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A GENERALIZED N-LEVEL N-1-MODE J-C MODEL

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1. Introduction

In recent years, much interest has been paid to the Jaynes--Cummings (J-C) model (Jaynes and Cummings 1963) exactly solvable in the rotating wave approximation (RWA) of a two-level atom linearly interacting with a single mode of the quantized-field radiation (Eberly et al. 1980, 1981, Knight and Radmore 1982, Singh 1982). A distinctive, from a physical point of view, feature of the J-C model is an infinite sequence of quantum collapses and revivals of Rabi oscillations revealed by Eberly et al. (1980, 1981) and Knight and Radmore (1982). There have been considered various possible generalizations of the J-C model. In particular, Buck and Sukumar (1981a, 1981b, 1984a, 1984b) have found an exact solution to the equations of motion for an atomic system with an interaction nonlinear in bosonic variables. Singh (1982) has studied photon statistical properties of such nonlinear systems. Exact wave functions and energy levels have been found for various J-C-type systems nonlinear both in bosonic and spin variables (Kochetov 1987). There also should be mentioned a recent series of papers by Agarwal and Puri (1986a, 1986b) devoted to the generalization of the J-C model to include the effect of cavity damping. All of these rather simple models describe, however, the essential physics of radiation-matter interaction. Apart from the above vacuum-field Rabi oscillations, these models are capable to describe such very interesting phenomena now intensively studied theoretically and experimentally as photon antibunching and sub-Poissonian distribution (Paul 1982, Mandel 1986, Zakharov and Tynterev 1987) and squeezing (Paul 1982, Knight 1986) effects.

Another form of the generalization of the J-C model deals with adding other levels leading, in particular, to the appearance of new branches of the Rabi frequency as compared to the standard two-level J-C model. Li and Bei (1984) have extended the J-C model to the case of a three-level atom interacting with a two-mode radiation field. Multiphoton transitions in such a system have been considered as well (Shumovsky 1985). Very recently, photon antibunching and squeezing effect have been revealed in a three-level system (Bogolubov (jr) et al. 1986a, 1986b). Mention is also to be made of a recent paper by Abdel-Hafez et al. (1987a) containing detailed investigations of the phenomena of collapses and revivals, squeezing and sub-Poissonian photon statistics in three-level systems.

There are two possible ways to generalize the J-C model to an N-level system: considering an N-level atom interacting either with a) a single mode or b) N-1 modes of the radiation field. Both the cases are exactly soluble due to the existence of conserved charges and completely independent of each other except the case N=2. In particular, the mean photon numbers and average values of the atomic level occupations have been studied by Kozierowski and Shumovsky (1987) for an N-level atom coupled with N-1 modes of the radiation field. There have also been considered multiphoton transitions in such a system (Abdel-Hafez et al.1987b), both the papers dealing with the operator equations of motion.

In the present note, I would like to consider an N-level atom immersed in a lossless cavity and interacting with N-1 field modes in a nonlinear way. Owing to the conservation laws, the state space of the system can be decomposed into a direct sum of finite-dimensional subspaces corresponding to fixed values of constants of motion. An eigenstate vector of the system parametrized by a fixed set of these constants is determined as an expansion over the basis of an appropriate finite-dimensional subspace, with the expansion coefficient being defined from the Schrödinger equation that is reduced on every subspace to a finite system of algebraic equations.

In sec.2, a multiboson variant of the N-level N-1 mode J-C model is formulated; the structure of state space is thoroughly analysed, exact wave functions and energy levels of a system are found, and completeness of the obtained system of eigenfunctions is discussed. Secs. 3 and 4 illustrate the application of the obtained spectrum to examine the sub-Poissonian distribution and squeezing effects.

