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U.Behn*, V.A.Zagrebnov

ONE-DIMENSIONAL RANDOM FIELD ISING MODEL AND DISCRETE STOCHASTIC MAPPINGS

* Sektion Physik der Karl-Marx-Universität
Leipzig, Karl-Marx-Platz, Leipzig, 7010, DDR

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1. INTRODUCTION

In this paper we consider discrete stochastic mappings which appear when one studies the one-dimensional Ising chain in a (frozen) random external magnetic field,

$$H_N = -J\sum_{n=1}^{N} s_n s_{n+1} - \sum_{n=1}^{N} h_n s_n \quad , \quad s_n = \pm 1 \; , \; s_{N+1} = 0 \; , \; J > 0 \; . \tag{1.1}$$

These mappings are originated, e.g., from the reduction of the problem of calculating the partition function for N spins in the external field $\{h_n\}_{n=1}^{N}$ to the equivalent problem of <u>only one</u> spin in some auxiliary (local) random field governed by a probability distribution depending on the probability distribution of the external field as well as on the parameters of the system.

To demonstrate the main idea, we explain how the partition function of the Ising chain (1.1) can be calculated. According to the identity ^{x}

$$\sum_{s_n=\pm 1} \exp\left(J s_n s_{n+1} + \xi_n s_n\right) = \exp\beta\left[A(\xi_n)s_{n+1} + B(\xi_n)\right] \quad , \tag{1.2}$$

where $\beta = (k_B T)^{-1}$, T beeing the temperature, and

$$A(\xi_n) = (2\beta)^{-1} \ln\left[ch\,\beta(\xi_n + J)/ch\,\beta(\xi_n - J)\right] \quad , \tag{1.3}$$

$$B(\xi_n) = (2\beta)^{-1} \ln\left[4\, ch\,\beta(\xi_n + J)\, ch\,\beta(\xi_n - J)\right] \quad , \tag{1.4}$$

the partition function Z_N can be summed up step by step starting from the site $n = 1$. In the $(N-1)$-th step the partition function is obtained as

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^{x} Galam and Salinas /1/ are incorrect at this point (see their formula (4)).

$$Z_N=\sum_{S_N=\pm1}\exp\beta\left[\xi_N S_N+\sum_{n=1}^{N-1}B(\xi_n)\right] \quad . \tag{1.5}$$

Thus, the partition function of the whole system is reduced to that for <u>one</u> spin in the auxiliary field ξ_N which is defined by the recursion formula

$$\xi_n=h_n+A(\xi_{n-1})\equiv f(h_n,\xi_{n-1}),\ \xi_{n=0}=0\ \ ,n=1,2,\dots,N. \tag{1.6}$$

If $\{h_n\}_{n\geq1}$ is a random field, then (1.6) is nothing but a stochastic equation (discrete stochastic mapping) which is the main object of our investigation, and the main problem is to find the density $p_n(x)$ of the probability measure $\mu_n(dx)$ of the auxiliary random field ξ_n or its weak limits $\{\mu\}$ for $n\to\infty$.

This probability density is useful for calculating physical observables. For example, from (1.5) we obtain the free energy density in the thermodynamic limit

$$f(\beta)=-\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N-1}B(\xi_n)+(\beta N)^{-1}\ln 2\operatorname{ch}\beta\xi_N\right)= \tag{1.7}$$
$$=-\lim_{N\to\infty}\int\mu_N(dx)B(x)=-\int\mu(dx)B(x) \quad .$$

These equalities suppose some ergodic properties of the random sequence $\{\xi_n\}_{n\geq1}$ and convergence of $\mu_n(dx)$ to the stationary measure $\mu(dx)$ and hold with μ-$Pr=1$, e.g.,the second term in the bracket tends to zero only with these restrictions. A second example is the magnetization per spin m in the thermodynamic limit. We consider the expectation value for a spin on the site k of the chain. Applying the recursion procedure described above from both ends of the chain up to this site we obtain:

$$\langle S_k\rangle_{H_N}=Z_N^{-1}\exp\left[\beta\sum_{n=1}^{k-1}B(\xi_n)\right]\left\{\sum_{S_k=\pm1}S_k\exp[\beta(\xi_k+\tilde{\eta}_k)S_k]\right\}\times \tag{1.8}$$
$$\times\exp\left[\beta\sum_{n=N}^{k+1}B(\eta_n)\right]=\operatorname{th}\beta\left[\xi_k+A(\eta_k)\right],$$

where ξ_n is governed by (1.6), η_n is governed in a similar way by $\eta_{n-1}=h_{n-1}+A(\eta_n)$, $\eta_N=h_N$, $n=N,N-1,\dots,k+1$ and $\tilde{\eta}_k=A(\eta_k)$. In the thermodynamic limit we obtain for the magnetization (with the same restrictions as hold for (1.7)):

$$m(\beta)=\lim_{N\to\infty}N^{-1}\sum_{k=1}^{N}\langle S_k\rangle_{H_N}= \tag{1.9}$$
$$=\int\mu(dx)\int\mu(dy)\operatorname{th}\beta\left[x+A(y)\right] \quad .$$

Similarly, one obtains for the Edwards-Anderson parameter q_{EA} the following:

$$q_{EA}=\lim_{N\to\infty}N^{-1}\sum_{n=1}^{N}\langle S_n\rangle_{H_N}^2=\int\mu(dx)\int\mu(dy)\left[\operatorname{th}\beta(x+A(y))\right]^2 \quad . \tag{1.10}$$

The idea to reduce the system with many degrees of freedom to a fictitious one-particle system in an auxiliary field substituting the influence of the surrounding is a common approach on the level of an approximation (e.g.,the Bethe approximation). Only in the last few years this approach is used to obtain exact results for a rather general class of Ising models /2-4/ . In the one-dimensional case this idea was applied as well to the random field as to the random exchange Ising model /5-9/ . Stochastic mappings like (1.6) are investigated only for uncorrelated driving fields /8,9/ .

