ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E17-87-46

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ON THE JAYNES - CUMMINGS MODEL WITH MULTI-PHOTON TRANSITIONS IN A CAVITY

Submitted to "Journal of Physics A"

1987

The Jaynes - Cummings model [1-4] of a two-level atom interacting with the electromagnetic field in a lossless cavity is one of the few exactly soluble models in quantum optics. It enables one to calculate all the quantum-mechanical properties of a system. It predicts many interacting effects such as vacuum field Rabi oscillations, in the presence of a coherent field [2-6], etc. It is now becoming possible to test experimentally many of the predictions of this model [4,7]. In recent papers, Agarwal and Puri [8,9], Barnett and Knight [10] and Filipowicz et al. [11] have studied the effects of dissipation in the Jaynes - Cummings model and their influence on revivals and other quantum features; in particular, the absorption and emission spectra have been calculated. Single-mode m-photon absorption and m-photon emission processes in a two-level atomic system have been considered by Zubairy and Yeh [12]. Other multi-photon processes in a lossless cavity have recently been extensively investigated in a number of papers [3,13-17].

In this paper we considered the Jaynes - Cummings model with multi-photon transitions in the presence of cavity-relaxation effects. In order to solve the problem we follow the procedure presented by Agarwal and Puri [8] and Shumovsky et al. [16].

Solution for density-matrix elements

The Jaynes - Cummings model with multi-photon transitions describes the interaction of a single-model electromagnetic field with a two-level atom via m-photon processes. The Hamiltonian for this model in the RWA and electric dipole approximation is

$$H=\hbar\omega_0 S^z+\hbar\omega a^+a+\hbar g\left(a^{+m}S^-+a^mS^+\right), \tag{1}$$

where $S^{\pm,z}$ are the spin-1/2 operators, a (a^+) is the annihilation (creation) operator of the radiation field. The parameter g is the constant of atom-mode coupling. Here ω_0 is the transition frequency of the atom and ω is the model frequency, and they obey the condition :

$$\omega_0 - m\omega = \Delta, \tag{2}$$

where Δ is the detuning parameter.

Further, we shall assume that a field can decay at the rate 2κ. The density matrix for the combined atom-field system by the standard master-equation techniques is [8,10]

$$\frac{\partial \rho}{\partial t} = -i[H,\rho] - \kappa\left(a^+a\rho - 2a\rho a^+ + \rho a^+ a\right) \equiv L\rho. \qquad (3)$$

The Hamiltonian H causes transitions between the states $|n,e\rangle$ and $|n+m,g\rangle$. Field and atom occupation numbers change at the same time. The relaxation in the cavity changes only the photon number. For example, if the initial state of the system is $|n,g\rangle$, then the system can be found in any of the states

$$|p,g\rangle;\quad p = 0,1,\dots,n;$$

$$|q,e\rangle,\quad q = 0,1,\dots,n-m.$$

For the initial state $|0,e\rangle$, the states to be considered are $|m,g\rangle, |m-1,g\rangle, \dots, |1,g\rangle, |0,g\rangle$. The density-matrix elements now satisfy

$$\langle 0,g|\dot\rho|0,e\rangle = i\left(m\omega+\Delta\right)\langle 0,g|\rho|0,e\rangle + ig\sqrt{m!}\,\langle 0,g|\rho|m,g\rangle, \qquad (4)$$

$$\langle 0,g|\dot\rho|m,g\rangle = \left(im\omega - \kappa m\right)\langle 0,g|\rho|m,g\rangle + ig\sqrt{m!}\,\langle 0,g|\rho|0,e\rangle. \qquad (5)$$