2. The N-1 mode nonlinear N-level J-C problem

The Hamiltonian for an N-level N-1 mode system in RWA is

$$\mathcal{H} = \sum_{j=1}^{N} \Omega_j R_{jj} + \sum_{j=1}^{N-1} \omega_j a_j^+ a_j + \sum_{j=1}^{N-1} \lambda_j (a_j^{\nu_j} R_{Nj} + a_j^{+\nu_j} R_{jN}) \qquad (1)$$

the $R_{ij} = |i\rangle\langle j|$ are the transition ($i \neq j$) and projections $(i=j)$ operators with the commutation relations

$$[R_{ij}, R_{kl}] = R_{il}\delta_{jk} - R_{kj}\delta_{il}$$

$|j\rangle$ and Ω_j, $j=1,2,\ldots,N$ are the eigenstates and eigenvalues of an N-level atom. The photon creation a_j^+ and annihilation a_j operators $j=1,2,\ldots,N-1$ describe N-1 modes of the radiation field with frequencies ω_j and λ_j are the corresponding atom-mode coupling constants. Note that

$$[a_i, a_j^+] = \delta_{ij} \qquad [a_i, R_{kl}] = [a_i^+, R_{kl}] = 0.$$

According to (1) the N-th level is coupled with any other j-th level by ν_j -photon dipole transiton ($\nu_j \geq 0$ are integers, $1 \leq j \leq N-1$), whereas the mutual transitions between the remaining N-1 levels are forbidden. Note that (1) is at N=2 and $\nu=1$ reduced to a standard J-C problem whose exact solution is known since 1963 (Jaynes and Cummings 1963, Lee 1973, Rupasov 1982).

2.1. The Structure of State Space

The state space of system (1) is generated by the basis

$$\{ |\Phi_m^{\{n_i\}}\rangle = |\{n_i\}\rangle |m\rangle \ , n_i \geq 0 \ , 1 \leq m \leq N \},$$

where by $\{n_i\}$ one should understand the set of the arguments:

$$\{n_i\} = (n_1, n_2, \ldots, n_{N-1}) \quad , \quad |\{n_i\}\rangle = |n_1\rangle|n_2\rangle\ldots|n_{N-1}\rangle,$$

where

$$a_j^+ a_j |n_j\rangle = n_j |n_j\rangle \quad , \quad R_{ii}|m\rangle = \delta_{im}|m\rangle.$$

Model (1) possesses N-1 independent constants of motion

$$\hat{M}_j = a_j^+ a_j - \nu_j R_{jj} \quad , \quad 1 \leq j \leq N-1 .$$

Conservation of the charges of a system $\hat{M}_j$ limits the range of variation of numbers $\{n_i\}$ and m by the N-1 conditions

$$M_j = n_j - \nu_j \delta_{jm} \quad , \quad 1 \leq j \leq N-1 ,$$

where ($M_1, M_2, \ldots, M_{N-1}$) $\equiv \{M_i\}$ are common eigenvalues of charges $\hat{M}_i$. Owing to the conservation laws, the state space can be represented by $\sum \oplus \mathcal{H}_{\{M_i\}}$, where every subspace $\mathcal{H}_{\{M_i\}}$ corresponding to a fixed set of charges $\{M_i\}$ is generated by the basis

$$\left\{ |\Phi_m^{\{M_i\}}\rangle = |\{M_i + \vartheta_i \delta_{mi}\}\rangle |m\rangle , \quad m \in \mathcal{P}_{\{M_i\}} \right\}$$
$$\hat{M}_j |\Phi_m^{\{M_i\}}\rangle = M_j |\Phi_m^{\{M_i\}}\rangle \tag{2}$$

and m runs over the set

$$\mathcal{P}_{\{M_i\}} = \{m | 1, 2, \ldots, N\} \cap \{m | M_1 + \vartheta_1 \delta_{m1} \geq 0\} \cap \{m | M_2 + \vartheta_2 \delta_{m2} \geq 0\}$$
$$\cap \ldots \cap \{m | M_{N-1} + \vartheta_{N-1} \delta_{m\,N-1} \geq 0\}$$

the state vector $|\psi^{\{M_i\}}\rangle$ obeying the Schrödinger equation

$$\hat{\mathcal{H}} |\psi^{\{M_i\}}\rangle = \mathcal{E} |\psi^{\{M_i\}}\rangle \tag{3}$$

with Hamiltonian (1) and corresponding to the set $\{M_i\}$ can be represented by the following expansion over the basis (2) of the subspace $\mathcal{H}_{\{M_i\}}$:

$$|\psi^{\{M_i\}}\rangle = \sum_{m \in \mathcal{P}_{\{M\}}} c_m |\Phi_m^{\{M_i\}}\rangle . \tag{4}$$

