-1};\, h_1,\ldots,h_N)$ and $S_{i-1}(K_1,\ldots,K_{N-1};\, h_1,\ldots,h_N)$. Using the simple technique given in section 2 we have

$$
\begin{aligned}
Z_N(K_1,\ldots K_{N-1};h_1,\ldots h_N) &= 2\cosh K_{i-1}\cosh K_i \cosh h_i\,\{1 +\\
&+ \tanh h_i \tanh K_i \langle s_{i+1}\rangle_i + \tanh h_i \tanh K_{i-1}\langle s_{i-1}\rangle_i + \tanh K_{i-1}\,\cdot\\
&\tanh K_i \langle s_{i-1} s_{i+1}\rangle_i\} Z_{N-1}(K_1,\ldots K_{i-2}, K_{i+1},\ldots K_{N-1};\, h_1,\ldots h_{i-1}, h_{i+1},\ldots h_N)\,.
\end{aligned}
\tag{3.1a}
$$

$$
\begin{aligned}
S_i(K_1,\ldots K_{N-1};h_1,\ldots h_N) &= 2\cosh K_{i-1}\cosh K_i\cosh h_i\,\cdot\\
&\{\tanh h_i + \tanh K_i \langle s_{i+1}\rangle_i + \tanh K_{i-1}\langle s_{i-1}\rangle_i + \tanh h_i\,\cdot\\
&\tanh K_{i-1}\tanh K_i \langle s_{i-1} s_{i+1}\rangle_i\} Z_{N-1}(K_1,\ldots K_{i-2}, K_{i+1},\ldots K_{N-1};\\
&h_1,\ldots h_{i-1}, h_{i+1},\ldots h_N)\,,
\end{aligned}
\tag{3.1b}
$$

$$
\begin{aligned}
S_{i+1}(K_1,\ldots K_{N-1};h_1,\ldots h_N) &= 2\cosh K_{i-1}\cosh K_i \cosh h_i\\
&\{\langle s_{i+1}\rangle_i + \tanh h_i \tanh K_i + \tanh h_i \tanh K_{i-1}\langle s_{i-1} s_{i+1}\rangle_i +\\
&+ \tanh K_{i-1}\tanh K_i \langle s_{i-1}\rangle_i\} Z_{N-1}(K_1,\ldots K_{i-2}, K_{i+1},\ldots K_{N-1};\\
&h_1,\ldots h_{i-1}, h_{i+1},\ldots h_N)\,,
\end{aligned}
\tag{3.1c}
$$

$$
\begin{aligned}
S_{i-1}(K_1,\ldots K_{N-1};h_1,\ldots h_N) &= 2\cosh K_{i-1}\cosh K_i\cosh h_i\,\cdot\\
&\{\langle s_{i-1}\rangle_i + \tanh h_i \tanh K_i \langle s_{i-1} s_{i+1}\rangle_i + \tanh h_i \tanh K_{i-1} +\\
&+ \tanh K_{i-1}\tanh K_i \langle s_{i+1}\rangle_i\} Z_{N-1}(K_1,\ldots K_{i-2}, K_{i+1},\ldots K_{N-1};\\
&h_1,\ldots h_{i-1}, h_{i+1},\ldots h_N)\,.
\end{aligned}
\tag{3.1d}
$$

The spins localized at sites $i-1$, $i+1$ are statistically independent in the absence of the i th spin and, therefore, $\langle s_{i-1}s_{i+1}\rangle_i = \langle s_{i-1}\rangle_i \langle s_{i+1}\rangle_i$. Then, using the definitions $\tanh a_i = \tanh K_i \langle s_{i+1}\rangle_i$, $\tanh b_i = \tanh K_{i-1}\langle s_{i-1}\rangle_i$ and the general formulae for computing the spin excitation value, we obtain from (3.1a-d),

$$
\langle s_i\rangle = \frac{\tanh h_i + \tanh a_i + \tanh b_i + \tanh h_i \tanh a_i \tanh b_i}{1 + \tanh h_i \tanh a_i + \tanh h_i \tanh b_i + \tanh a_i \tanh b_i}\,,
\tag{3.2a}
$$

