объединенный
институт
ядерных
исследований
дубна



E17-87-885

A.A.Bakasov

# PHYSICAL REASONS FOR LIAPUNOV'S STABILITY OF RADIATION REGIME OF A PREVIOUSLY INVERTED SYSTEM

Submitted to "Physics Letters"

1987

## 1. "Superradiant threshold" and Liapunov's stability

The superradiation theory, as the laser theory, contains such values of some physical quantities that are sometimes assumed to correspond to the transition of a system from one regime to another. In the laser theory the intensity of pumping may be a parameter taking critical values. Upon exceeding the first critical level of pumping laser emits a sinusoidal wave that is replaced by ultrashort pulses when the second threshold is reached [1,2]. The solutions of differential equations describing dynamics of such a laser become unstable at critical values of the pumping intensity: first, a stable focus is realised, then it is replaced by a limit cycle and further by the torus [3]. In this case the pumping intensity is the master external parameter of a system.

For superradiant systems one can observe a somewhat different situation. An object of the investigation is usually a previously inverted system that may go over to the ground state in different ways. In this case a superradiant system does not possess an external parameter similar to the intensity of pumping for a laser. The regime of radiation depends on preparation of the system containing emitters. This is reflected in the initial conditions for the system of differential equations describing dynamics of such a system.

One of the main quantities is the number of previously inverted emitters $K(t_0)$ where $t_0$ is the initial time moment. It is sometimes assumed that if $K(t_0) > K_{thr}$, where $K_{thr}$ is some critical value, then the regime of cooperative radiation is realised [4], otherwise if $K(t_0) < K_{thr}$ radiation will be spontaneous. Another example is the Arecchi-Courtens criterion [5], in particular, for a number of emitters $N$. Thus, if $N < N_c$, where $N_c$ is the critical value, then as is sometimes assumed, the superradiation pulse has a smooth secant-shaped form, but if $N > N_c$, then the cooperative emission becomes oscillatory.

In this paper we investigate the problem of the nature of changes in the radiation regime of a previously inverted system of two-level atoms by the rigorous methods of the stability theory of solutions to ordinary differential equations. If there are such critical values of $K_{thr}$ or $N_c$ at which the solutions of the corresponding evolution equations become unstable, then the radiation regime changes step-wise: a spontaneous radiation changes to cooperative or the secant-shaped pulse changes to oscillatory pulses. If the solutions remain stable, the passage over the "critical" values $K_{thr}$ and $N_c$ is smooth and continuous.

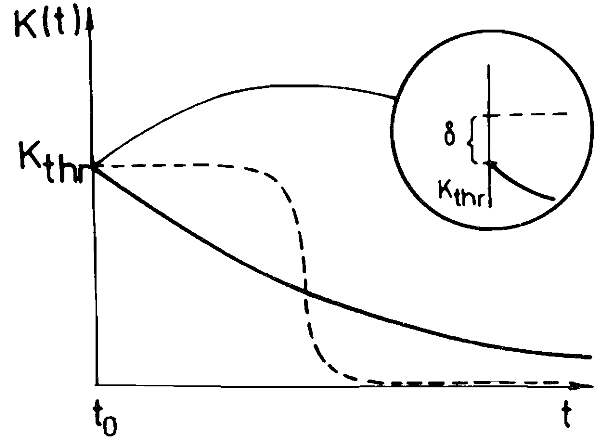
Our analysis will be based on the following system of ordinary differential equations describing the radiation of a pencil-shaped system of two-level atoms interacting with two resonant modes [6]:

$$\begin{aligned}
&\frac{dn}{dt} + \frac{n}{\tau} = F, \\
&\frac{dF}{dt} + \frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{T_2}\right)F = \frac{1}{T_0^{2}}\left(2nK - Nn + K + S\right), \qquad (1)\\
&\frac{dS}{dt} + \frac{S}{T_2} = F\,(2K - N), \\
&\frac{dK}{dt} + \frac{K}{T_{non}} = -F.
\end{aligned}$$



Here $n$ is the number of photons inside the sample, $K$ is the number of inverted atoms, $S$ is the correlation between the dipole moments of transitions of two-level atoms, $F$ is the energy exchange rate between the field and atoms, $\tau$ is the field relaxation time owing to the escape of photons from the sample, $T_2$ is the time of homogeneous relaxation of a macroscopic dipole moment, $T_{non}$ is the time of relaxation of inverted atoms owing to radiationless mechanisms, $N$ is the number of atoms in the system, $T_0^{-1} = \sqrt{2}\,|g|/\hbar$, where $g$ is the interaction constant in the Dicke Hamiltonian.