2.2. Energy levels and eigenfunctions

It may be verified that feasible common eigenvalues of charges $\hat{M}_i$, $i = 1, 2, \ldots, N-1$ form the sets

$$(n_1 - \vartheta_1, n_2, n_3, \ldots, n_{N-1})$$
$$(n_1, n_2 - \vartheta_2, n_3, \ldots, n_{N-1})$$
$$\vdots$$
$$(n_1, n_2, \ldots, n_{N-2}, n_{N-1} - \vartheta_{N-1}), \quad n_i \geq 0 . \tag{5}$$

Due to (5) expansion (4) assumes the form

$$|\psi^{\{M_i = n_i\}}\rangle \equiv |\psi_{\{n\}}\rangle$$
$$= c_1 |n_1 + \vartheta_1\rangle |n_2\rangle \ldots |n_{N-1}\rangle |1\rangle + c_2 |n_1\rangle |n_2 + \vartheta_2\rangle |n_3\rangle \ldots |n_{N-1}\rangle |2\rangle$$
$$+ \ldots + c_{N-1} |n_1\rangle |n_2\rangle \ldots |n_{N-2}\rangle |n_{N-1} + \vartheta_{N-1}\rangle |N-1\rangle$$
$$+ c_N |n_1\rangle |n_2\rangle \ldots |n_{N-1}\rangle |N\rangle , \quad n_i \geq 0, \quad 1 \leq i \leq N \tag{6}$$

and

$$|\psi^{\{M_{i \neq j} = n_i\},\, M_j = n_j - \vartheta_j}\rangle \equiv |\tilde{\psi}_{\{n\}}^{(j)}\rangle = |n_1\rangle |n_2\rangle \ldots |n_{N-1}\rangle |j\rangle . \tag{6'}$$

where

$$0 \leq n_j < \vartheta_j , \quad n_{i \neq j} \geq 0 , \quad 1 \leq i, j \leq N-1 .$$

Inserting (6) and (6') into the Schrödinger equation (3) and introducing the detuning parameter

$$\Delta = \Omega_j - \Omega_N + \omega_j \vartheta_j , \quad j = 1, 2, \ldots, N-1 \tag{7}$$

we obtain the following solutions:

i)

$$|\psi_{\{n\}}^{(\pm)}\rangle = \left(Q_{N-1}^2 + y_{(\pm)}^2\right)^{-1/2} \times$$
$$\left(\sum_{j=1}^{N-1} g_j |\{n_{i \neq j}\}\rangle |n_j + \vartheta_j\rangle |j\rangle + y_{\pm} |\{n_i\}\rangle |N\rangle \right),$$
$$n_i \geq 0 , \quad 1 \leq i \leq N-1 \tag{8}$$

$$\mathcal{E}^{(\pm)}_{\{n\}} = \Omega_N + \sum_{i=1}^{N-1} \omega_i n_i + y_\pm + \Delta \,, \quad n_i \geq 0. \tag{9}$$

Here and in what follows we use the notation

$$|\{n_{i\neq j}\}\rangle |n_j+\nu_j\rangle = |n_1\rangle|n_2\rangle\ldots|n_{j-1}\rangle|n_j+\nu_j\rangle|n_{j+1}\rangle\ldots|n_{N-1}\rangle$$

$$y_\pm = -\frac{\Delta}{2} \pm \sqrt{(\Delta/2)^2 + Q^2_{N-1}} \,, \qquad Q^2_{N-1} = \sum_{j=1}^{N-1} g_j^2$$

$$g_j^2 = \lambda_j^2 \frac{(n_j+\nu_j)!}{n_j!},$$

ii) apart from this, N-2 eigenfunctions

$$|\Psi^{(k)}_{\{n\}}\rangle = N_k^{-1}\Big(-\sum_{i=1}^{k-1} g_i\, |\{n_{j\neq i}\}\rangle|n_i+\nu_i\rangle|i\rangle \tag{10}$$