The results following from (4) and (5) are

$$\langle 0,g|\rho|0,e\rangle = \frac{1}{Z_1 - Z_2}\Big\{\Big[\left(Z_1 - im\omega + \kappa m\right)\langle 0,g|\rho(0)|0,e\rangle + ig\sqrt{m!}\,\langle 0,g|\rho(0)|m,g\rangle\Big]e^{Z_1 t} - \Big[\left(Z_2 - im\omega + \kappa m\right)\langle 0,g|\rho(0)|0,e\rangle + ig\sqrt{m!}\,\langle 0,g|\rho(0)|m,g\rangle\Big]e^{Z_2 t}\Big\}, \qquad (6)$$

$$\langle 0,g|\rho|m,g\rangle = \frac{1}{Z_1 - Z_2}\Big\{\Big[\left(Z_1 - im\omega - i\Delta\right)\langle 0,g|\rho(0)|m,g\rangle + ig\sqrt{m!}\,\langle 0,g|\rho(0)|0,e\rangle\Big]e^{Z_1 t} - \Big[\left(Z_2 - im\omega - i\Delta\right)\langle 0,g|\rho(0)|m,g\rangle + ig\sqrt{m!}\,\langle 0,g|\rho(0)|0,e\rangle\Big]e^{Z_2 t}\Big\}, \qquad (7)$$

$$Z_{1,2} = i\left(m\omega + \frac{\Delta}{2}\right) - \frac{\kappa m}{2} \pm \frac{1}{2}\left[\left(\kappa m + i\Delta\right)^2 - 4g^2 m!\right]^{\frac{1}{2}}. \qquad (8)$$

Using these solutions we can calculate an absorption spectrum for the model, assuming additionally that our model interacts with a weak-probe field. Then, the master-equation (3) will be

$$\frac{\partial \rho}{\partial t} = L\rho - i\left[\left(GS^+e^{-i\nu t} + H.c.\right), \rho\right], \qquad (9)$$

where the coupling constant G is

$$G = -\frac{\vec{d}\vec{\mathcal{E}}}{\hbar}. \qquad (10)$$

The time-average rate of absorption W is

$$W = \frac{d}{dt}\langle p\rangle\vec{\mathcal{E}} = i\nu(\vec{d}\vec{\mathcal{E}})\langle S^+\rangle e^{-i\nu t} + c.c. = -2\nu\left|\frac{\vec{d}\vec{\mathcal{E}}}{\hbar}\right|^2 \mathrm{Re}\int_0^\infty d\tau\, e^{-i\nu\tau}\, \mathrm{Tr}\left(S^+e^{L\tau}[S^-, \rho^{(0)}]\right), \qquad (11)$$

where $\langle S^+\rangle$ can be calculated in the usual manner by evaluating the induced dipole moment to the first order in G.

In the case, when our cavity is at zero temperature, the initial density matrix $\rho^{(0)}$ is

$$\rho^{(0)} = |0,g\rangle\langle 0,g| \qquad (12)$$

and hence

$$W = 2\nu\left|\frac{\vec{d}\vec{\mathcal{E}}}{\hbar}\right|^2 \mathrm{Re}\int_0^\infty d\tau\, e^{-i\nu\tau}\, \mathrm{Tr}\left[S^+e^{L\tau}\left(|0,g\rangle\langle 0,e|\right)\right]. \qquad (13)$$

The operator $e^{L\tau}|0,g\rangle\langle 0,e|$ satisfies (4) and hence

$$e^{L\tau}|0,g\rangle\langle 0,e| = \alpha(\tau)\,|0,g\rangle\langle 0,e| + \beta(\tau)\,|0,g\rangle\langle m,g|, \qquad (14)$$

where

$$\alpha(\tau) = \frac{Z_1 - im\omega + \kappa m}{Z_1 - Z_2}e^{Z_1\tau} - \frac{Z_2 - im\omega + \kappa m}{Z_1 - Z_2}e^{Z_2\tau}, \qquad (15)$$

$$\beta(\tau) = \frac{ig\sqrt{m!}}{Z_1 - Z_2}e^{Z_1\tau} - \frac{ig\sqrt{m!}}{Z_1 - Z_2}e^{Z_2\tau} \qquad (16)$$

Substituting (14) in (13) and simplifying (13) we get

$$W = 2\nu \left|\frac{\vec{d}\vec{\varepsilon}}{\hbar}\right|^2 \operatorname{Re} \hat{\alpha}(i\nu), \tag{17}$$

where

$$\hat{\alpha}(i\nu) = \int_0^\infty e^{-i\nu\tau} \alpha(\tau)\, d\tau .$$

We will consider the case of exact resonance when $\Delta = \omega_0 - m\omega = 0$ and

$$Z_{1,2} = i\omega_0 - \frac{1}{2} km \pm \frac{1}{2} \left(k^2 m^2 - 4 g^2 m!\right)^{\frac{1}{2}} . \tag{18}$$

Here, we can consider the following cases:

1) $k^2 m^2 - 4 g^2 m! > 0$.