If the regime of radiation changes step-wise when the initial number of inverted atoms passes through the value $K_{thr}$ the corresponding solution of equations (1) becomes unstable. This is illustrated in the figure. If $K(t_0)$ exceeds $K_{thr}$ by an infinitesimal value of $\delta > 0$, the solutions corresponding to the solid and dashed lines are a finite distance apart from each other and consequently become unstable a la Liapunov /7/ Therefore, our aim is to investigate some nontrivial solutions of eq. (1) for stability. We shall also find out what conditions will be imposed on the investigated solutions.



## 2. Proof of stability

Now we introduce dimensionless variables and time

$$y_1 = n,\ y_2 = \tau F,\ y_3 = S,\ y_4 = K, \quad \theta = \frac{t}{\tau} \tag{2}$$

and dimensionless constants

$$\alpha_1 = \tau/T_2\ ,\ \alpha_2 = \tau/T_0\ ,\ \alpha_3 = \tau/T_{non}\ . \tag{3}$$

System (1) acquires the form

$$\begin{aligned}
\dot{y}_1 &= -y_1 + y_2, \\
\dot{y}_2 &= -\tfrac{1}{2}(1+\alpha_1)y_2 + \alpha_2^2(2y_1y_4 - Ny_1 + y_3 + y_4), \\
\dot{y}_3 &= -\alpha_1 y_3 + 2y_2y_4 - Ny_2, \\
\dot{y}_4 &= -\alpha_3 y_4 - y_2,
\end{aligned} \tag{4}$$

The dot means the differentiation with respect to the dimensionless time $\theta$ .

Let $A(\theta) = \{A_1(\theta), A_2(\theta), A_3(\theta), A_4(\theta)\}$ be some arbitrary solution of system (4). Let us investigate its stability. Writing down

$$y(\theta) = A(\theta) + x(\theta) \tag{5}$$

we get for the variations $x(\theta) = \{x_1(\theta), x_2(\theta), x_3(\theta), x_4(\theta)\}$ the reduced nonautonomous quasilinear system of equations

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2, \\
\dot{x}_2 &= -\tfrac{1}{2}(1+\alpha_1)x_2 + \alpha_2^2(2x_1x_4 + 2A_1(\theta)x_4 + \\
&\quad + 2A_4(\theta)x_1 - Nx_1 + x_3 + x_4), \\
\dot{x}_3 &= -\alpha_1 x_3 - Nx_2 + 2x_2x_4 + 2A_2(\theta)x_4 + 2A_4(\theta)x_2, \\
\dot{x}_4 &= -\alpha_3 x_4 - x_2 .
\end{aligned} \tag{6}$$



The concise form is

$$\dot{x} = (B + D(\theta))x + f(x), \tag{7}$$

where $B$ is constant matrix

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -\alpha_2^2 N & -\frac{1}{2}(1+\alpha_1) & \alpha_2^2 & \alpha_2^2 \\ 0 & -N & -\alpha_1 & 0 \\ 0 & -1 & 0 & -\alpha_3 \end{pmatrix} \tag{8}$$