$$+ \mathcal{L}_k\, |\{n_{i\neq k}\}\rangle|n_k+\nu_k\rangle|k\rangle\Big) \,, \qquad 2 \leq k \leq N-1$$

correspond to the eigenvalues

$$\mathcal{E}_{\{n_i\}} = \Omega_{N-1} + \sum_{j=1}^{N-2}\omega_j n_j + \omega_{N-1}(n_{N-1}+\nu_{N-1}) \,, \tag{11}$$

where

$$N_k^2 = \frac{Q^2_{k-1}\, Q^2_k}{g_k^2} \,, \qquad \mathcal{L}_k = \frac{Q^2_{k-1}}{g_k},$$

iii)

$$|\tilde{\Psi}^{(j)}_{\{n\}}\rangle = |\{n_i\}\rangle|j\rangle \,, \quad n_i \geq 0, \quad 0 \leq n_j < \nu_j \tag{12}$$

$$\tilde{\mathcal{E}}^{(j)}_{\{n\}} = \Omega_j + \sum_{i=1}^{N-1}\omega_i n_i \,, \quad 1 \leq j \leq N-1 \tag{13}$$

Formulae (8)-(13) at N=2, ν =1 go over into the well-known results of the standard two-level J-C model, whereas at N=2 and $\nu \geq 2$ we arrive at the spectrum of the non-linear two-level J-C model to be found in the paper by Kochetov (1987). At N=3 and Δ =0 formulae (8)-(13) reproduce the spectrum of the two-mode three-level J-C model with multiphoton transitions first obtained by Shumovsky et al. (1986).

2.3. Completeness of the system of eigenfunction.

Let us verify the completeness of the orthonormalized system of functions (8), (10), (12). Composing of these functions the operator $\sum |\Psi_i\rangle\langle\Psi_i|$ we get the $N \times N$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{pmatrix}.$$

where, for instance,

$$a_{NN} = \sum_{\{n_i\}\geq 0} |\{n_i\}\rangle\langle\{n_i\}| \left\{ \frac{y_+^2}{Q^2_{N-1} + y_+^2} + \frac{y_-^2}{Q^2_{N-1} + y_-^2} \right\}.$$

Due to the relation $y_\pm = -\frac{\Delta}{2} \pm \sqrt{(\Delta/2)^2 + Q^2_{N-1}}$, the expression in the braces is equal to unity and consequently, a_{NN} =1. One can easily verify also that $a_{Nj} = a_{jN} = 0$, $j=1,2,\ldots,N-1$. Further, we have $a_{kk} = b_k^{(1)} + b_k^{(2)}$, where the operator

$$b_k^{(2)} = \sum_{\{n_{i\neq k}\}\geq 0,\; n_k < \vartheta_k} |\{n_\gamma\}\rangle\langle\{n_\gamma\}|$$

is constructed from the functions belonging to system (12); and $b_k^{(1)}$, from the appropriate functions belonging to system (8) and (10):

$$b_k^{(1)} = \sum_{\{n_j\}\geq 0} |\{n_{i\neq k}\}\rangle\langle\{n_{i\neq k}\}|\,|n_k+\vartheta_k\rangle\langle n_k+\vartheta_k| \left\{ \frac{g_k^2}{Q_{N-1}^2} + \frac{Q_{k-1}^2}{Q_k^2} + g_k^2\sum_{i=k+1}^{N-1} 1/N_i^2 \right\}.$$

Taking into account the relationship

$$\frac{1}{Q_{N-1}^2} + \sum_{i=k+1}^{N-1} 1/N_i^2 = \frac{1}{Q_k^2} \qquad 1\leq k\leq N-1$$

that can be easily proved by induction, we obtain

$$b_k^{(1)} = \sum_{\{n_\gamma\}\geq 0} |\{n_{i\neq k}\}\rangle\langle\{n_{i\neq k}\}|\,|n_k+\vartheta_k\rangle\langle n_k+\vartheta_k| \left\{\frac{Q_{k-1}^2}{Q_k^2} + \frac{g_k^2}{Q_k^2}\right\} = \sum_{\{n_{i\neq k}\}\geq 0,\; n_k\geq\vartheta_k} |\{n_i\}\rangle\langle\{n_i\}|$$

and finally

$$a_{kk} = \sum_{\{n_{i\neq k}\}\geq 0}' |\{n_{i\neq k}\}\rangle\langle\{n_{i\neq k}\}| \left\{\sum_{n_k<\vartheta_k} |n_k\rangle\langle n_k| + \sum_{n_k\geq\vartheta_k}|n_k\rangle\langle n_k|\right\} = \sum_{\{n_i\}\geq 0}' |\{n_i\}\rangle\langle\{n_i\}| = 1.$$

