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SOLITON-LIKE "BUBBLES" IN THE SYSTEM OF INTERACTING BOSONS

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1. Introduction. In the Hartree approximation the Schrödinger equation for the boson system with 2-body attractive and 3-body repulsive δ-function interaction reduces [1] to the $\psi^3-\psi^5$ non-linear Schrödinger equation (NLSE):

$$i\psi_t + \Delta\psi - \alpha\psi + \psi|\psi|^2 - \psi|\psi|^4 = 0. \qquad (1)$$

In this letter we study Eq.(1) under non-vanishing boundary conditions $|\psi(\vec{x},t)| \to |\psi_0|$. Physically, this implies analysis of nonlinear excitations in the constant density condensate [1].

Apart from the mentioned problem, $\psi^3-\psi^5$ NLSE arises in a number of independent fields including nuclear hydrodynamics [2], nonlinear optics [3], ferromagnets and molecular crystals [4]. Next, in the static case Eq. (1) is the Euler-Lagrange equation for the functional $F = \int [|\vec{\nabla}\psi|^2 + V(|\psi|^2)]\, d^D\vec{x}$, $V = \alpha|\psi|^2 - \frac{1}{2}|\psi|^4 + \frac{1}{3}|\psi|^6$, which may be thought of as the Landau expansion of free energy in powers of the order parameter ψ and its gradient (see e.g. [5,6]). The situations when $|\psi|^6$ term should be kept in the expansion are discussed in literature [6]. Finally, the static solutions to (1) obey simultaneously the $\psi^3-\psi^5$ nonlinear <u>Klein-Gordon</u> equation also belonging to extensively applied class of models [7].

The nonvanishing boundary conditions admit definite physical interpretation in terms of all these fields. However, we postpone the interpretation to a more detailed publication and concentrate on solutions to (1). Our main goal is to show that $\psi^3-\psi^5$ NLSE possesses a new type of soliton solutions. These nontopological bubble-like solitons turn out to be characteristic just for the three-well potentials V describing competing interactions (to compare, note that the repulsive ψ^3 NLSE does not have solutions of this kind). Also we display several integrable limits of (1).

2. Useful form. It is convenient to make a substitution $\varphi(\vec{x},t) = \beta\psi(\sqrt{3}\beta^2\vec{x}, 3\beta^4 t)$ with $\beta^2 = \frac{2}{3}(A+2\rho_0)$, $\rho_0 > 0$,

$$A/\rho_0 = -2 + \frac{3}{4}\alpha^{-1}[1-(1-4\alpha)^{1/2}] \qquad (2)$$

and pass from (1) to the equation

$$i\varphi_t + \Delta\varphi - (2A+\rho_0)\rho_0\varphi + 2(A+2\rho_0)|\varphi|^2\varphi - 3|\varphi|^4\varphi = 0 \qquad (3)$$

with Hamiltonian

$$E = \int [|\vec{\nabla}\varphi|^2 + (|\varphi|^2-\rho_0)^2(|\varphi|^2-A)]\, d^D\vec{x}. \qquad (4)$$

Eq.(3) admits a homogeneous solution $\varphi \equiv \rho_0^{1/2}$ ("condensate"). Linearizing (3) about it, one obtains the dispersion relation $\omega^2 = \vec{k}^2[\vec{k}^2 + 4\rho_0(\rho_0 - A)]$ and, consequently, the velocity of sound:

$$c = [4\rho_0(\rho_0 - A)]^{1/2}. \qquad (5)$$

We shall be interested in the nonlinear, localized excitations of the condensate, i.e., in solutions to (3) with the boundary conditions

$$|\varphi(\vec{x},t)| \to \rho_0^{1/2}, \quad \vec{\nabla}\varphi(\vec{x},t) \to 0 \quad \text{as} \quad \vec{x}^2 \to \infty. \qquad (6)$$