In the present paper the previous results are generalized to a Markovian random magnetic field. We construct the corresponding stochastic mapping and investigate different limit cases for the transient probability of the driving process both for zero and non-zero temperatures. It is shown that for T = 0 all results can be obtained in the frame of the standard theory of finite-state Markov chains. The main results here concern the description of the essential states and their dependence (besides on the Markovian parameter) on the parameters J , h and h_0 . The same approach is developed for T > 0 (an infinite-state Markov chain) including the evaluation of the fractal (Hausdorff) dimensionality d_f of the support S of the unique stationary measure $\mu(dx)$. The dependence of S and d_f on J , h , h_0 and the h-T phase diagram for $h_0=0$ are also discussed.

The paper is organized as follows. In Section 2 the general properties of the discrete stochastic mapping (1.6) are discussed and the Chapman-Kolmogorov equation for the corresponding probability density $p_n(x)$ is derived for a Markovian random external magnetic field. In the following parts we consider only binary random external fields $\{h_n = h_0 \pm h\ , h_0 \geq 0\ , h > 0\}_{n\geq 1}$. In Sections 3 and 4 we consider the important case of zero temperature where the mapping (1.6) is piecewise-linear and the support of the stationary (invariant) measure consists of a finite set of points. In Section 5 we consider the nonzero temperature case, in which the support of the stationary measure has a fractal structure with a nonzero fractal (Hausdorff) dimension depending on the physical parameters of the system. The possible changes in the support of the stationary probability measure are so drastic that we would like to call them "phase transitions" characterized by the fractal dimension of the support as the "order parameter".

2. THE STOCHASTIC MAPPING

The properties of the stochastic mapping (1.6) depend obviously on the properties of the driving process $\{h_n\}_{n\geq 1}$. For driving processes with continuous support of its probability density $\rho_n(x)$ the support of the measure $\mu_n(dx)$ is also continuous. However, for driving processes with a discrete support of $\rho_n(x)$ a drastic change of the structure of the support of $\mu_n(dx)$ is possible. Therefore, we consider in the following as a model for the driving process the two-valued homogeneous, stationary Markov chain.

The properties of (1.6) are further determined by the behaviour of the function $A(x)$ given by (1.3) (cf. Fig.1).

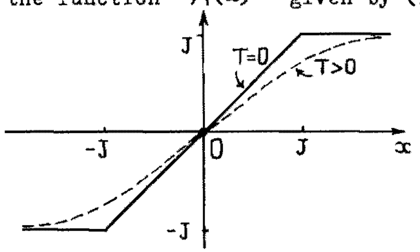


Fig.1.
The function $A(x)$ for zero and nonzero temperatures.

Since $A(x)$ is monotonous and bounded (from below and above) we are not faced with the problem of intrinsic chaos which appears for discrete mappings with nonmonotonous $A(x)$ (see, e.g., Ref. 10). For zero temperature the function $A(x)$ is piecewise-linear, whereas for nonzero temperature $A(x)$ is infinitely many differentiable. As will be shown in the following Sections, for the former case the support of the stationary measure $\mu(dx)$ consists of a finite set of points, whereas in the latter case it is an uncountable set of points which constitute a fractal.

To calculate the probability density for the driven process $\{\xi_n\}_{n\geq 1}$ we remark, that if the driving process is a first-order Markov chain, the driven one is of second order. Therefore, we introduce the vector (ξ_n, h_n) with the joint probability density $p_n(x,\eta)$ which is governed by a first-order Chapman-Kolmogorov equation (see, e.g., Ref. 11 Ch. V, § 3). From (1.6) we obtain

$$p_n(x,\eta) = \int d\eta' \int dx'\, T(\eta|\eta')\, p_{n-1}(x',\eta')\, \delta(x-\eta-A(x')) \ , \qquad (2.1)$$

where we introduced the transient probability density $T(\eta|\eta')$ for the driving Markov chain and its stationary distribution density $\rho(\eta)$.

If we restrict ourself to an uncorrelated driving provess, then $T(\eta|\eta') = \rho(\eta)$, and we obtain

$$p_n(x) = \int d\eta\, p_n(x,\eta) = \int d\eta \int dx'\, \rho(\eta)\, p_{n-1}(x')\, \delta(x-\eta-A(x')) \ , \qquad (2.2)$$

which is nothing but the Chapman-Kolmogorov equation for the first-order Markov chain.

In the opposite case of a constant field, $h_n = h$ (purely correlated case), where $T(\eta|\eta') = \delta(\eta-\eta')$ and $\rho(\eta) = \delta(\eta - h)$, we obtain from (2.1)

$$p_n(x,h) = \int dx'\, p_{n-1}(x',h)\, \delta(x-h-A(x')) \qquad (2.3)$$

with the fixed point solution

$$p^*(x,h) = \delta(x - x^*(h)) \ , \quad x^* = h + A(x^*) \ . \qquad (2.4)$$

Thus, we reduced the investigation of the model (1.1) to the study of the stochastic mapping (1.6) and finally to the Chapman-Kolmogorov equation (2.1) for a driven Markov process $\{\xi_n\}_{n\geq 1}$. The stationary solutions of (2.1) (fixed point probability densities $p(x,\eta)$) give us a complete information about the thermodynamic properties of the model (1.1). Hence, our further strategy follows the Markov chain theory /11/ and consists of two steps. Firstly, using the mapping (1.6) we describe the space of states of the Markov chain $\{\xi_n\}_{n\geq 1}$. Secondly, specifying the initial conditions (distributions) we classify the states into essential (support) and inessential ones and using (2.1) we calculate the invariant (stationary) measures which have this support.