$$
\tanh K_i \langle s_{i+1}\rangle = \frac{\tanh a_i + \tanh h_i \tanh a_i \tanh b_i + \tanh^2 K_i(\tanh h_i + \tanh b_i)}{1 + \tanh h_i \tanh a_i + \tanh h_i \tanh b_i + \tanh a_i \tanh b_i}\,,
\tag{3.2b}
$$

$$
\tanh K_{i-1}\langle s_{i-1}\rangle = \frac{\tanh b_i + \tanh h_i \tanh a_i \tanh b_i + \tanh^2 K_{i-1}(\tanh h_i + \tanh a_i)}{1 + \tanh h_i \tanh a_i + \tanh h_i \tanh b_i + \tanh a_i \tanh b_i}\,.
\tag{3.2c}
$$

This set of equations contains three unknown h_i, a_i, b_i depending on the fixed average spin values $\langle s_{i-1}\rangle$, $\langle s_i\rangle$, $\langle s_{i+1}\rangle$ and coupling constants K_{i-1}, K_i. The solution of the inverse problem is then given by eliminating a_i and b_i from the above set. For this purpose we rewrite (3.2a-c) as follows,

$$\langle s_i\rangle = \tanh(h_i + a_i + b_i)\,, \tag{3.3a}$$

$$\frac{\langle s_i\rangle - \tanh K_i \langle s_{i+1}\rangle}{(1-\tanh^2 K_i)\langle s_i\rangle} = \frac{\tanh(h_i + b_i)}{\tanh(h_i+b_i) + \tanh a_i}\,, \tag{3.3b}$$

$$\frac{\langle s_i\rangle - \tanh K_{i-1} \langle s_{i-1}\rangle}{(1-\tanh^2 K_{i-1})\langle s_i\rangle} = \frac{\tanh(h_i + a_i)}{\tanh(h_i+a_i) + \tanh b_i}\,, \tag{3.3c}$$

By combining (3.3a) with (3.3b) and (3.3c), we find the respective second-degree equations for $\tanh a_i$ and $\tanh b_i$,

$$[\langle s_i\rangle - \tanh K_i \langle s_{i+1}\rangle]\tanh^2 a_i - (1-\tanh^2 K_i)\tanh a_i + \tanh K_i\,[\langle s_{i+1}\rangle - \tanh K_i \langle s_i\rangle] = 0, \tag{3.4a}$$

$$[\langle s_i\rangle - \tanh K_{i-1} \langle s_{i-1}\rangle]\tanh^2 b_i - (1-\tanh^2 K_{i-1})\tanh b_i + \tanh K_{i-1}\,[\langle s_{i-1}\rangle - \tanh K_{i-1} \langle s_i\rangle] = 0\,. \tag{3.4b}$$

Consequently,

$$\tanh a_i = x_i^+ = \frac{(1-\tanh^2 K_i) - \sqrt{\Delta_i^+}}{2[\langle s_i\rangle - \tanh K_i \langle s_{i+1}\rangle]}\,, \tag{3.5a}$$

$$\Delta_i^+ = (1-\tanh^2 K_i)^2 - 4\tanh K_i\,[\langle s_i\rangle - \tanh K_i\langle s_{i+1}\rangle]\cdot[\langle s_{i+1}\rangle - \tanh K_i\langle s_i\rangle], \tag{3.5b}$$

and

$$\tanh b_i = x_i^- = \frac{(1-\tanh^2 K_{i-1}) - \sqrt{\Delta_i^-}}{2[\langle s_i\rangle - \tanh K_{i-1} \langle s_{i-1}\rangle]}\,, \tag{3.6a}$$

$$\Delta_i^- = (1-\tanh^2 K_{i-1})^2 - 4\tanh K_{i-1}\,[\langle s_i\rangle - \tanh K_{i-1}\langle s_{i-1}\rangle]\cdot[\langle s_{i-1}\rangle - \tanh K_{i-1}\langle s_i\rangle], \tag{3.6b}$$

the sign of the square root being fixed by the conditions

$$\lim_{K_i\to 0} a_i(K_i, \langle s_i\rangle, \langle s_{i+1}\rangle) = 0\,,$$

$$\lim_{K_{i-1}\to 0} b_i(K_{i-1}, \langle s_i\rangle, \langle s_{i-1}\rangle) = 0\,.$$