In view of (6), the appropriate form of the second conserved quantity ("number of particles") is

$$N = \int (|\varphi|^2 - \rho_0)\, d^D\vec{x}. \qquad (7)$$

3. Exact solution in D=1. In the one-dimensional space the soliton solution to (3), (6) may be found explicitly[1)]:

$$\varphi_s = (2\rho_0)^{1/2}\cosh(\tilde{x} - i\mu)[(2\rho_0 - A)(A^2+v^2)^{-1/2} + \cosh 2\tilde{x}]^{-\frac{1}{2}}, \qquad (8)$$

where $\tilde{x} = \frac{1}{2}(c^2-v^2)^{1/2}(x - vt)$, $\cos 2\mu = (A\rho_0 + \frac{1}{2}v^2)\rho_0^{-1}(A^2+v^2)^{-1/2}$ and $0 < 2\mu < \pi$. Eq. (8) represents a rarefaction domain propagating at velocity $v < c$ (note that the shape of $|\varphi_s|^2$ depends on v). The number of particles (7) displaced from the domain is given by

$$N = \mathrm{Arccosh}[(2\rho_0 - A)\cdot(A^2+v^2)^{-1/2}] < 0. \qquad (9)$$

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1) Details will be published elsewhere.

Let v=0. Then at $A<0$ we have $\mu=\frac{1}{2}\pi$ and the solution (8) is kink-shaped: $\varphi_s(\infty)=-\varphi_s(-\infty)$ (this soliton resembles the well-known φ^3 NLSE kink [5]). Passing to the D=2 space, the kink is transformed into vortex [5], but it has no any stationary counterpart in D=3. On the other hand, at v=0, $0<A<\rho_0$ we have $\mu=0$ and φ_s looks like a "one-dimensional bubble": $\varphi_s(\infty)=\varphi_s(-\infty)$. As we shall see, the remarkable feature of this latter soliton is that it survives passing both to 2 and 3 dimensions.

4. Static limit, D>1. For static real rotationally-symmetric solutions $\varphi(\vec{x},t)=\varphi(r)$ (where $r^2\equiv\vec{x}^2$) Eq. (3) reduces to

$$\varphi_{rr}+\frac{D-1}{r}\varphi_r-(2A+\rho_0)\rho_0\varphi+2(A+2\rho_0)\varphi^3-3\varphi^5=0\,. \quad (10)$$

In view of (2) solutions with the same value of A/ρ_0 are similar and we may fix $\rho_0=1$. Analyzing (10) on the phase plane, one can verify that for each $A\in(0,1)$ the solution exists with the boundary conditions $\varphi_r(0)=0$, $\varphi(r)\to 1$ as $r\to\infty$. We have obtained this bubble-like soliton numerically (fig. 1). The number of displaced particles [Eq. (7)] is exposed in fig. 2 [for A→1 we have used Eq. (19) below].

5. Transonic limit. There is one more limit when solitons of φ^3–φ^5 NLSE are easy to describe. Let us introduce real ρ, ϑ and $\vec{u}=\{u_1,\dots,u_D\}$ such that $\varphi=\rho^{1/2}e^{i\vartheta}$ and $\vec{u}=\vec{\nabla}\vartheta$. Then (3) implies

$$\rho_t+2\vec{\nabla}(\rho\vec{u})=0 \quad (11)$$

$$w_t+\left[\vec{u}^2+\frac{1}{4}(\vec{\nabla}\rho)^2\rho^{-2}-\frac{1}{2}\rho^{-1}\Delta\rho+(\rho-\rho_0)(3\rho-\rho_0-2A)\right]_x=0 \quad (12)$$

with $w\equiv u_1$, $x\equiv x_1$. Passing to the variables $\vec{\eta}$ and τ: $\eta_1\equiv\xi=\varepsilon^{1/2}(x-ct)$, $\vec{\eta}_\perp\equiv\{\eta_2,\dots,\eta_D\}=\varepsilon\vec{x}_\perp$, $\tau=\varepsilon^{\frac{3}{2}}t$ (ε = small parameter) we confine ourselves to the transonic waves weakly depending on transversal coordinates. Let us expand solution with asymptotics (6) in power se-