3. ZERO TEMPERATURE AND ZERO MEAN EXTERNAL FIELD

3.1. The Support

For zero temperature the function $A(x)$ which governs the mapping (1.6) is piecewise-linear

$$A(x) = \begin{cases} -J & x < -J \\ x & \text{for} \quad |x| \leq J \\ J & x > J \end{cases} \quad . \tag{3.1}$$

As a consequence, for a finite-state driving process the mapping (1.6) generates for a given J only a finite number of values $X = \{x_i\}$, which constitute together with the possible values of the driving process $\{h_n\}_{n\geq 1}$ the space of states of a finite-state (second-order) Markov chain: $\{z_i\} = \{x_i, h_i\}$.

Assuming that the $\{h_n\}_{n\geq 1}$ can take only the values $\pm h$, $h>0$, one shows straightforward that the $\{\xi_n\}_{n\geq 1}$ can take only the values

$$x(m, \pm J) = m h \pm J \quad , \text{ and} \tag{3.2}$$

$$x(m, 0) = m h . \tag{3.3}$$

In both cases $m = 0, \pm 1, \pm 2, \ldots$ has to be chosen such that

$$x_i \in [h-J, h+J] \cup [-h+J, -h-J] . \tag{3.4}$$

Thus, the space of the states X as a function of J can be found in Fig.2.

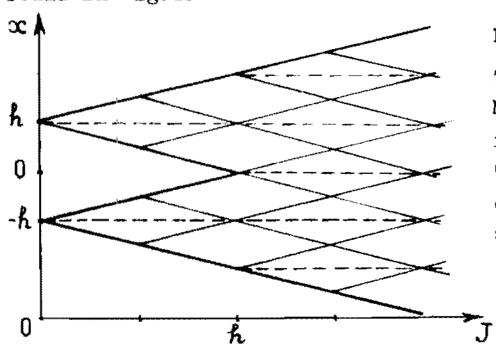


Fig.2.

The space of states X of the Markov chain (2.1) as function of J for $T = 0$ and $0 < \alpha < 1$. Dashed lines correspond to inessential states.

Specifying the transient probability density T in (2.1) for the two-valued driving process by

$$T(\eta|\eta') = \alpha\,\delta(\eta+\eta') + (1-\alpha)\,\delta(\eta-\eta') \tag{3.5}$$

we can distinguish between essential and inessential states in dependence of the value of α.

For $0<\alpha<1$ the $\{x(m,\pm J)\}$ are the essential states S which map exclusively into themselves. For example, for $0<J<h/2$ we have four essential states as can be seen in the corresponding flow diagram:

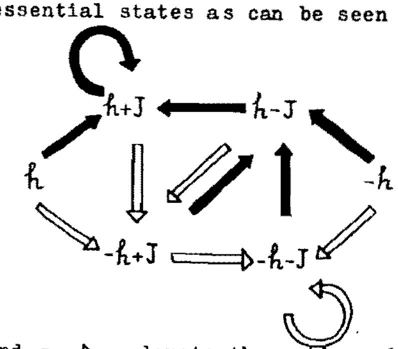


Diagram 1.

Here, ➡ and ⇨ denote the action of the mapping (1.6) with realization $h_n = h$ and $-h$, respectively. The $\{x(m,0)\}$ are the inessential states, since there is a net outflow into essential states. This can be seen in the above diagram as well as, e.g., in the part of the diagram for $h < J < 3h/2$ which contains these states:

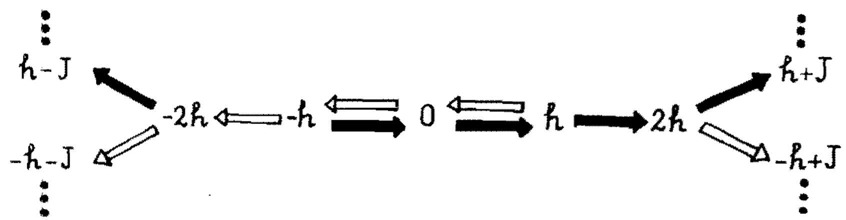


Diagram 2.

Thus, in the limit $n \to \infty$ the probability that we find the system in the states $\{x(m,0)\}$ vanishes and the support S consists only of the states $\{x(m,\pm J)\}$, the number of which is $2q+6$ for $q h/2 < J < (q+1) h/2$, $1 \leq q$.

Now it may be worthwhile to consider special values of α.

Obviously, for $\alpha = 0$ (homogeneous field, cf. (3.5)) the states $x_\pm = \pm(h+J)$ for $h_n = \pm h$ are trapping states corresponding to the fixed point solutions (2.4).

For $\alpha = 1$ (alternating field with period one, cf. (3.5)) we study different initial conditions $h_1 = \pm h$ and odd (even) number n of iterations which correspond to coincidence (noncoincidence) of the field on the site n under consideration with the initial one. We denote the corresponding state by $x^{h_n = \pm h}_{h_1 = \pm h} \equiv x^\pm_\pm$.
As above we obtain from (1.6)

$$
\begin{aligned}
& x^+_+ = x^+_- = -x^-_+ = -x^-_- = h - J && \text{for} \quad 0 \leq J < h/2 \ ; \\
& \left.\begin{aligned} x^+_+ &= -x^-_- = J \\ x^+_- &= -x^-_+ = h - J \end{aligned}\right\} && \text{for} \quad h/2 \leq J \leq h \ ; \qquad (3.6) \\
& \left.\begin{aligned} x^+_+ &= -x^-_- = h \\ x^+_- &= x^-_+ = 0 \end{aligned}\right\} && \text{for} \quad h \leq J \quad ,
\end{aligned}
$$

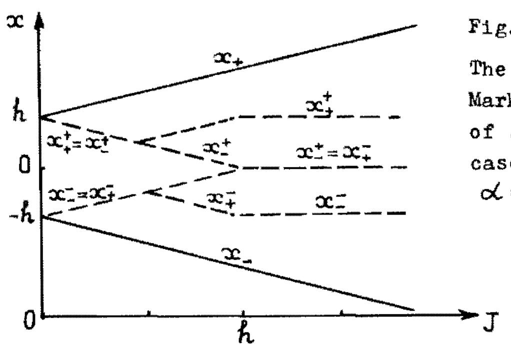


Fig.3.
The space of states X of the Markov chain (2.1) as function of J for $T = 0$ in the limit cases: $\alpha = 0$ (solid lines) and $\alpha = 1$ (dashed lines).

see also Fig.3. From (3.6) we see that for $h/2 \leq J$ we have at least two disconnected sets of essential states, so that the mixing property is destroyed and the ergodicity of the corresponding Markov chain is broken. For example, in the case $h \leq J$ we find the sets ($x^+_+ \rightleftarrows x^-_+$) and ($x^-_- \rightleftarrows x^+_-$), see also Diagram 2. These sets are disconnected because there is no possibility to arrive at x^-_- starting from x^+_+ .