In this case, from (15), (17) and (18) we get

$$W = 2\nu \left|\frac{\vec{d}\vec{\varepsilon}}{\hbar}\right|^2 \frac{g^2 m!}{\sqrt{k^2m^2 - 4g^2m!}} \left\{ \frac{1}{(\nu-\omega_0)^2 + \frac{1}{4}\left(km - \sqrt{k^2m^2 - 4g^2m!}\right)^2} - \right. \tag{19}$$

$$\left. - \frac{1}{(\nu-\omega_0)^2 + \frac{1}{4}\left(km + \sqrt{k^2m^2 - 4g^2m!}\right)^2} \right\} .$$

In the case of bad-cavity $k^2 m^2 \gg 4 g^2 m!$ Eq. (8) becomes approximately equal to

$$i\omega_0 - km + O\left(\frac{g^2 m!}{k^2 m^2}\right) \quad ; \qquad i\omega_0 - O\left(\frac{g^2 m!}{k^2 m^2}\right) \tag{20}$$

and the spectra have only a single peak in the position $\nu = \omega_0$.

2) $k^2 m^2 - 4 g^2 m! < 0$.

In this case by substituting Eqs. (15) and (18) into Eq. (17), we obtain

$$W = 2\nu \left|\frac{\vec{d}\vec{\varepsilon}}{\hbar}\right|^2 \frac{\frac{1}{2} km}{\sqrt{4g^2m! - k^2m^2}} \left\{ \frac{\nu - \omega_0 + \sqrt{4g^2m! - k^2m^2}}{\left(\nu - \omega_0 + \frac{1}{2}\sqrt{4g^2m! - k^2m^2}\right)^2 + \frac{1}{4}k^2m^2} - \right. \tag{21}$$

$$\left. - \frac{\nu - \omega_0 - \sqrt{4g^2m! - k^2m^2}}{\left(\nu - \omega_0 - \frac{1}{2}\sqrt{4g^2m! - k^2m^2}\right)^2 + \frac{1}{4}k^2m^2} \right\} .$$

For the good cavity case $k^2 m^2 \ll 4 g^2 m!$ Eq. (21) shows that the spectrum is a doublet $\nu = \omega_0 \pm g\sqrt{m!}$ the width of each doublet being $km/2$. It should be noted that in the case of multi-photon absorption the widths of lines are proportional to m.

Emission spectra for multi-photon processes

Following Agarwal and Puri [8] , we define the transient spectrum of the radiation that leaks out as

$$S(\nu, T) = 2\Gamma\beta \operatorname{Re} \sum_{ij} A_{ij} \left(2\Gamma + \eta_j\right)^{-1} \Big[\left(\Gamma + \eta_j + i\nu - \lambda_i\right)^{-1} \times \tag{22}$$

$$\times \left(e^{\eta_j T} - e^{-T(\Gamma - \lambda_i + i\nu)}\right) - \left(\Gamma + \lambda_i - i\nu\right)^{-1} \left(e^{-(\Gamma + i\nu - \lambda_i)T} - e^{-2\Gamma T}\right) \Big],$$

where we assume that the correlation function has the structure

$$\langle a^+(t+\tau)\, a(t) \rangle = \sum_{ij} A_{ij}\, e^{\lambda_i \tau + \eta_j t} \tag{23}$$

Γ is the bandwidth of the detector, T is the time at which the spectrum is evaluated, and β is a measure of the leakage of the field energy.

Using regression theorem, one can show that

$$\langle a^+(t+\tau)\, a(t) \rangle = \operatorname{Tr}\left[a^+ e^{L\tau} a\, e^{Lt} \rho(0) \right], \tag{24}$$

where the initial density matrix $\rho(0)$ is $|0,e\rangle\langle 0,e|$. This initial state is chosen keeping in view the problem of pure spontaneous emission.