Finally, taking into account (3.3a), (3.5), (3.6), we have respectively

$$h_i + a_i + b_i = \frac{1}{2}\ln\left(\frac{1+\langle s_i\rangle}{1-\langle s_i\rangle}\right), \tag{3.7a}$$

$$a_i = \frac{1}{2}\ln\left(\frac{1+x_i^+}{1-x_i^+}\right), \tag{3.7b}$$

$$b_i = \frac{1}{2}\ln\left(\frac{1+x_i^-}{1-x_i^-}\right), \tag{3.7c}$$

from which the desired field variable h_i is obtained as

$$h_i = \frac{1}{2}\ln\left\{\left(\frac{1+\langle s_i\rangle}{1-\langle s_i\rangle}\right)\left(\frac{1-x_i^+}{1+x_i^+}\right)\left(\frac{1-x_i^-}{1+x_i^-}\right)\right\}. \tag{3.8}$$

Here, the edge effects are reflected through $x_1^- = 0$, $x_N^+ = 0$. It can be shown after lengthy calculations that the present symmetric solution of the one-dimensional inverse problem coincides with the previous one by Tejero [7]. However, as we will see, the present formulation permits us to solve the inverse problem for more complicated structures.

4. THE INVERSE PROBLEM FOR THE ISING MODEL ON THE BETHE LATTICE

Now, we will extend the method to the Bethe lattice of N spins with the coordination number q = 3. A typical situation is drawn in fig.1, where the reference spin at site 0 is coupled to the surrounding 1 th, 2 th, 3 th spins by the dimensionless interactions K_1, K_2, K_3. The remaining bounds, whose values and positions on the Bethe lattice are irrelevant in the problem, will be denoted by K_4, ... Our objective is to find the external field at site 0, h_0, needed to produce the magne-

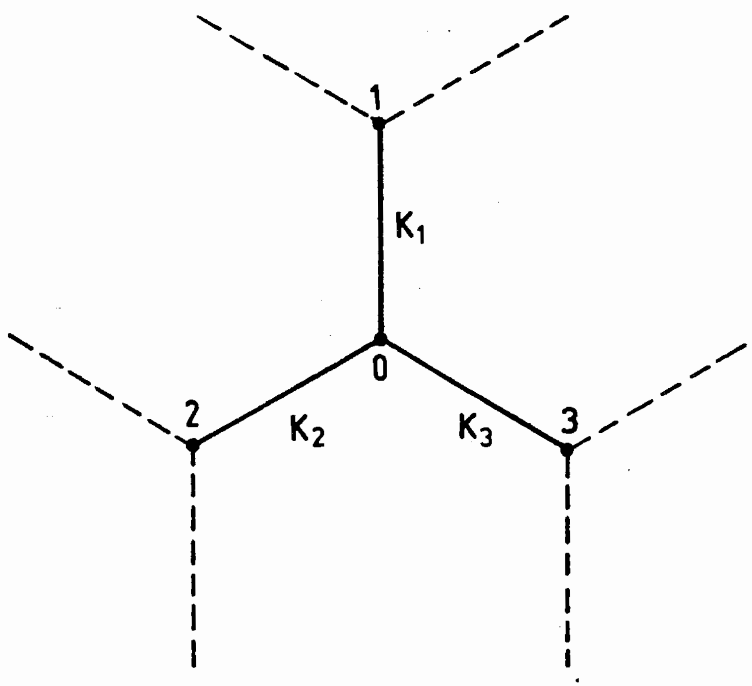

Fig. 1

tizations $\langle s_0\rangle$, $\langle s_1\rangle$, $\langle s_2\rangle$, $\langle s_3\rangle$, ... for the fixed nonconstant nearest-neighbour interactions K_1, K_2, K_3, ...