3.2. The Invariant Measure

For zero temperature the mapping (1.6) generates a finite-state (second-order) Markov chain, the probability density $p_n(x,\eta)$ of which consists of a sum of δ-functions with relative weights $\{w_i\}$ located at the points $\{x_i\}$ which constitute the space of states (cf. (3.2-4)). Inserting

$$p_n(x,\eta) = \sum_i w_i^{(n)} \, \delta(\eta - h_i) \, \delta(x - x_i) \tag{3.7}$$

into the Chapman-Kolmogorov equation (2.1) and using (3.5) we obtain

$$
\begin{aligned}
\sum_i w_i^{(n)} \delta(\eta - h_i)\delta(x - x_i) = \sum_j \{ & \alpha w_j^{(n-1)} \delta(\eta + h_j) \delta(x + h_j - A(x_j)) + \\
+ & (1-\alpha) w_j^{(n-1)} \delta(\eta - h_j) \delta(x - h_j - A(x_j)) \} \ .
\end{aligned} \tag{3.8}
$$

Having in mind that the $\pm h_j + A(x_j)$ are nothing but certain points of the support, we may reorder the sum on the right-hand side of (3.8) as

$$\sum_{i,j} D_{ij} w_j^{(n-1)} \delta(\eta - h_i) \delta(x - x_i) \tag{3.9}$$

with

$$
D_{ij} = \begin{cases} \alpha & \text{if} \quad x_i = f(h, x_j = f(-h, \cdot)) \\ \gamma = 1 - \alpha & \text{if} \quad x_i = f(h, x_j = f(h, \cdot)) \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}
$$

Introducing the vector $\vec{w}^{(n)} = \{w_i^{(n)}\}$ we may rewrite (3.8) using (3.9) shortly as follows:

$$\vec{w}^{(n)} = D \, \vec{w}^{(n-1)} \ . \tag{3.11}$$

The invariant probability measure densities

$$p^*(x,\eta) = \sum_i w_i^* \, \delta(\eta - h_i) \delta(x - x_i) \ , \tag{3.12}$$

where the $\{w_i^*\}$ are the components of the fixed point vector of (3.11), can be found by solving the linear equation

$$(1 - D) \vec{w}^* = 0 \ . \tag{3.13}$$

If the state space consists of only one connected set of essential states, the invariant measure is unique and should coincide with the limit value

$$\vec{w}^* = \lim_{n\to\infty} D^n \vec{w}^{(0)} \qquad (3.14)$$

for arbitrary initial vector (distribution) $\vec{w}^{(0)}$ (see, e.g., Ref. 11).

The number of independent solutions of (3.13) is equal to the number of disconnected sets of essential states. These solutions can be found also from (3.14) starting with different initial distributions with support on the corresponding subsets of connected essential states /11/ .

For example we first consider the case $0 < J < h/2$. Then the essential states, as can be seen in Diagram 1, are

$$\{z_i\}_{i=1}^{4} = \{(h+J,h),(h-J,h),(-h+J,-h),(-h-J,-h)\} = S \times \{\pm h\} \qquad (3.15)$$

The one-step transition matrix D according to (3.10) has the form:

$$D = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ \alpha & \alpha & 0 & 0 \\ 0 & 0 & \gamma & \gamma \end{pmatrix} \qquad (3.16)$$

Solving (3.13) we obtain for $0 < \alpha < 1$ the unique fixed point distribution

$$\vec{w}^* = \tfrac{1}{2}(\gamma, \alpha, \alpha, \gamma)^T , \qquad (3.17)$$

which can also be obtained from (3.14), observing that

$$\lim_{n\to\infty} D^n = \frac{1}{2}\begin{pmatrix} \gamma & \gamma & \gamma & \gamma \\ \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha \\ \gamma & \gamma & \gamma & \gamma \end{pmatrix} \qquad (3.18)$$

and starting from arbitrary initial weights $\vec{w}^{(0)}$.

For $\alpha \to 0$ the states z_1 and z_4 become trapping and have the same weight.

For $\alpha \to 1$ we have an oscillation between z_2 and z_3 which both occur with the same weight.

For $\alpha = 0$ the transition matrix D becomes idempotent and has two different fixed points corresponding to the trapping states z_1 and z_4 , cf. Diagram 1.

For $\alpha = 1$ the transition matrix D describes oscillations between z_2 and z_3 . Formally, this corresponds to a fixed point solution $\vec{w}^* = (0,\ 1/2,\ 1/2,\ 0)^T$. We remark, that $\lim_{n\to\infty} D^n$ does not exist, but $D^2 = D^{2n}$ (n= 1,2, ...) has two different eigenvectors (fixed points) $(0,\ 1,\ 0,\ 0)^T$ and $(0,\ 0,\ 1,\ 0)^T$.

As a second example, we consider the case $h < J < 3/2\, h$. Here we should take into account also those states which are for $0<\alpha<1$ inessential, because part of them become essential for $\alpha = 1$.The full space of states can be found in the Table. From the second column of this table we can obtain the elements of the transition matrix D . The matrix elements corresponding to solid (broken) lines are γ (α) . Disconnected points correspond to zero matrix elements. For instance, $D_{81} = \alpha$, $D_{11} = \gamma$ and $D_{21} = 0$. In the next column one can find the weights of the corresponding invariant measure.