$D(\theta)$ is the time-dependent matrix

$$D(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\alpha_2^2 A_4(\theta) & 0 & 0 & 2\alpha_2^2 A_1(\theta) \\ 0 & 2A_4(\theta) & 0 & 2A_2(\theta) \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

and nonlinearity is

$$f(x) = \{0,\ 2\alpha_2^2 x_1x_4,\ 2x_2x_4,\ 0\}. \tag{10}$$

To apply the Liapunov criterion for nonautonomous quasilinear systems [8], one should show that the nonautonomous system of the first approximation

$$\dot{\xi} = (B + D(\theta))\,\xi \tag{11}$$

is correct à la Liapunov [7]. We require that the investigated solution $A(\theta)$ should satisfy the condition

$$\int_{\theta_0}^{\infty} \|D(\theta)\| d\theta < \infty, \qquad \theta_0 = t_0/\tau. \tag{12}$$

Making the change $\xi = C\eta$ in system (11) where $C$ is nonsingular matrix diagonalizing the matrix $B$

$$\Lambda = C^{-1}BC = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{13}$$

we get the linear system of equations

$$\dot{\eta} = \Lambda\eta + C^{-1}D(\theta)C\eta, \tag{14}$$

where the time-independent coefficients $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ on the right-hand side are along the diagonal. It follows from condition (12) that

$$\int_{\theta_0}^{\infty} \| C^{-1} D(\theta) C \| d\theta < \infty.$$

Then system (14) is correct à la Liapunov and the numbers $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ form its total spectrum [9]. Since nonlinearity (10) satisfies the conditions of the Liapunov criterion for nonautonomous quasilinear systems, the investigation of stability of a trivial solution of the reduced system (6)-(10) or, which is the same, of the nontrivial $A(\theta)$ solution of initial system (4), is reduced in virtue of the condition (12) to a simple problem of negative definiteness of the constant matrix $B$. Using the Sylvester criterion, one can easily see that at all positive values of $\tau$, $T_2$, $T_0$, $T_{non}$ and $N$ in Eq. (1) the principal minors of the matrix $-B$ are positive

$$\begin{aligned} &\Delta_1(-B) = 1 > 0, \\ &\Delta_2(-B) = \tfrac{1}{2}(1+\alpha_1) + \alpha_2^2 N > 0, \\ &\Delta_3(-B) = \alpha_1 \Delta_2(-B) + \alpha_2^2 N > 0, \\ &\Delta_4(-B) = \alpha_3 \Delta_3(-B) + \alpha_1 \alpha_2^2 > 0 \end{aligned} \qquad (15)$$

and, consequently, the matrix $B$ is negatively definite, i.e. $\lambda_i < 0$, i = 1,2,3,4. Then, according to the Liapunov criterion for nonautonomous quasilinear systems, a trivial solution of the reduced system (6)-(10) is stable à la Liapunov. This means that the nontrivial $A(\theta)$ solution of the primary system (1) or (4) satisfying the only condition (12) is stable. Moreover, as it follows from this criterion, the solution is exponentially stable à la Liapunov [10].

The stability of the $A(\theta)$ solution, that is an arbitrary one (except for condition (12)), physically implies that there are no such initial conditions, in particular, there is no such initial number of inverted atoms $K_{thr}$ at which the regime of



radiation of a previously inverted system described by eq. (1) could change step-wise. Consequently, in a given system the transition from the regime of spontaneous radiation to superradiation is continuous. The exponential stability of the $A(\theta)$ solution means that during the time comparable with $max(\tau, T_2, T_0, T_{non})$ the evolution of variables $R$, $F$, $S$ and $K$ undergoes a stage of their exponential tending to zero values. Consequently, eq.(1) adequately describes a dissipative nature of radiation of a previously inverted system of two-level atoms.

From a point of view of a mathematician sufficient condition (12) is very strong. However, a physicist should require fulfillment of this condition, since it is equivalent to the simple physically necessary conditions. Let us show the validity of the statement.