Using (3) we define equations of motion for the operator

$$e^{Lt} |0,e\rangle\langle 0,e| \equiv \left(|0,e\rangle\langle 0,e|\right)_t$$

$$\frac{d}{dt} \left(|0,e\rangle\langle 0,e|\right)_t = -ig\sqrt{m!} \left(|m,g\rangle\langle 0,e|\right)_t + ig\sqrt{m!} \left(|0,e\rangle\langle m,g|\right)_t . \tag{25}$$

Resulting is a closed set of equations

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$$\left[\frac{d}{dt} + \begin{bmatrix} 0 & ig\sqrt{m!} & -ig\sqrt{m!} & 0 \\ ig\sqrt{m!} & (km-i\Delta) & 0 & -ig\sqrt{m!} \\ -ig\sqrt{m!} & 0 & (km+i\Delta) & ig\sqrt{m!} \\ 0 & -ig\sqrt{m!} & ig\sqrt{m!} & 2km \end{bmatrix}\right]\begin{bmatrix} |0,e\rangle\langle 0,e| \\ |m,g\rangle\langle 0,e| \\ |0,e\rangle\langle m,g| \\ |m,g\rangle\langle m,g| \end{bmatrix} = 0. \qquad (26)$$

These equations are solved by the Laplace transforms, the results of which are given by the matrix relation

$$\hat{\psi}(z) = P^{-1}(z)\begin{vmatrix} z_0(z_0^2-k^2m^2+4g^2m!) & i\Delta(z_0^2-k^2m^2) & 2\Delta kmg\sqrt{m!} & 2\Delta z_0 g\sqrt{m!} \\ i\Delta(z_0^2-k^2m^2) & z_0(z_0^2-k^2m^2) & 2iz_0kmg\sqrt{m!} & -2iz_0g\sqrt{m!} \\ 2\Delta kmg\sqrt{m!} & -2iz_0kmg\sqrt{m!} & z_0(z_0^2+\Delta^2+4g^2m!) & km(z_0^2+\Delta^2) \\ 2\Delta z_0 g\sqrt{m!} & -2iz_0^2g\sqrt{m!} & km(z_0^2+\Delta^2) & z_0(z_0^2+\Delta^2) \end{vmatrix}\psi(0) \qquad (27)$$

$$z_0 = z + mk \;; \qquad \psi_{\substack{1\\2}} = \langle m,g|\rho|0,e\rangle \pm \langle 0,e|\rho|m,g\rangle \;;$$

$$\psi_{\substack{3\\4}} = \langle 0,e|\rho|0,e\rangle \pm \langle m,g|\rho|m,g\rangle ,$$

where the polynomial $P(z)$ is

$$P(z) = (z+mk)^4 + (z+mk)^2(\Delta^2+4g^2m!-k^2m^2) - k^2m^2\Delta^2 . \qquad (28)$$

If we denote by M the 4x4 square matrix in (26), then it can be shown that

$$e^{L\tau}a\,e^{Lt}\rho(0) = \left(e^{-Mt}\right)_{12} e^{L\tau}\sqrt{m}\,|m-1,g\rangle\langle 0,e| + $$
$$+ \left(e^{-Mt}\right)_{14} e^{L\tau}\sqrt{m}\,|m-1,g\rangle\langle m,g| . \qquad (29)$$

We can further show that for the operators

$$e^{L\tau}|m-1,g\rangle\langle 0,e| = \left(|m-1,g\rangle\langle 0,e|\right)_\tau$$ and
$$e^{L\tau}|m-1,g\rangle\langle m,g| = \left(|m-1,g\rangle\langle m,g|\right)_\tau$$ we have

a closed set of equations

$$\left[\frac{d}{d\tau} + \begin{bmatrix} -i(\omega+\Delta)+(m-1)k & -ig\sqrt{m!} \\ -ig\sqrt{m!} & (2m-1)k-i\omega \end{bmatrix}\right]\begin{bmatrix} |m-1,g\rangle\langle 0,e| \\ |m-1,g\rangle\langle m,g| \end{bmatrix} = 0 . \qquad (30)$$