As in the previous case one should distinguish the cases $\alpha \to 0$ and $\alpha = 0$.

For $\alpha \to 1$ we have oscillations between the four pairs of states z_4 and z_8 , z_6 and z_{12} , z_7 and z_{13} , z_{11} and z_{15} , but with different weights.

For $\alpha = 1$ we have in addition oscillations between z_5 and z_{10} , z_9 and z_{14} , which were former inessential. Accordingly, we have six independent formal fixed points of D , cf. the Table. The matrix D^{2n} (n = 1,2, ...) has twelve different fixed points.

4. ZERO TEMPERATURE AND NONZERO MEAN EXTERNAL FIELD

In this ohapter we shortly consider the case of a nonzero mean external field, i.e., $\{h_n\}_{n\geq 1}$ takes now the values $h_0 \pm h$ ($h_0, h > 0$). As in the previous case the mapping (1.6) generates a finite state (second order) Markov chain.

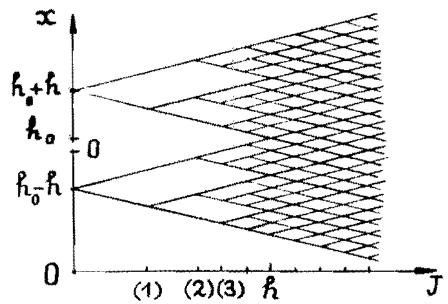
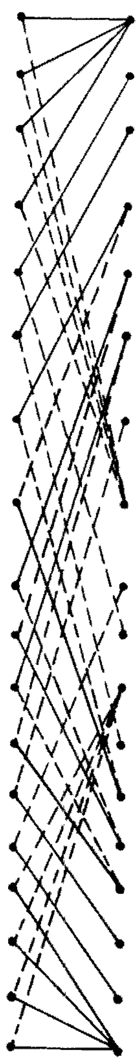


Fig.4.

The space of states X of the Markov chain (2.1) for nonzero mean external field ho = = h/4 as function of J for T=0, 0 < α < 1. Only the essential states are shown.

TABLE



State		Mapping		w_i^*		
i	z_i	n	n+1	$0 < \alpha < 1$	$\alpha = 0$	$\alpha = 1$
1	(h+J, h)			$\frac{\gamma}{2(1+2\alpha)}$	$w_1 = 1, 0$	0
2	(2h, h)			0	0	0
3	(3h−J, h)			$\frac{\alpha\gamma}{2(1+\alpha)(1+2\alpha)}$	0	0
4	(J, h)			$\frac{\alpha^2}{(1+\alpha)(1+2\alpha)}$	0	w_8
5	(h, h)			0	0	w_{10}
6	(2h−J, h)			$\frac{\alpha}{2(1+\alpha)(1+2\alpha)}$	0	w_{12}
7	(−h+J, h)			$\frac{\alpha^2}{2(1+\alpha)(1+2\alpha)}$	0	w_{13}
8	(−h+J, −h)			$\frac{\alpha}{2(1+2\alpha)}$	0	w_4
9	(0, h)			0	0	w_{14}
10	(0, −h)			0	0	w_5
11	(h−J, h)			w_8	0	w_{15}
12	(h−J, −h)			w_7	0	w_6
13	(−2h+J, −h)			w_6	0	w_7
14	(−h, −h)			0	0	w_9
15	(−J, −h)			w_4	0	w_{11}
16	(−3h+J, −h)			w_3	0	0
17	(−2h, −h)			0	0	0
18	(−h−J, −h)			w_1	$1-w_1$	0

In Fig.4 we present the dependence of the essential states for $0 < \alpha < 1$ as a function of J for h_0 less than h, namely for $h_0 = h/4$. For this value of h_0 the support shows a behaviour similar to that for $h_0 = 0$ with the difference that the symmetry with respect to $X = 0$ is broken, and that for J not too small the states are denser because the bifurcations at (1), (2),... have a smaller period compared with the zero mean case.

For $h \leq h_0$ and $0 < \alpha \leq 1$ we find a completely different behaviour. We first consider the flow diagram for $h = h_0$ and $2qh < J < 2(q+1)h$, $(q = 1, 2, \dots)$:

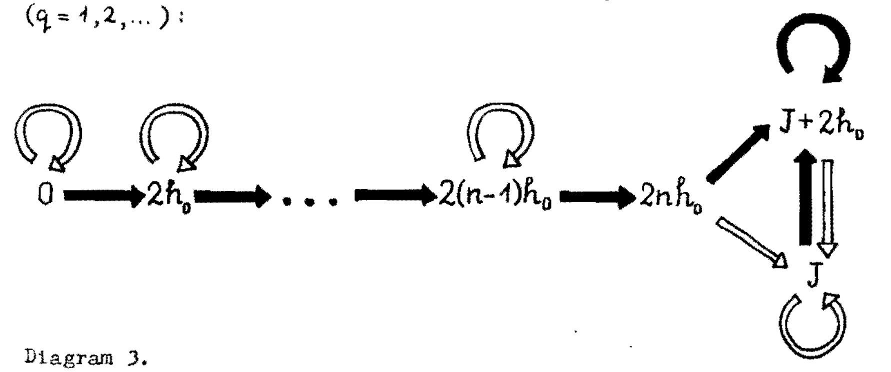


Diagram 3.

It shows that the only essential states S are J and $J+2h_0$. A similar analysis gives the result that these states are the only essential ones also for $h < h_0$. Note that the space of essential states is the same both for stochastic ($0 < \alpha < 1$) and periodic ($\alpha = 1$) external fields.

This drastic reduction of the space of states in dependence on the mean value of the external field can also be found for nonzero temperature, see below.