## 3. Physical reasons for stability

### 3.1. Sufficient conditions of stability

For a solution $A(\theta)=\{A_1(\theta), A_2(\theta), A_3(\theta), A_4(\theta)\}$ of the system of evolution equations (4) sufficient stability condition (12) has the form:

$$\int_{\theta_0}^{\infty} \| D(\theta) \| d\theta = 2 \int_{\theta_0}^{\infty} \sqrt{\alpha_2^4 A_1^2(\theta) + A_2^2(\theta) + (1+\alpha_2^2)^2 A_4^2(\theta)}\, d\theta < \infty. \qquad (16)$$

The Euclidian norm is used. The function $A_3(\theta)$ is seen to be out of this condition. The following upper estimate is valid for the improper integral from (16):

$$\begin{aligned} &\int_{\theta_0}^{\infty} \sqrt{\alpha_2^4 A_1^2(\theta) + A_2^2(\theta) + (1+\alpha_2^2)^2 A_4^2(\theta)}\, d\theta \leq \\ &\leq \int_{\theta_0}^{\infty} \left(\alpha_2^2 |A_1(\theta)| + |A_2(\theta)| + (1+\alpha_2^2)|A_4(\theta)|\right) d\theta. \end{aligned} \qquad (17)$$

If the improper integral in the right-hand side of inequality (17) exists and converges, the improper integrals

$$I_1 = \int_{\theta_0}^{\infty} |A_1(\theta)|\, d\theta, \tag{18}$$

$$I_2 = \int_{\theta_0}^{\infty} |A_2(\theta)|\, d\theta, \tag{19}$$

$$I_4 = \int_{\theta_0}^{\infty} |A_4(\theta)|\, d\theta \tag{20}$$

also exist and converge in virtue of the non-negative character of the functions under the integral from (17).

So, if $I_j < \infty$, $j$ = 1,2,4, then sufficient stability condition (16) is obviously valid.

3.2. Non-negative definiteness and finiteness of the energy of a radiating system; existence of its equilibrium state at infinitely long time.

First let us show that the condition of non-negative and finite energy implies convergence of the integrals $I_1$ and $I_4$. The functions $A_1(\theta)$ and $A_4(\theta)$ are the number of photons in the volume containing emitters and the number of excited emitters respectively. They determine the energy of the system, therefore

$$0 \leq A_1(\theta) < \infty, \tag{21}$$

$$0 \leq A_4(\theta) < \infty. \tag{22}$$

For any two moments of the dimensionless time $\theta_2 > \theta_1 \geq \theta_0$ one can write down

$$A_1(\theta_2) - A_1(\theta_1) = \int_{\theta_1}^{\theta_2} \dot{A}_1(\theta)\, d\theta. \tag{23}$$

$$A_4(\theta_2) - A_4(\theta_1) = \int_{\theta_1}^{\theta_2} \dot{A}_4(\theta)\, d\theta. \tag{24}$$

Summing (23) and (24) and taking into account evolution equations (4), we obtain

$$\int_{\theta_1}^{\theta_2} (A_1(\theta) + \alpha_3 A_4(\theta))\, d\theta = A_1(\theta_1) - A_1(\theta_2) + A_4(\theta_1) - A_4(\theta_2). \tag{25}$$

Let limits for $A_1(\theta)$ and $A_4(\theta)$ at $\theta \to \infty$ exist. It implies



that the open system of emitters described by eqs. (4) or (1) goes to an equilibrium state at infinitely long time. Then we have in virtue of (21) and (22)

$$\int_{\theta_0}^{\infty} (|A_1(\theta)| + \alpha_3 |A_4(\theta)|)\, d\theta \leq |A_1(\theta_0)| + |A_1(\infty)| + |A_4(\theta_0)| + |A_4(\infty)| < \infty. \tag{26}$$

From this it follows immediately that the improper integrals $I_1$ and $I_4$ exist and converge:

$$I_1 = \int_{\theta_0}^{\infty} |A_1(\theta)|\, d\theta < \infty, \tag{27}$$

$$I_4 = \int_{\theta_0}^{\infty} |A_4(\theta)|\, d\theta < \infty. \tag{28}$$

3.3. Finiteness of the intensity of radiation or absorbtion

Let us now estimate the integral $I_2$. Note that the following inequality is valid in virtue of the first evolution equation (4)

$$|A_2(\theta)| \leq |A_1(\theta)| + |\dot{A}_1(\theta)|. \tag{29}$$

Further on we shall discriminate two cases. The first is a monotonous asymptotic behaviour of the field. Let there be a moment of time $\theta' \geq \theta_0$ when the function $A_1(\theta)$ is monotonous on the semi-axis $[\theta', \infty)$. This function is also continuously differentiated in virtue of the theorem of existence and uniqueness [8]. Then it follows from convergence of $I_1$ in (27) that $A_1(\theta)$ is a monotonously non-increasing function, such that

$$I_1' = \int_{\theta_0}^{\infty} |\dot{A}_1(\theta)|\, d\theta < \infty. \tag{30}$$