The following results are obtained from (30):

$$\left(|m-1,g\rangle\langle 0,e|\right)_\tau = \frac{1}{x_1-x_2}\Big\{\Big[\big(x_1-i\omega+(2m-1)k\big)|m-1,g\rangle\langle 0,e| + $$
$$+ ig\sqrt{m!}\,|m-1,g\rangle\langle m,g|\Big]e^{x_1\tau} - \Big[\big(x_2-i\omega+(2m-1)k\big)|m-1,g\rangle\langle 0,e| + $$
$$+ ig\sqrt{m!}\,|m-1,g\rangle\langle m,g|\Big]e^{x_2\tau}\Big\} , \qquad (31)$$

$$\left(|m-1\rangle\langle m,g|\right)_\tau = \frac{1}{x_1-x_2}\Big\{\Big[\big(x_1-i(\omega+\Delta)+(m-1)k\big)|m-1,g\rangle\langle m,g| + $$
$$+ ig\sqrt{m!}\,|m-1,g\rangle\langle 0,e|\Big]e^{x_1\tau} - \Big[\big(x_2-i(\omega+\Delta)+(m-1)k\big)|m-1,g\rangle\langle m,g| + \qquad (32)$$
$$+ ig\sqrt{m!}\,|m-1,g\rangle\langle 0,e|\Big]e^{x_2\tau}\Big\} ;$$

$$x_{1,2} = i\left(\omega+\frac{\Delta}{2}\right) + k\left(1-\frac{3m}{2}\right) \pm \frac{1}{2}\left[(km+i\Delta)^2 - 4g^2m!\right]^{\frac{1}{2}} . \qquad (33)$$

Let us denote the 2x2 square matrix in (30) by N . Then, using the solution of (30) in (29), one can show that

$$\langle a^+(t+\tau)a(t)\rangle = m\left(e^{-Mt}\right)_{12}\left(e^{-N\tau}\right)_{12} + m\left(e^{-Mt}\right)_{14}\left(e^{-N\tau}\right)_{22} . \qquad (34)$$

The relevant elements of $e^{-N\tau}$ are given by (31) and (32)

$$\left(e^{-N\tau}\right)_{12} = \frac{ig\sqrt{m!}}{x_1-x_2}\left(e^{x_1\tau}-e^{x_2\tau}\right) , \qquad (35)$$

$$\left(e^{-N\tau}\right)_{22} = \frac{1}{x_1-x_2}\Big[\big(x_1-i(\omega+\Delta)+(m-1)k\big)e^{x_1\tau} - \big(x_2-i(\omega+\Delta)+(m-1)k\big)e^{x_2\tau}\Big].$$

Complete spectrum of spontaneous emission can now be obtained using (34) and (23) in (22). In the long-time limit $\Gamma T \gg 1$ the spontaneous emission spectra consist of several lines whose positions and widths are determined by $\mathrm{Im}(\lambda_i-\eta_j)$, $\Gamma+Re(\eta_j-\lambda_i)$.

For the case of a good cavity on resonance, Eq. (33) shows that the emission spectrum has a form of doublet the lines of which are positioned at $\nu = \omega \pm g\sqrt{m!}$ and have the width $\Gamma + k\left(\frac{3m}{2} - 1\right)$. On the other hand, for large Δ spontaneous-emission lines occur at the positions $\omega + \Delta$, ω and their widths are $\Gamma + k(m-1)$ and $\Gamma + k(2m-1)$, respectively.

It should be noted in the case m=1 our results reduce to those obtained by Agarwal and Puri [8].

The authors are grateful to M.Kozierowski for useful discussions.

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Received by Publishing Department
on January 29, 1987.

Алискендеров Э.И. и др. E17-87-46
Модель Джейнса - Каммингса с многофотонными переходами в резонаторе

Исследована модель Джейнса - Каммингса с многофотонными переходами в резонаторе. Найдено точное решение уравнения типа Master Equation. Изучены спектры поглощения и излучения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Aliskenderov E.I. et al. E17-87-46
On the Jaynes - Cummings Model with Multi-Photon Transitions in a Cavity

Jaynes - Cummings model with multi-photon transitions in a cavity is examined. Exact solution of master equation for the density-matrix is found. Absorption and emission spectra are investigated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987