5. THE MAPPING FOR NONZERO TEMPERATURE

5.1. The Space of the States and the Invariant Measure

To describe the space of states of the stochastic mapping (1.6) for nonzero temperature, we introduce the following notation. We denote the result of the iteration of the mapping (1.6) starting from the arbitrary initial value $\xi_0 = y$ by

$$x_{\epsilon_n, \epsilon_{n-1}, \dots, \epsilon_1; y} = f(h_n, f(h_{n-1}, f(\dots, f(h_1, y)\dots))) , \qquad (5.1)$$

where $\{h_1,\dots,h_n\}$ is a given realization of the binary driving process and $\{\sigma_1,\dots,\sigma_n\}$ is the corresponding sequence of signs. It is clear that the space of states of the driven Markov process consists of all points which can be represented in this way.

As for the zero temperature case, we call the sets of states which are invariant under the mapping (1.6) the sets of essential states. Since for nonzero temperature $f \in C^\infty(\mathbb{R}^2)$ and $0<\partial_x f(x,h)<1$, the sets of essential states are given by $\{\lim_{n\to\infty} x_{\sigma_1,\dots,\sigma_n;y}\}_{y\in\mathbb{R}^1}$. The existence of these limits and their independence of the starting point $y\in\mathbb{R}^1$ are provided by the property $\partial_x f(x,h)<1$, see, e.g., Ref. /10/. We denote these limits by $S=\{x_{\underset{\sim}{\sigma}}\}$ where $\underset{\sim}{\sigma}$ corresponds to an infinite realization $\underset{\sim}{h}$ of the driving process.

As a consequence of the independence of $\{x_{\underset{\sim}{\sigma}}\}$ on the starting points $y\in\mathbb{R}^1$ one has: (i) for the stochastic mapping (1.6) the set $S=\{x_{\underset{\sim}{\sigma}}\}$ is an attractor whose basin of attraction is $\mathbb{R}^1$; (ii) for $0<\alpha<1$ any two points $x_{\underset{\sim}{\sigma}'}, x_{\underset{\sim}{\sigma}''}\in S$ can be connected by the mapping (1.6), so that there are no disconnected invariant subsets in S. Therefore, the attractor S is a unique set of essential states (support) for the driven Markov process (1.6) for $T>0$ and $0<\alpha<1$.

We can construct the invariant measure for this process by iteration of the corresponding Chapman-Kolmogorov equation (2.1) starting from an arbitrary nontrivial probability density $p_0(x,\eta)$ on $\mathbb{R}^1\times\{h_0\pm h\}$. Because the basin of attraction of S is $\mathbb{R}^1$, there exists a compact $K=K_1\times K_2$ such that

$$\int_{K_1\times K_2} dP_n(x,\eta)=1\ ,\ n\geq 1\quad ;\quad dP_n(x,\eta)=p_n(x,\eta)\,dx\,d\eta\ . \qquad (5.2)$$

Then by Prohorov's theorem /12/ the sequence of probability measures $\{P_n\}_{n\geq 1}$ is compact with respect to weak convergence, i.e., there are subsequences $\{P_{n_k}\}_{n_k\geq 1}$ such that for arbitrary $g\in C(\mathbb{R}^1\times\mathbb{R}^1)$ one has

$$\lim_{n_k\to\infty}\int dP_{n_k}(x,\eta)\,g(x,\eta)=\int dP_k^*(x,\eta)\,g(x,\eta)\ . \qquad (5.3)$$

By construction (5.3) the invariant measures $\{P_k^*\}$ have the same support coinciding with the attractor S which is the only set which is invariant with respect to the mapping (1.6) (transitivity).

This means that the measures $\{P_k^*\}$ are ergodic. But on the same support there exists only one ergodic invariant measure /13/, i.e., the sequence $\{P_n\}_{n\geq 1}$ converges to the unique invariant probability measure P^*.

5.2. The Geometrical Structure of the Support and the Fractal Order Parameter

We now consider the geometrical structure of the attractor S on $\mathbb{R}^1$. As it follows from 5.1, for the binary driving process there is a one-to-one correspondence between the points $x_{\underset{\sim}{\sigma}}\in S$ and the infinite sequences $\underset{\sim}{\sigma}$ of + and −. Therefore, the cardinality of the set S is the continuum.

Further, by construction (5.1) the points of the support S are located in the interval $[x_{\underset{\sim}{\sigma}^-}, x_{\underset{\sim}{\sigma}^+}]$, where $\underset{\sim}{\sigma}^{\pm}=\{\sigma_n=+ \text{ or } \sigma_n=-\}_{n=1}^{\infty}$. As it follows from (5.1), the two points $x_{\underset{\sim}{\sigma}^\pm}$ are the trapping states (fixed points) for $\alpha=0$, the case of the constant field (see Section 2). Thus, in this limit the attractor reduces to two disjoint parts concentrated at the points $S_+=x_{\underset{\sim}{\sigma}^+}$ and $S_-=x_{\underset{\sim}{\sigma}^-}$ (cf. 5.1). To discuss the details we restrict ourselves in the following to the case of a zero mean random field. Then one has (see Fig.5)

$$x_{\underset{\sim}{\sigma}^\pm}=\pm x^*\ . \qquad (5.4)$$

For $\alpha=1$ (alternating external field) the attractor S (see (5.1)) also reduces to a two-point set $\{\pm x_p\}$, where $x_p=x_{\underset{\sim}{\sigma}^{+-}}$ ($\underset{\sim}{\sigma}^{+-}$ is the infinite alternating sequence starting with +), or equivalently, the solution of the equation $x_p=f(h,-x_p)$. But now $S=\{-x_p, x_p\}$ is a connected invariant set corresponding to an attractive orbit of the mapping (1.6) for $\alpha=1$ (cf. Fig.5).