Analogously, it is easy to see that $a_{ki} = 0,\; k, i = 1,2,\dots, N-1;\; k\neq i$.

3. Sub-Poissonian photon statistics

Here we apply the above results to analyse the sub-Poissonian photon distribution and squeezing effects in the considered model.

Both the sub-Poissonian distribution and squeezing are aspects of the quantum nature of light: the former being a particle effect and the latter, a wave one. These unique properties are related to the decrease of quantum fluctuations in the photon number and phase below those of coherent light. The photons are said to be sub-Poissonian if there is a nonclassical state of the field in which the variance of the number of photons is less than the mean value of photons.

To characterize the photon statistical properties of the j^{th} mode $(1\leq j\leq N-1)$, we introduce the normally ordered variance $V_j(t)$ of the photon number in the j^{th} mode

$$V_j(t) = \langle \hat{n}_j^2(t)\rangle - \langle \hat{n}_j(t)\rangle - \langle \hat{n}_j(t)\rangle^2 \qquad (14)$$

$$\hat{n}_j(t) = e^{i\mathcal{H}t}\, a_j^{+} a_j\, e^{-i\mathcal{H}t}.$$

The quantity (14) is proportional to the excess of coincidences in counting rates measured in the Hanbury Brown and Twiss-type experiment (Hanbury Brown and Twiss 1956, Glauber 1965). The sign (+) or (−) of $V_j(t)$ shows that the photon statistics of the field are super or sub-Poissonian, respectively (Walls 1979, Ficek et al 1984). Usually, to examine the number and phase-dependent field fluctuations in the J-C-type model, the radiation field is initially assumed to be in a coherent state interacting with an atom initially excited to a certain level. Very recently, Knight (1986) has considered the two-level J-C model for an atom interacting at $t=0$ not with a coherent state but with a vacuum field, provided the atom is excited to a coherent superposition of upper and lower states. To generalize Knight's procedure to an N-level J-C model, we assume

$$|\Psi(t=0)\rangle \equiv |\Phi\rangle = \sum_{j=1}^{N} c_j\, |\{0_i\}\rangle |j\rangle , \tag{15}$$

where $\sum |c_j|^2 = 1$, $|\{0_i\}\rangle = |0\rangle_1 |0\rangle_2 \dots |0\rangle_{N-1}$ is the field-vacuum state and $|j\rangle$ is the pure atomic state with the energy Ω_j. We shall neglect the effect of detuning (i.e., write $\Delta = 0$) and assume also that all $\nu_j > 0$; $j = 1, 2, \dots, N-1$. Then using the complete set of functions (8), (10), (12) one can easily obtain the state vector for $t \geq 0$

$$|\Psi(t)\rangle = e^{-i\mathcal{H}t} |\Phi\rangle = \sum_{\{n_i\} \geq 0} |\Psi^{(+)}_{\{n\}}\rangle \langle \Psi^{(+)}_{\{n\}} | \Phi\rangle e^{-it\mathcal{E}^{(+)}_{\{n\}}}$$

$$+ \sum_{\{n_i\} \geq 0} |\Psi^{(-)}_{\{n\}}\rangle \langle \Psi^{(-)}_{\{n\}} | \Phi\rangle e^{-it\mathcal{E}^{(-)}_{\{n\}}} + \sum_{j=1}^{N-1} \sum_{\{n_{i\neq j}\} \geq 0,\ n_j < \nu_j} |\tilde{\Psi}^{(j)}_{\{n\}}\rangle \langle \tilde{\Psi}^{(j)}_{\{n\}} | \Phi\rangle e^{-it\tilde{\mathcal{E}}^{(j)}_{\{n\}}}$$