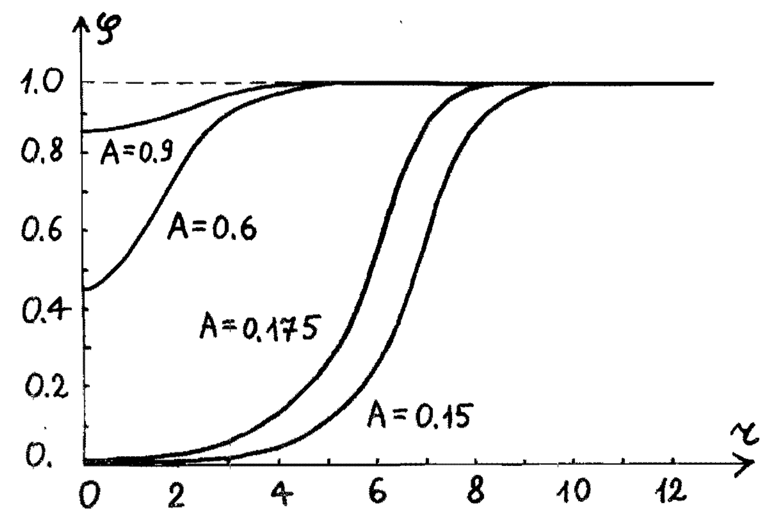

FIG.1

The spherically-symmetric bubble-like soliton for $\rho_0=1$ and different values of A [at D=2 the behaviour of $\varphi_s(r)$ is qualitatively the same.]

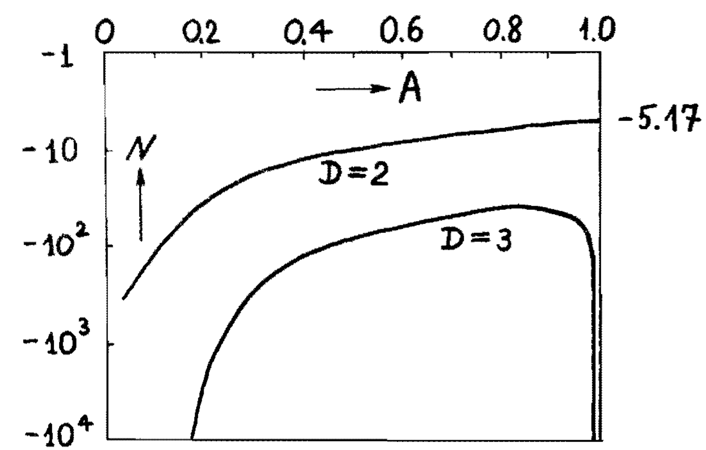

FIG.2

The number of particles displaced from the "bubble" (computed numerically) for D=2 and 3 ($\rho_0=1$). For D=3 the maximal value of N, N_{max}= -40.51 is attained at A= 0.82 .

ries: $\rho=\rho_0+\varepsilon\rho^{(1)}+\varepsilon^2\rho^{(2)}+\ldots$, $w=\varepsilon w^{(1)}+\varepsilon^2 w^{(2)}+\ldots$, $\vec{u}_\perp=\varepsilon^{3/2}\vec{u}_\perp^{(1)}+\ldots$, where $\rho^{(1)}, w^{(1)}\to 0$ as $|\xi|\to\infty$, and substitute these into (11), (12). Then we find:

$$c\rho^{(1)}_{\xi}=2\rho_0 w^{(1)}_{\xi} \qquad (13)$$

$$\rho^{(1)}_\tau-c\rho^{(2)}_\xi+2(\rho_0 w^{(2)}+\rho^{(1)}w^{(1)})_\xi+2\rho_0\,\partial\vec{u}_\perp^{(1)}/\partial\vec{\eta}_\perp=0 \qquad (14)$$

$$w^{(1)}_\tau-cw^{(2)}_\xi+\left[w^{(1)2}-\tfrac{1}{2}\rho_0^{-1}\rho^{(1)}_{\xi\xi}+3\rho^{(1)2}+2(\rho_0-A)\rho^{(2)}\right]_\xi=0, \qquad (15)$$

whence one obtains $[\Delta_\perp\equiv(\partial/\partial\vec{\eta}_\perp)^2]$:

$$\left[2c\rho^{(1)}_\tau-\rho^{(1)}_{\xi\xi\xi}+6(2\rho_0-A)(\rho^{(1)2})_\xi\right]_\xi=-c^2\Delta_\perp\rho^{(1)}. \qquad (16)$$

At D=1 ($\Delta_\perp\rho^{(1)}\equiv 0$) Eq. (16) is recognized as KdV, while at D=2 it is the Kadomtsev-Petviashvili (KP-1) equation. Both the systems possess stable soliton solutions[2)]. The explicit form of these is well-known [8,9] and will not be given here. At D=3 the soliton[2)] solution to (16) has been found numerically and shown to be unstable [10]. Summarising, for any $A<\rho_0$ Eq. (3) possesses localized solutions in the form of transonic weak rarefaction domains (flattened in the direction of motion), stable at D=1 and 2, and unstable at D=3.