As can be seen from (5.1) (cf. also Fig.5) there are no states from S between the points

$$x_{+,\underset{\sim}{\sigma}^-}=f(h,x_{\underset{\sim}{\sigma}^-})\qquad\text{and}\qquad x_{-,\underset{\sim}{\sigma}^+}=f(-h,x_{\underset{\sim}{\sigma}^+})\ . \qquad (5.5)$$

Therefore, the set S has a gap of the length

$$\Delta=x_{+,\underset{\sim}{\sigma}^-}-x_{-,\underset{\sim}{\sigma}^+}=2\,(2h-x^*)\ . \qquad (5.6)$$

After applying the mapping (1.6) this gap produces two gaps of the next generation, whose end-points are (see Fig.5):

$$x_{\sigma_2,\sigma_1,\underset{\sim}{\sigma}^{\pm}} = f\,(h_2, f\,(h_1, x_{\underset{\sim}{\sigma}^{\pm}}))\,.$$

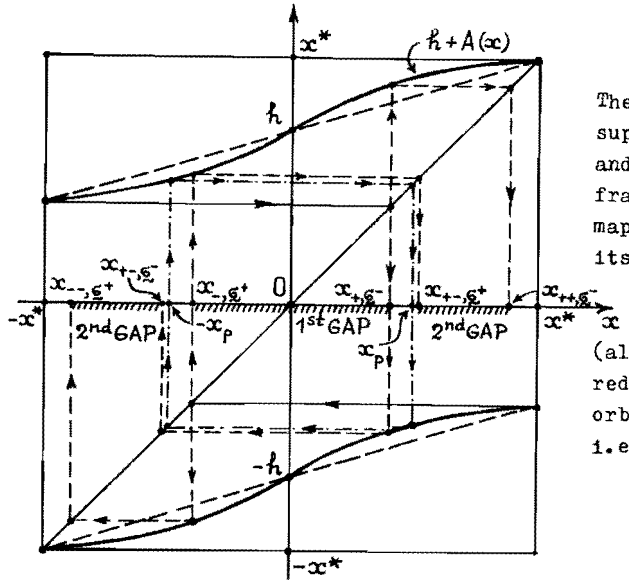


Fig.5.

The construction of the support (attractor) S and the origin of its fractal structure for mapping (1.6) and for its linearised version (bold dashed lines). For $\alpha = 1$ (alternating field) S reduces to an attracting orbit (dot-dashed lines), i.e. $S = \{-x_p, x_p\}$.

In the same way one can construct the end-points of the gaps in the n-th generation as

$$x_{\sigma_n,\sigma_{n-1},\dots,\sigma_1,\underset{\sim}{\sigma}^{\pm}} = f\,(h_n, \dots, f\,(h_1, x_{\underset{\sim}{\sigma}^{\pm}})\dots)\,. \qquad (5.7)$$

This procedure allows one to construct all gaps in the attractor S. We call the finite sequence of n (different) signs "head", and the infinite sequence of identical signs "tail". The two end-points of one of the gaps in the n-th generation can be represented by two infinite sequences consisting of a head of n signs which differ only in the first sign and an infinite tail of signs opposite to the first one of the head.

Hence the set of all end-points is obviously countable. On the other hand, it is dense in the support S : in an arbitrary neighbourhood of a point $x_{\underset{\sim}{\sigma}} \in S$ one can find an end-point (an end-point is as closer to $x_{\underset{\sim}{\sigma}}$ as longer its "head" is which coincides with the corresponding first signs of $\underset{\sim}{\sigma}$). Vice versa, the set S is nowhere dense. Therefore, the support S constitutes a Cantor-type-fractal /14/, but in contrast to the Cantor set it has no simple self-similar structure.

To elucidate the latter, let us linearize the mapping (1.6) on the interval $[-x^*, x^*]$ substituting the function $f(\pm h, x)$ by $\pm h + x(x^* - h)/x^*$, see Fig.5. Then the above procedure (5.4-7) gives us instead of S the standard Cantor set C_Δ with the largest gap equal to Δ (5.6). Now it is clear that S is nothing but a smooth deformation of C_Δ , and deviations of the support S from the Cantor set C_Δ are due to the nonlinearity of the function $A(x)$, see (1.6) and Fig.5.

Now we give a qualitative analysis of the Hausdorff (fractal) dimension /13/ $d_f = d_H(S)$ of the support S in dependence on the physical parameters of the system (1.1).

For the zero mean external random field we represent this dependence in Fig.6 (the phase diagram). The condition $\Delta(h, x^*(J,h,T>0)) = 0$ (5.6) defines the boundary between two

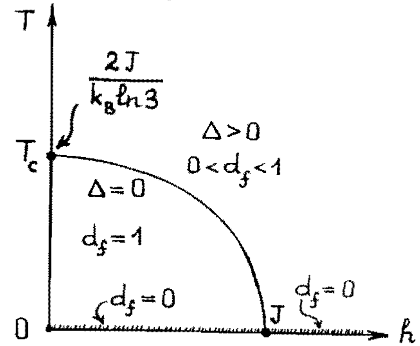


Fig.6.

The phase diagram for the model (1.1) (zero mean external field) with fractal order parameter d_f .

essentially different regions: for $\Delta > 0$ the support S has a fractal structure with $0 < d_f < 1$, whereas for $\Delta = 0$ (and formally for $\Delta < 0$, see (5.6)) the support has no gap, i.e. $d_f = 1$. To discuss the behaviour of d_f as a function of (J, h, T) we use the approximation $d_f \simeq d_H(C_\Delta)$, whose accuracy depends on the accuracy of the linearized mapping considered above. Because the Hausdorff dimension $d_H(C_\Delta)$ of the Cantor set C_Δ is well-known /14/, one gets

$$d_f(J,h,T) = \begin{cases} 1 & \text{for } \{(h,T):\ \Delta(h,x^*) \le 0\} \\ \dfrac{\ln 2}{\ln \dfrac{x^*}{x^*-h}} & \text{for } \{(h,T):\ \Delta(h,x^*) > 0\}\,. \end{cases} \qquad (5.8)$$

The result (5.8) establishes the phase diagram presented in Fig.6 and gives a reason to consider $d_f = d_f(J, h, T)$ as a fractal order parameter [x].