$$= -\frac{i c_N e^{-it\Omega_N}}{S_{N-1}} \sin t S_{N-1} \Big(\lambda_1 \sqrt{\nu_1!}\, |\nu_1\rangle |0\rangle \dots |0\rangle |1\rangle + \lambda_2 \sqrt{\nu_2!}\, |0\rangle |\nu_2\rangle \dots |0\rangle |2\rangle$$

$$+ \dots + \lambda_{N-1} \sqrt{\nu_{N-1}!}\, |0\rangle \dots |0\rangle |\nu_{N-1}\rangle |N-1\rangle \Big) + c_N e^{-it\Omega_N} \cos t S_{N-1}$$

$$|0\rangle \dots |0\rangle |N\rangle + \sum_{j=1}^{N-1} c_j e^{-it\Omega_j} |0\rangle \dots |0\rangle |j\rangle . \tag{16}$$

where S^2_{N-1} stands for $\left. Q^2_{N-1} \right|_{\{n_i\} = \{0_i\}} = \sum_{i=1}^{N-1} \lambda_i^2 \nu_i!$.

The average value of the time-dependent function $f_t(a_j^+ a_j) = e^{i\mathcal{H}t} f(a_j^+ a_j) e^{-i\mathcal{H}t}$ is then

$$\langle f_t \rangle \equiv \langle \Phi | f_t | \Phi \rangle = \frac{|c_N|^2 \sin^2 t S_{N-1}}{S^2_{N-1}} \lambda_j^2 \nu_j! f(\nu_j) \tag{17}$$

provided that $f(x=0) = 0$. To characterize the departure from Poisson statistics, it is convenient to introduce the parameter (Mandel 1986)

$$\frac{V_j}{\langle n_j \rangle} = \frac{\langle (\Delta \hat{n}_j)^2 \rangle - \langle \hat{n}_j \rangle}{\langle \hat{n}_j \rangle} .$$

which has some value between 0 and -1 when the variance V_j is sub-Poissonian. With the help of eq. (17) for $\langle f(\hat{n}_j) \rangle$ it is not difficult to show that

$$\frac{V_j}{\langle n_j \rangle} = \nu_j - 1 - \frac{|c_N|^2 \sin^2 t S_{N-1}}{S^2_{N-1}} \lambda_j^2 \nu_j! \nu_j , \quad |c_N| \sin t S_{N-1} \neq 0 . \tag{18}$$

This implies that the j^{th} -mode light field generated from the initial vacuum by an atomic-field interaction has the sub-Poissonian photon statistics for all times t satisfying the condition

$$|c_N|^2 \sin^2 t S_{N-1} > \frac{S^2_{N-1}}{\lambda_j^2 \nu_j!} \frac{\nu_j - 1}{\nu_j} \tag{19}$$

Eq. (19) reveals an interesting ν, N -dependence of the time behaviour of the sub-Poissonian distribution effect. Namely if the r.h.s. of eq. (19) is less than unity, there exists, within any sufficiently large fixed time interval (T_1, T_2) , $T_2 > T_1$ a sequence of such time subintervals that in each of them photons exhibit the sub-Poissonian statistics. As S^2_{N-1} increases, but ν_j remains fixed, the total length of these subintervals decreases.

So, as N and (or) ν_i, $i \neq j$ increase the time duration of the effect within (T_1, T_2) becomes shorter. The same is also true if ν_j increases, provided ν_j is sufficiently large. To illustrate the point, let us consider the special case N=2 ($\nu_1 \equiv \nu$, $\lambda_1 \equiv \lambda$), the so-called nonlinear two-level J-C model. Then eq.(19) goes over into

$$|c_N|^2 \sin t\lambda\sqrt{\nu!} > \nu - 1/\nu$$

and the effect asymptotically dies out as $\nu \to \infty$.