Lastly, it is appropriate to mention that both KdV and KP-1 are completely integrable systems. Thus, in the transonic limit much more information is available at D=1 and 2, including explicit N-soliton [8] and finite-gap [9] solutions, the Cauchy problem asymptotics [9], etc.

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2) Here only the solitons are meant decaying to zero in all directions, i.e., lumps.

6. <u>Small c approximation</u>. Allowing $A\to\rho_0$ we obtain another informative limit. Let us assume that $\chi\equiv\rho_0^{1/2}-\varphi$ is of order (ρ_0-A) in this case. Keeping only up to χ^2 in (3) we have

$$i\chi_t+\Delta\chi+2\rho_0(A-\rho_0)(\chi+\chi^*)+3\rho_0^{3/2}(\chi+\chi^*)^2=0. \qquad (17)$$

In terms of $n\equiv\chi+\chi^*$ and $m\equiv i(\chi^*-\chi)$, (17) reduces to $m_t=$ $=\Delta n-c^2 n+6\rho_0^{3/2}n^2$ and $n_t=-\Delta m$ [recall (5)]. Elimination of m yields

$$n_{tt}+\Delta\left(\Delta n-c^2 n+6\rho_0^{3/2}n^2\right)=0. \qquad (18)$$

Lastly, if $\tilde{n}(\vec{x},t)$ is a solution to equation $\tilde{n}_{tt}+\Delta(\Delta\tilde{n}-\tilde{n}+6\tilde{n}^2)=0$, then $n(\vec{x},t)=c^2\rho_0^{-3/2}\tilde{n}(c\vec{x},c^2t)$ solves (18), thereby justifying the above assumption.

At D=1 Eq. (18) is the Boussinesq equation (BqE) integrable through the Inverse Scattering method (N-soliton solution is in [11]). The soliton of BqE is stable or not, depends on its velocity [12]. Applying results of [12] to Eq. (18) and, subsequently, to (3), we find that at $A\to\rho_0$ the "bubble" (8) is stable only for $v>\frac{1}{2}c$. Also this suggests some critical velocity to exist for general A.

At D=2 and 3 we can employ the scaling $n\to\tilde{n}$ to specify the dependence of the number of displaced particles (7) on A for $A\to\rho_0$. Indeed, for static bubble we may choose $\chi=\chi^*$ and, therefore, $\chi(\vec{x})=\frac{1}{2}n(\vec{x})=\frac{1}{2}c^2\rho_0^{-3/2}\tilde{n}(c\vec{x})$. Then

$$N_D(A)=-c^{2-D}\rho_0^{-1}\int\tilde{n}(\vec{x})\,d^D\vec{x}, \qquad (19)$$

with $\tilde{n}(\vec{x})$ verifying $\Delta\tilde{n}-\tilde{n}+6\tilde{n}^2=0$. Thus, as $c\to 0$ we have $N_1\to 0$, $N_2\to$ const and $N_3\to-\infty$.

7. <u>Stability</u>. Let us examine the stability of the static real rotationally-symmetric "bubble" solution $\varphi_s(r)$ of Eq. (3) with respect to small perturbations $\delta\varphi(\vec{x})$. We shall require that these $\delta\varphi$ disturb neither the total momentum integral, $\vec{P}=i\int(\varphi^*\vec{\nabla}\varphi-\varphi\vec{\nabla}\varphi^*)\,d^D\vec{x}$,

nor the number of particles (7). For infinitesimal $\delta\varphi = f + ig$ $(f, g \in \mathbb{R})$ this amounts to $\int g(\vec{x})\vec{\nabla}\varphi_s(r)\,d^D\vec{x} = 0$ and

$$\int f(\vec{x})\varphi_s(r)\,d^D\vec{x} \equiv \mathcal{F}[f] = 0. \qquad (20)$$