In the spirit of our approach, we eliminate the 0 th spin from the thermodynamic quantities of interest, namely

$$\begin{aligned} Z_N(K_1,\dots;h_0,\dots) &= 2\cosh K_1 \cosh K_2 \cosh K_3 \cosh h_0 \cdot \\ &\{1 + \tanh h_0 \tanh K_1 \langle s_1\rangle_0 + \tanh h_0 \tanh K_2 \langle s_2\rangle_0 + \tanh h_0 \cdot \\ &\tanh K_3 \langle s_3\rangle_0 + \tanh K_1 \tanh K_2 \langle s_1 s_2\rangle_0 + \tanh K_1 \tanh K_3 \cdot \\ &\langle s_1 s_3\rangle_0 + \tanh K_2 \tanh K_3 \langle s_2 s_3\rangle_0 + \tanh h_0 \tanh K_1 \tanh K_2 \cdot \\ &\tanh K_3 \langle s_1 s_2 s_3\rangle_0 \} Z_{N-1}(K_4,\dots;h_1,\dots)\,, \end{aligned} \tag{4.1a}$$

$$\begin{aligned} S_0(K_1,\dots;h_0,\dots) &= 2\cosh K_1 \cosh K_2 \cosh K_3 \cosh h_0 \cdot \\ &\{\tanh h_0 + \tanh K_1 \langle s_1\rangle_0 + \tanh K_2 \langle s_2\rangle_0 + \tanh K_3 \langle s_3\rangle_0 + \\ &\tanh h_0 \tanh K_1 \tanh K_2 \langle s_1 s_2\rangle_0 + \tanh h_0 \tanh K_1 \tanh K_3 \cdot \\ &\langle s_1 s_3\rangle_0 + \tanh h_0 \tanh K_2 \tanh K_3 \langle s_2 s_3\rangle_0 + \tanh K_1 \tanh K_2 \cdot \\ &\tanh K_3 \langle s_1 s_2 s_3\rangle_0 \} Z_{N-1}(K_4,\dots;h_1,\dots)\,, \end{aligned} \tag{4.1b}$$

$$\begin{aligned} S_1(K_1,\dots;h_0,\dots) &= 2\cosh K_1 \cosh K_2 \cosh K_3 \cosh h_0 \cdot \\ &\{\langle s_1\rangle_0 + \tanh h_0 \tanh K_1 + \tanh h_0 \tanh K_2 \langle s_1 s_2\rangle_0 + \tanh h_0 \cdot \\ &\tanh K_3 \langle s_1 s_3\rangle_0 + \tanh K_1 \tanh K_2 \langle s_2\rangle_0 + \tanh K_1 \tanh K_3 \cdot \\ &\langle s_3\rangle_0 + \tanh K_2 \tanh K_3 \langle s_1 s_2 s_3\rangle_0 + \tanh h_0 \tanh K_1 \tanh K_2 \cdot \\ &\tanh K_3 \langle s_2 s_3\rangle_0 \} Z_{N-1}(K_4,\dots;h_1,\dots)\,, \end{aligned} \tag{4.1c}$$

$$\begin{aligned} S_2(K_1,\dots;h_0,\dots) &= 2\cosh K_1 \cosh K_2 \cosh K_3 \cosh h_0 \cdot \\ &\{\langle s_2\rangle_0 + \tanh h_0 \tanh K_1 \langle s_1 s_2\rangle_0 + \tanh h_0 \tanh K_2 + \tanh h_0 \cdot \\ &\tanh K_3 \langle s_2 s_3\rangle_0 + \tanh K_1 \tanh K_2 \langle s_1\rangle_0 + \tanh K_1 \tanh K_3 \cdot \\ &\langle s_1 s_2 s_3\rangle_0 + \tanh K_2 \tanh K_3 \langle s_3\rangle_0 + \tanh h_0 \tanh K_1 \tanh K_2 \cdot \\ &\tanh K_3 \langle s_1 s_3\rangle_0 \} Z_{N-1}(K_4,\dots;h_1,\dots)\,, \end{aligned} \tag{4.1d}$$