It is worth noting also that there is a more usual way to choose the initial state vector considering the so-called pumping mode $i \neq N$ in a coherent state $|z\rangle$; all the remaining modes, in vacuum; and the atom, on an i^{th} level:

$$|\Psi_i(t=0)\rangle = |\{0_{j\neq i}\}\rangle |z\rangle_i |i\rangle_{at}, \quad i - \text{fixed}.$$

With the help of the complete set of functions (8), (10), (12) for any signal mode $\alpha \neq i$ being initially in vacuum it is not difficult to show that

$$\langle\Psi_i(t=0)| \hat{n}_\alpha(t) |\Psi_i(t=0)\rangle \equiv \langle \hat{n}_\alpha\rangle_i$$

$$= 4\sum_{n_i\geq 0} \frac{|\langle n_i+\nu_i|z\rangle|^2}{S^4_{N-1}} g_i^2 g_\alpha^2 \nu_\alpha \sin^2 t S_{N-1}/2,$$

which for $\nu_j = 1$ and $\lambda_j = \lambda$, $1 \le j \le N-1$ goes over into the expression

$$\langle \hat{n}_a\rangle_i = 4\sum_{n_i\geq 1}\frac{|\langle n_i|z\rangle|^2}{[n_i+N-2]^2} n_i \sin^2\frac{\lambda t}{2}\sqrt{n_i+N-2}$$

first obtained by Kozierowski (1987) by means of an exact solution of operator equations of motion.

4. Squeezing effect

Another nonclassical effect, the so-called squeezing effect, has recently become a subject of intense studies. As for the sub--Poissonian distribution effect, the light field in a squeezed state has no classical counterpart. Strictly speaking, squeezing is characterized by a field state in which the variance of one of two noncommuting observables is less than half the absolute value of their commutator (Stoler 1970, Yuen 1975, Hollenhorst 1979, Knight 1986). We define the Hermitian field operators of the j^{th} mode $a_j^{(1)}$ and $a_j^{(2)}$ by (Knight 1986)

$$a_j^{(1)} = \frac{1}{2}(a_j + a_j^+), \quad a_j^{(2)} = \frac{1}{2i}(a_j - a_j^+)$$

so that $[a_j^{(1)}, a_j^{(2)}] = i/2$ and the variance $(\Delta a_j^{(i)})^2 = \langle a_j^{(i)2}\rangle - \langle a_j^{(i)}\rangle^2$, $i = 1, 2$ satisfies the uncertainty relation

$$(\Delta a_j^{(1)})^2 (\Delta a_j^{(2)})^2 \geq 1/16.$$

The field in the j^{th} mode is squeezed if $(\Delta a_j^{(i)})^2 < 1/4$ for either $i = 1$ or 2. With the aid of the state-vector (16) it is easy to obtain

$$\langle a_j^p\rangle \equiv \langle\Psi(t)|a_j^p|\Psi(t)\rangle$$

$$= -\frac{i c_N c_j^*}{S_{N-1}} \sin t S_{N-1}\, e^{-itp\omega}\,\delta_{\nu_j,p}\,\lambda_j\, p!, \quad p > 0 \qquad (20)$$

provided that as befor $\Delta = 0$ and all $\nu_j > 0$. Taking into account the relations

$$\langle a_j^{(1)2}\rangle = \frac{1}{4} + \frac{1}{2}\langle a_j^+ a_j\rangle + \frac{1}{4}\langle a_j^2\rangle + \frac{1}{4}\langle a_j^{+2}\rangle$$

$$\langle a_j^{(2)2}\rangle = \frac{1}{4} + \frac{1}{2}\langle a_j^+ a_j\rangle - \frac{1}{4}\langle a_j^2\rangle - \frac{1}{4}\langle a_j^{+2}\rangle$$

with the help of eq.(17) and (20) we have for the variances

$$(\Delta a_j^{(1)})^2 = \frac{1}{4} + \frac{|c_N|^2 \sin^2 t S_{N-1}}{2 S_{N-1}^2} \lambda_j^2 \nu_j! \nu_j$$

$$-\frac{i\sin tS_{N-1}}{2S_{N-1}}\lambda_j\delta_{\nu_j,2}\left(c_N c_j^* e^{-2it\omega}-c_N^* c_j e^{2it\omega}\right)+\frac{\sin^2 tS_{N-1}}{4S_{N-1}^2}\lambda_j^2\delta_{\nu_j,1} \tag{21}$$