For instance, we obtained above that at T = 0 the support S consists of a finite number of states (see Sections 3 and 4)), i.e. $d_f(J, h, T=0) = 0$. Therefore, with $T \to 0$ one should observe a transition of d_f to zero (cf. (5.8)) which is continuous in the gap region, but should be discontinuous in the gapless one (see Fig.6). On the other hand, the function $A(x)$ tends to zero for $T \to \infty$ (cf. (1.3)). Then the support S reduces to the two-point set $\{-h, h\}$ because for $A(x) \to 0$ the first gap increases and its end-points converge to $\pm x^* = \pm h$ (cf. (5.6) and Fig.5). Consequently, $d_f(J, h, T \to \infty) = 0$ and the border line on the phase diagram should behave for $T \to \infty$ as it is presented in Fig.6.

Finally, for a nonzero mean external random field, $\{h_n = h_0 \pm h\}_{n=1}^{\infty}$ we observe that for small $h_0 > 0$ $(h_0 \leqslant J)$ the fractal structure of the support S becomes nonsymmetric. It contracts to the left end-point x_ℓ^* and stretches near x_r^* (cf. Fig.7) while in the linear approximation the fractal structure is similar to that for $h_0 = 0$. For large $h_0 > 0$ $(h_0 \gg J)$ the first gap rapidly increases and the fractal structure approaches that in the linear approximation because the curvature of $A(x)$ for $x \geqslant x_\ell^*$ ($x_\ell^* \to \infty$ for $h_0 \to \infty$) tends to zero. Therefore, in this case one can again utilize the approximation $d_H(S) \simeq d_H(C_\Delta)$ to evaluate the dimensionality of the support.

6. CONCLUSIONS

The present paper is devoted to the study of the one-dimensional random field Ising model (RFIM) by the stochastic mappings method. We restrict ourselves to investigate the support S of the fixed points (or stationary) measure rather than calculating physical observables. As it follows from above, the knowledge of S and $\mu(dx)$ allows us to calculate the free-energy density (1.7), the magnetization per site (1.9)and the Edwards-Anderson parameter (1.10). These calculations can be easily performed for T=0 (e.g. using Table) ; but not so easy for $T > 0$. Starting from the Chapman-Kolmogorov

x Very recently, the fractal dimension of the RFIM was calculated avoiding this approximation /21/ . The results support the phase diagram in Fig.6.

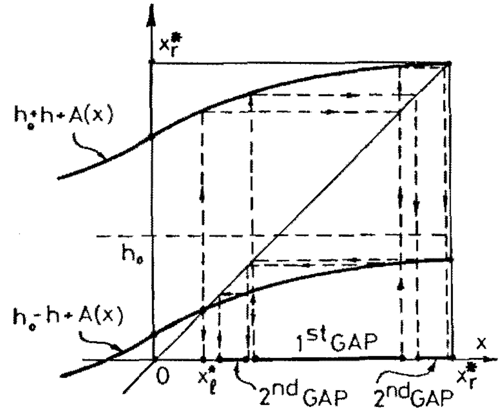


Fig.7.

The construction of the support (attractor) S and arising of its fractal structure for the mapping (1.6) with nonzero mean external field: $0 < h_0 < J$.

equation (2.1) (for uncorrelated random field see Ref. /8/) we can do these calculations as a systematic expansion in terms of the momenta $\langle h^m \rangle$ of the stationary probability measure corresponding to the Markov chain driven by the external field, see (3.5). Then for zero mean external field we get for the Edwards-Anderson parameter susceptibility the following:

$$\lim_{T \to 0} \frac{\partial}{\partial \langle h^2 \rangle} q_{EA} = \infty \ , \tag{6.1}$$

where the divergence is exponential for $0 < \alpha \leqslant 1$.

On the other hand, the most instructive information about thermodynamics of the model is obviously contained in the behaviour of the support S including the "phase transitions" which we propose to characterize by the "order parameter" d_f . Simultaneously as it follows from Section 4 we have $\lim_{h_0 \to \pm 0} m(h_0) = 0$ for T = 0 (see also Ref./15/). Therefore, the phase transitions we discussed have no connection with the controversy on the lower critical dimension of RFIM (partially settled in Refs. /16-18/) where the magnetization is the order parameter.

As it is shown above, the local field in the one-dimensional RFIM may have a very peculiar distribution with a nontrivial fractal dimensionality of its support. The natural question is whether this is also characteristic of other simple models, e.g. of mean-field ones. In recent papers /19,20/ the mean-field RFIM is considered for an independent and identically distributed random external field. From Ref. /20/ , Section 3 it follows that in this case the distribution of the local magnetization only mimics that of the external field.

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Бен У., Загребнов В.А. E17-87-138

Одномерная модель Изинга в случайном поле
и стохастические дискретные отображения

Одномерная модель Изинга в случайном внешнем поле исследована с помощью стохастических отображений. Рассмотрен случай /замороженного/ марковского внешнего поля. Показано, что вся информация о термодинамических свойствах модели содержится в инвариантной мере, которая соответствует некоторому марковскому процессу, управляемому внешним полем, и в частности, в носителе S этой меры. При нулевой температуре S содержит конечное число точек, а при ненулевой является нехаотическим /странным/ аттрактором с фрактальной структурой канторовского типа. Показано, что фрактальная размерность d_f носителя S играет роль параметра порядка для данной модели. Исследована зависимость S и d_f от параметров модели и температуры.

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Behn U., Zagrebnov V.A. E17-87-138

One-Dimensional Random Field Ising Model and Discrete
Stochastic Mappings

The one-dimensional random field Ising model is studied using stochastic mappings. The case of a (frozen) Markovian external random field is considered. We show that all information about thermodynamic properties of the model is contained in an invariant (stationary) measure corresponding to some Markov process driven by the external field and particularly in the support S of this measure. For zero temperature it contains a finite number of points, but for nonzero ones it is a nonchaotic (strange) attractor with a Cantor-type fractal structure. The fractal dimensionality d_f of S is proposed as an "order parameter" for the model. The dependence of S and d_f on the model parameters and the temperature is studied in detail.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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