Then the integral $I_2 < I_1 + I_1'$ (due to estimate (29)) and, consequently, exists and converges. Thus if the field is asymptotically monotonous, condition (16) is fulfilled owing to (27), (28), (30). So, the stability of the radiation regime in this case is again determined by the non-negative and finite character or the energy of the system.

Let us consider another possible case - oscillatory asymptotic behaviour of the field, when the number of photons oscillates in time. Let $\theta_1, \theta_2, \ldots$ be a sequence of isolated extrema of the function $A_1(\theta)$ on the semi-axis $[\theta_0, \infty)$. If this sequence is bounded, the consideration is reduced to the case of the asymptotically monotonous behaviour of the field. Actually, it is sufficient to consider the field evolution for the time surpassing the upper limit of the sequence $\{\theta_i\}$.

Let the increasing sequence $\{\theta_i\}$ be unbounded. Consider the following definite integral:

$$\bar{I}_1''(m) = \int_{\theta_0}^{\theta_m} |\dot{A}_1(\theta)|\, d\theta = \sum_{i=1}^{m} \int_{\theta_{i-1}}^{\theta_i} |\dot{A}_1(\theta)|\, d\theta. \tag{31}$$

According to the mean-value theorem

$$\bar{I}_1''(m) = \sum_{i=1}^{m} |\dot{A}_1(\theta_i^*)|\,(\theta_i - \theta_{i-1}), \quad \theta_{i-1} < \theta_i^* < \theta_i. \tag{32}$$

Let $p_i = max|\dot{A}_1(\theta)|$, where $\theta \in ]\theta_{i-1}, \theta_i[$. Then

$$\bar{I}_1''(m) \le \sum_{i=1}^{m} p_i(\theta_i - \theta_{i-1}). \tag{33}$$

Let us now demand that the following inequality should be valid starting from some $\dot{i}$:

$$\dot{p}_i \le M\bar{A}_1(i) = \frac{M}{\theta_i - \theta_{i-1}} \int_{\theta_{i-1}}^{\theta_i} A_1(\theta)\, d\theta, \tag{34}$$

where $M$ is an arbitrary constant independent of $i$, $\bar{A}_1(i)$ is the mean value of $A_1(\theta)$ in the interval $]\theta_{i-1}, \theta_i[$.

Then the integral which converges owing to (27)

$$\bar{I}_1 = \sum_{i=1}^{\infty} \int_{\theta_{i-1}}^{\theta_i} A_1(\theta)\, d\theta = \sum_{i=1}^{\infty} \bar{A}_1(i)(\theta_i - \theta_{i-1}) < \infty \tag{35}$$

is a series which dominates the series

$$\bar{I}_1'' = \lim_{m \to \infty} \bar{I}''(m) = \sum_{i=1}^{\infty} |\dot{A}_1(\theta_i^*)|(\theta_i - \theta_{i-1}). \tag{36}$$



Consequently, in virtue of convergence of $\bar{I}_1$ and condition (34), the integral $\bar{I}_1''$ exists and converges, furthermore there is a convergent integral

$$\bar{I}_2 = \int_{\theta_0}^{\infty} |A_2(\theta)|\, d\theta \le \bar{I}_1 + \bar{I}_1'' < \infty, \tag{37}$$

which was to be shown. Then the oscillatory radiation regime is stable, as the monotonous regime investigated before.

## 4. Short commentary

We enumerate once more the physical conditions, imposed on the solutions considered.