$$\left(c_N^* c_j e^{it\omega}-c_N c_j^* e^{-it\omega}\right)^2, \qquad \left(\Delta a_j^{(2)}\right)^2=\frac{1}{4}+\frac{|c_N|^2\sin^2 tS_{N-1}\lambda_j^2\nu_j!}{2S_{N-1}^2}\nu_j$$

$$+\frac{i\sin tS_{N-1}}{2S_{N-1}}\lambda_j\delta_{\nu_j,2}\left(c_N c_j^* e^{-2it\omega}-c_N^* c_j e^{2it\omega}\right)-\frac{1}{4}\frac{\sin^2 tS_{N-1}}{S_{N-1}^2}\lambda_j^2\delta_{\nu_j,1}$$

$$\left(c_N^* c_j e^{it\omega}+c_N c_j^* e^{-it\omega}\right)^2. \tag{22}$$

As a consequence of relations (21)-(22) there is no squeezing in the j^{th} mode if $\nu_j \geqslant 3$. The squeezing might accur provided ν_j = 1 or 2.

Let us put, for instance ν_j =1, then (20), (21) go over into

$$\left(\Delta a_j^{(1)}\right)^2=\frac{1}{4}+\frac{\lambda_j^2 r_N^2\sin^2 tS_{N-1}}{S_{N-1}^2}\left\{\frac{1}{2}-r_j^2\sin^2(t\omega+\varphi_j)\right\} \tag{23}$$

$$\left(\Delta a_j^{(2)}\right)^2=\frac{1}{4}+\frac{\lambda_j^2 r_N^2\sin^2 tS_{N-1}}{S_{N-1}^2}\left\{\frac{1}{2}-r_j^2\cos^2(t\omega+\varphi_j)\right\}. \tag{24}$$

Where we use the parametrization $c_N = r_N$, $c_j = r_j e^{i\varphi_j}$. Note that for N=2 and $r_2 = \cos\theta/2$, $r_1 = \sin\theta/2$ formulae (23), (24) reduce to the appropriate results obtained by Knight (1986). Due to eqs. (23), (24) $\left(\Delta a_j^{(1,2)}\right)^2$ are squeezed for all times satisfying the relations

$$\left\{\begin{matrix}\sin^2(t\omega+\varphi_j)\\ \cos^2(t\omega+\varphi_j)\end{matrix}\right\} > \frac{1}{2r_j^2}, \qquad r_N\sin tS_{N-1}\neq 0$$

It is worth noting that the squeezing effect is diminishing as N increases, due to the increase of S_{N-1}^2. The case ν_j =2 can be considered in a similar way. The squeezing considered above is due to the initially excited atomic states. There is a number of papers dealing with that type of squeezing (Walls and Zoller 1981, Wodkiewicz 1984), including the models, where many atoms coherently interact with the field (Heidmann et al. 1985a, 1985b, 1985c).

Thus, we have constructed the complete set of wave functions and energy levels of the nonlinear ($\{\nu\}$ -dependent) N-level N-1 mode J-C model and then used the obtained spectrum to investigate the sub--Poissonian distribution and squeezing effects. There have been revealed the ν - and N-dependences of parameters characterizing these effects.

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Обобщенная N-уровневая N-1-модовая
модель Джейнса-Каммингса

Рассмотрена квантовая модель, описывающая нелинейное взаимодействие N-уровневой системы с N-1 модой поля излучения. Получены полная система волновых функций и уровней энергии. Обсуждаются приложения модели для исследования эффектов суб-пуассоновского распределения фотонов и сжатия света.

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Kochetov E.A. E17-87-736
A Generalized N-Level N-1-Mode
J-C Model

A quantum statistical model for the interaction of an N-level atom with N-1-mode radiation field in a nonlinear way is presented, and a complete system of eigenfunctions and eigenvalues is found. Application of this model to the investigation of photon sub-Poissonian distribution and squeezing effects is also discussed.

The investigation has been perfprmed at the Laboratory of Theoretical Physics, JINR.

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