$$\begin{aligned} S_3(K_1,\dots;h_0,\dots) &= 2\cosh K_1 \cosh K_2 \cosh K_3 \cosh h_0 \cdot \\ &\{\langle s_3\rangle_0 + \tanh h_0 \tanh K_1 \langle s_1 s_3\rangle_0 + \tanh h_0 \tanh K_2 \langle s_2 s_3\rangle_0 + \\ &+ \tanh h_0 \tanh K_3 + \tanh K_1 \tanh K_2 \langle s_1 s_2 s_3\rangle_0 + \tanh K_1 \cdot \\ &\tanh K_3 \langle s_1\rangle_0 + \tanh K_2 \tanh K_3 \langle s_2\rangle_0 + \tanh h_0 \tanh K_1 \cdot \\ &\tanh K_2 \tanh K_3 \langle s_1 s_2\rangle_0 \} Z_{N-1}(K_4,\dots;h_1,\dots)\,. \end{aligned} \tag{4.1e}$$

Eliminating site 0, the 1 th, 2 th, 3 th spins belong to different lattice fragments which have no common bounds. They are consequently statistically independent and so $\langle s_1 s_2\rangle_0 = \langle s_1\rangle_0 \langle s_2\rangle_0$, $\langle s_1 s_3\rangle_0 = \langle s_1\rangle_0\langle s_3\rangle_0$, $\langle s_2 s_3\rangle_0 = \langle s_2\rangle_0 \langle s_3\rangle_0$, $\langle s_1 s_2 s_3\rangle_0 = \langle s_1\rangle_0 \langle s_2\rangle_0 \cdot \langle s_3\rangle_0$. Thus, introducing the auxilliary variables $\{a_n\}_{n=1}^{3}$ by $\tanh a_n = \tanh K_n \langle s_n\rangle_0$ and after some algebra, we find

$$\langle s_0\rangle = \tanh(h_0 + a_1 + a_2 + a_3)\,, \tag{4.2a}$$

$$\frac{\langle s_0\rangle - \tanh K_1 \langle s_1\rangle}{(1-\tanh^2 K_1)\langle s_0\rangle} = \frac{\tanh(h_0 + a_2 + a_3)}{\tanh(h_0 + a_2 + a_3) + \tanh a_1}\,, \tag{4.2b}$$

$$\frac{\langle s_0\rangle - \tanh K_2 \langle s_2\rangle}{(1-\tanh^2 K_2)\langle s_0\rangle} = \frac{\tanh(h_0 + a_1 + a_3)}{\tanh(h_0 + a_1 + a_3) + \tanh a_2}\,, \tag{4.2c}$$

$$\frac{\langle s_0\rangle - \tanh K_3 \langle s_3\rangle}{(1-\tanh^2 K_3)\langle s_0\rangle} = \frac{\tanh(h_0 + a_1 + a_2)}{\tanh(h_0 + a_1 + a_2) + \tanh a_3}\,. \tag{4.2d}$$

Analogously to section 3, by combining (4.2a) with (4.2b), (4.2c) and (4.2d) we get the second-order equations for $\tanh a_n$ $(n = 1,2,3)$,

$$[\langle s_0\rangle - \tanh K_n \langle s_n\rangle]\tanh^2 a_n - (1-\tanh^2 K_n)\tanh a_n + \tanh K_n[\langle s_n\rangle - \tanh K_n\langle s_0\rangle] = 0\,. \tag{4.3}$$

Respecting the right sign of the square root, their solutions are yielded by

$$\tanh a_n = x_n = \frac{(1-\tanh^2 K_n) - \sqrt{\Delta_n}}{2[\langle s_0\rangle - \tanh K_n\langle s_n\rangle]}\,, \qquad n = 1,2,3\,, \tag{4.4a}$$

$$\Delta_n = (1-\tanh^2 K_n)^2 - 4\tanh K_n[\langle s_0\rangle - \tanh K_n\langle s_n\rangle]\cdot[\langle s_n\rangle - \tanh K_n\langle s_0\rangle]\,. \tag{4.4b}$$