The soliton $\varphi_s(r)$ is a stationary point of the functional (4) under the conditions $\delta N = \delta\vec{P} = 0$, i.e., (10) may be written as $(\delta E)_{N,\vec{P}} = 0$, with the Lagrangian multipliers being equal to zero. Stability of φ_s depends on the positive definiteness of the second variation $(\delta^2 E)_{N,\vec{P}}$. We have

$$\delta^2 E = \int [g(\vec{x})L_1 g(\vec{x}) + f(\vec{x})L_2 f(\vec{x})]\,d^D\vec{x}, \qquad (21)$$

where

$$L_1 = -\Delta + \rho_0(\rho_0 + 2A) - 2(A + 2\rho_0)\varphi_s^2(r) + 3\varphi_s^4(r)$$

and

$$L_2 = L_1 - 4(A + 2\rho_0)\varphi_s^2(r) + 12\varphi_s^4(r) \equiv -\Delta + q(r).$$

Let us set D=3 and consider an eigenvalue problem $L_2 y(\vec{x}) = \lambda y(\vec{x}),\ y(0) < \infty,\ y(\infty) = 0$. Substituting $y(\vec{x}) = z(r)r^{-1}Y_{\ell m}(\theta\phi)$, this reduces to

$$-z_{rr} + [\ell(\ell+1)r^{-2} + q(r)]z = \lambda z,\quad z(0) = z(\infty) = 0.$$

As usual L_2 possesses 3 translational zero modes $y_1^i(\vec{x}) = \partial\varphi_s/\partial x_i$ with $\ell = 1$, the related radial eigenfunction $z_1(r) = r\,d\varphi_s/dr$ having no internal nodes in $(0,\infty)$. Consequently, at $\ell = 0$ the nodeless radial eigenfunction $z_0(r)$ exists with the eigenvalue $\lambda_0 < 0$.

Actually, L_2 operator has a negative eigenvalue λ_0 at any D[1]. The corresponding eigenfunction $y_0(\vec{x})$ being nowhere vanishing, it does not obey (20), i.e., $\mathcal{F}[y_0] \neq 0$. Thus, $f_0 = \varepsilon y_0$ $(\varepsilon \ll 1,\ \|y_0\| = 1)$ may not be used as a trial function in (21). However, since the functional $\mathcal{F}$ is not bounded in $L_2(\mathbb{R}^D)$[1], its kernel is <u>everywhere dense</u> in this space according to the well-known theorem [13]. This means that for any $\delta > 0$ the function $\tilde{f}(\vec{x})$ exists such that: i) it verifies (20); ii) $\int [\tilde{f}(\vec{x}) - f_0(\vec{x})]^2 d^D\vec{x} < \delta$. Using $\tilde{f}$ as a trial function in (21), and setting $g(\vec{x}) \equiv 0$, we obtain that $(\delta^2 E)_{N,\vec{P}}$ may be made arbitrarily close to $\lambda_0\varepsilon^2 < 0$. Therefore, the static "bubble" is unstable in any dimension.

8. <u>Discussion.</u> As we have already mentioned, the principal feature that distinguishes the bubble-like solitons from the topological ones is that the latter do not survive passing to 3 dimensions. In this respect the following question remains open. The static "bubbles" exist at $0 < A < \rho_0$, while the static kinks and vortices live at $A < 0$. Strange as it may seem, there does exist 3-dimensional (transonic) soliton at $A < 0$! It is unclear, what does it transform to under the velocity decrease.

Finally, let us comment on the distinctions between two types of nontopological solutions, i.e., between the "bubbles" and the lump solitons. The latter can generally be made stable ("Q-stable" [14]) by some integral fixation (at least, in certain parameter domain). The "bubbles", conversely, have turned out to be essentially unstable solutions. Next, it is well known that lumps may possess nodes, the stationary nodal lumps being normally interpreted as nonlinear excitations of the nodeless ones. Surprisingly, there exist no nodal "bubbles"[1]. The two mentioned facts are closely connected and imply these novel solitons should not be considered merely as "reversed lumps".

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Солитоноподобные "пузырьки"
в системе взаимодействующих бозонов

Рассматривается $\Psi^3 - \Psi^5$ нелинейное уравнение Шредингера, описывающее в приближении среднего поля систему бозонов с двух- и трехчастичным δ-образным взаимодействием. В одно-, двух- и трехмерном координатном пространстве получены солитоноподобные решения типа пузырьков и исследована их устойчивость.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Soliton-Like "Bubbles"
in the System of Interacting Bosons

We study the $\Psi^3 - \Psi^5$ NLS equation which arises as the mean-field approximation for the boson system with 2 and 3 body contact interactions. The bubble-like soliton solutions are found in 1, 2 and 3 dimensions and their stability is examined.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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