Conditions (21) and (22) ensure selection of the solutions with the finite and non-negative energy.

Assumption about the existence of limits for the functions $A_1(\theta)$ and $A_4(\theta)$ at $\theta \to \infty$ just ascertains an evident physical fact of transition of the open system described by eqs. (4) or (1) to an equilibrium state at infinitely long time.

It is easy to see that condition (34) is just a condition of a finiteness for an observable, namely for the intensity of radiation or absorbtion. Choosing the constant $M$ one can obtain any prescribed intensity, i.e. physical generality is not lost.

Thus, we have demanded that only the most necessary physical conditions are fulfilled. If it is so, the radiation regime of a previously inverted system of two-level atoms is stable. Chaos-exponential instability [11] - never occurs in this system; moreover, there are no threshold values of the initial inverse population or the number of emitters as well. This is a rigorous result for the considered system (1).

In conclusion, we should like to note that variables $n$, $F$, $S$ and $K$ do not allow one to use a direct energetic way of constructing the Liapunov's function for studying the stability of eq. (1), since the energy of the system is linear with respect to these variables [12]. We note also that eq.(1) which forms a basis of review [6] are quite general. Dropping some terms and introducing into these equations variables of the "action-angle" type one can obtain [6] nonautonomous pendulum equations considered in refs.[4,5]. There will be unstable solutions for a pendulum equation [7,8,10,12] and, consequently, the "superradiant threshold" will exist in this particular case. For the more general situation considered before, as we have shown, the "superradiant threshold" does not exist.

The author is grateful to P.E.Zhidkov for discussion of some mathematical aspects of the problem.

References

1. M.Sargent III, M.O.Scully, W.E.Lamb Jr. Laser Physics (Addison-Wesley, Reading, Mass., 1974).
2. H.Haken, Light, vol. 2, Laser Light Dynamics (North-Holland, Amsterdam, 1985).
3. H.Haken, H.Ohno. Opt.Comm. 16 (1976) 205.
4. R.Bonifacio, P.Schwendimann, F.Haake. Phys.Rev., A4 (1971) 302.
5. F.T.Arecchi, E.Courtens. Phys.Rev. A2 (1970) 1730.
6. A.V.Andreev, V.I.Emel'yanov, Yu.A.Il'inski. Sov.Usp. 23 (1980)493.
7. A.M.Liapunov. Stability of Motion (Academic Press, New York,1966).
8. I.G.Malkin. Theory of Stability of Motion (Atomic Energy Commission, Trans.No 3352, Dept.of Commerce, Washington, D.C., 1958).
9. N.Levinson, Duke Math.Journ. 15 (1948) 111.
10. N.N.Krasovski. Stability of Motion (Stanford Univ.Press, Stanford, 1963).
11. J.R.Ackerhalt, P.W.Milonni, M.-L.Shih. Phys.Rep., 128 (1985) 205.
12. J.La Salle, S.Lefschetz. Stability by Liapunov's Direct Method with Applications (Academic Press, New York-London, 1961).

Received by Publishing Department
on December 18, 1987.



Бакасов А.А. E17-87-885
Физические причины устойчивости режимов излучения предварительно инвертированной системы

Строго показано, что решения эволюционных уравнений для предварительно инвертированной системы двухуровневых атомов являются устойчивыми, если удовлетворяют лишь самым необходимым физическим условиям. Это означает, что не существует таких начальных условий, при которых режим излучения изменялся бы скачкообразно. В частности, переход от спонтанного излучения к сверхизлучению происходит непрерывным образом.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Bakasov A.A. E17-87-885
Physical Reasons for Liapunov's Stability of Radiation Regime of a Previously Inverted System

Evolution equations describing a previously inverted system of two-level atoms have been strictly investigated. Solutions obeying only the most necessary physical conditions are found to be stable. It implies physically that there are no such initial conditions at which the regime of radiation changes step-wise. Particularly, the transition from spontaneous radiation to superradiation proceeds continuously.

The investigation has been performed at the Laboratory of Theoretical Physics JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987