Since

$$h_0 + \sum_{n=1}^{3} a_n = \frac{1}{2}\ln\left(\frac{1+\langle s_0\rangle}{1-\langle s_0\rangle}\right)\,, \tag{4.5a}$$

$$a_n = \frac{1}{2}\ln\left(\frac{1+x_n}{1-x_n}\right)\,, \qquad n = 1,2,3\,, \tag{4.5b}$$

the field variable h_0 required to evoke the given magnetization profile is written

$$h_0 = \frac{1}{2}\ln\left(\frac{1+\langle s_0\rangle}{1-\langle s_0\rangle}\right) + \frac{1}{2}\sum_{n=1}^{3}\ln\left(\frac{1-x_n}{1+x_n}\right)\,. \tag{4.6}$$

As expected, it depends only on the spin thermal averages $\langle s_0\rangle$, $\langle s_1\rangle$, $\langle s_2\rangle$ and the couplings K_1, K_2, K_3.

The method can be generalized to the Bethe lattice with arbitrary coordination number q ($q = 2$ represents the ordinary one-dimensional chain), when the reference spin at site 0 interacts with the 1 th, 2 th, ... q th spins by $K_1, K_2, \ldots K_q$, respectively. The result is

$$h_0 = \frac{1}{2}\ln\left(\frac{1+\langle s_0\rangle}{1-\langle s_0\rangle}\right) + \frac{1}{2}\sum_{n=1}^{q}\ln\left(\frac{1-x_n}{1+x_n}\right)\,, \tag{4.7}$$

where the quantities x_n ($n = 1,\ldots,q$) are defined by (4.4a,b).

In conclusion, the present theory for solving the inverse problem can be formally extended to an arbitrary cluster of spins. In dependence on the situation, one must simultaneously eliminate a group of spins chosen so that their neighbouring spins become statistically independent. In our formulation this condition permits us to decouple two-, three-, ... spin correlations generated during the procedure and in this way to obatin a closed system of resulting equations. To be more specific, let us consider the cluster shown in fig.2. The spins are coupled to one another by nearest- as well as next-to-nearest-neighbour interactions. From the point of view of the present method it is inevitable to eliminate from the thermodynamic quantities of interest the spins at sites 2 and 4. Performing the whole procedure, we arrive at the final complete set of five equations which relates the magnetizations $\langle s_1\rangle$, $\langle s_2\rangle$, $\langle s_3\rangle$, $\langle s_4\rangle$, $\langle s_5\rangle$ to the unknowns h_2, h_4, $\langle s_1\rangle_2$, $\langle s_3\rangle_{2,4}$,

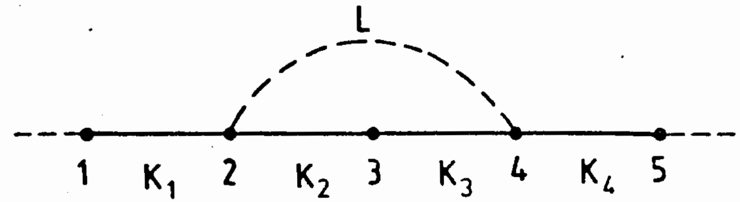

Fig. 2

$\langle s_5 \rangle_4$. The h_3 is determined simply by $\langle s_3 \rangle_{2,4} = \tanh h_3$, while the results for h_1, h_5 coincide with the ones obtained for the one-dimensional Ising chain.

We believe that the proposed approach will provide further new results for the inhomogeneous models not studied so far.

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Шамай Л. E17-88-380

Неоднородная модель Изинга на решетке Бете

Предложен новый подход к модели Изинга с непостоянными взаимодействиями и магнитным полем. Он представляет прямой путь для решений обратных проблем. Внешнее поле, которое создает данный магнетизационный профиль, получено для систем со спином 1/2 на решетке Бете с произвольным числом соседей q.

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Šamaj L. E17-88-380

Inhomogeneous Ising Model on the Bethe Lattice

A new approach to the Ising model with nonconstant interactions and a site-dependent external field is presented. It represents a straightforward way for solving the inverse problems. The external field required to produce a given magnetization profile is given explicitly for spin 1/2 system on the Bethe lattice with an arbitrary coordination number q.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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