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E17-85-945

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EXACT RESULTS IN THE PROBLEM OF INTERACTION OF A TWO-LEVEL ATOM WITH RADIATION

Submitted to "ДАН СССР"

1985

Studies of the quantum model of interaction of a two-level "atom" with a radiation resonance field were initiated by Jaynes and Cummings[1] who first succeeded in obtaining an exact solution to a model of the Dikke type. Later on various variants of multiboson processes were studied in two-level atoms. Here we mention papers by Buck and Sukumar[2,3,4] who have found an exact solution to the equations of motion describing an atomic system with the interaction nonlinear in bosonic variables. A distinctive, from a physical point of view, feature of the Jaynes-Cummings model and its multiboson modifications is the effect, observed for the first time by Eberly et al. [5], of periodic reproduction of atomic energy oscillations due to the initial coherent pumping.

In this note, we consider exactly solvable generalizations of the Jaynes-Cummings model nonlinear in bosonic variables. The obtained spectra may be used, for instance, to analyse the relaxation of a single spin to the thermostat temperature given by the boson field.

1. Let us consider the Hamiltonian

$$H = \omega a^+ a + \omega_0 S_3 + \lambda (a^+)^k a^\ell f(a^+ a) S_+ + \lambda f(a^+ a)(a^+)^\ell a^k S_- \qquad (1)$$

describing the interaction of a two-level atom with transition energy $2\omega_0$ with a single-mode phonon field a, $[a, a^+] = 1$. In model (1) atomic transitions occur through emission of ℓ or k phonons and absorption of k or ℓ phonons, the interaction intensity $\lambda f(a^+ a)$ being dependent on the phonon-field strength through the function f obeying the condition: $f(a^+ a)|n\rangle = f(n)|n\rangle$, where $a^+ a|n\rangle = n|n\rangle$. The Pauli operators are normalized so that $[S_3, S_\pm] = \pm 2 S_\pm$, $[S_+, S_-] = S_3$. The operator $\hat{N} = a^+ a + \frac{\ell - k}{2} S_3$ is a constant of motion: $[H, N] = 0$.

Consider the case $\ell \geq k$; we look for a combined system of eigenvectors H and $\hat{N}$ in the form

$$|\Psi_n(\alpha)\rangle = \exp\left(\alpha S_+ (a^+)^k a^\ell f(a^+ a)\right) |\varphi_n^{(-)}\rangle , \qquad (2)$$

where $\hat{N}|\varphi_n^{(\pm)}\rangle = \left(n \pm \frac{1}{2}(\ell - k)\right)|\varphi_n^{(\pm)}\rangle$ and α is a parameter to be determined. Because of the relation $[S_+ (a^+)^k a^\ell f(a^+ a), \hat{N}] = 0$ the

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vector $|\Psi_n(\alpha)\rangle$ corresponds to the same eigenvalue of the charge operator $N^{(-)} = n - \frac{1}{2}(\ell-k)$ as $|\Phi_n^{(-)}\rangle$. Inserting (2) into the Schrödinger equation

$$H|\Psi\rangle = \mathcal{E}|\Psi\rangle \qquad (3)$$

we determine parameters α and $\mathcal{E}$. The result is as follows:

When $n \geq \ell$,

$$|\Psi_n^{(\pm)}\rangle = \left(1 + \alpha_{nk\ell}^{(\pm)2} C_{nk\ell}^2 f^2(n)\right)^{-1/2} \begin{pmatrix} \alpha_{nk\ell}^{(\pm)} f(n) C_{nk\ell} |n-\ell+k\rangle \\ |n\rangle \end{pmatrix} \qquad (4)$$

$$\mathcal{E}_n^{(\pm)} = \omega n - \frac{\omega}{2}(\ell-k) \pm \sqrt{\Omega_{k\ell}^2 + \lambda^2 C_{nk\ell}^2 f^2(n)}, \quad C_{nk\ell}^2 = \frac{n!(n-\ell+k)!}{(n-\ell)!(n-\ell)!}$$

$$\alpha_{nk\ell}^{(\pm)} = \frac{-\Omega_{k\ell} \pm \sqrt{\Omega_{k\ell}^2 + \lambda^2 C_{nk\ell}^2 f^2(n)}}{\lambda C_{nk\ell}^2 f^2(n)}, \quad \Omega_{k\ell} = \frac{\omega(\ell-k)}{2} - \omega_0.$$

For $n<\ell$,

$$\alpha_{nk\ell}^{(+)} = \lambda/2\Omega_{k\ell}, \quad |\Psi_n^{(+)}\rangle = \begin{pmatrix}0\\ |n\rangle\end{pmatrix}, \quad \mathcal{E}_n^{(+)} = \omega n - \omega_0, \; n\geq 0 \qquad (5)$$

$$\alpha_{nk\ell}^{(-)} = \infty, \quad |\Psi_n^{(-)}\rangle = \begin{pmatrix}|n-\ell+k\rangle\\ 0\end{pmatrix}, \quad \mathcal{E}_n^{(-)} = \omega(n-\ell+k)+\omega_0, \; n\geq \ell-k.$$

In formulae (4) it is assumed that $f(n)\neq 0$. If, for a certain $n_0 \geq 0$, $f(n_0)=0$, the corresponding wave functions and energy levels will be

$$|\Psi_{n_0}^{(+)}\rangle = \begin{pmatrix}0\\ |n_0\rangle\end{pmatrix}, \quad \mathcal{E}_{n_0}^{(+)} = \omega n_0 - \omega_0$$

$$|\Psi_{n_0}^{(-)}\rangle = \begin{pmatrix}|n_0-\ell+k\rangle\\ 0\end{pmatrix}, \quad \mathcal{E}_{n_0}^{(-)} = \omega(n_0-\ell+k)+\omega_0, \; n_0\geq \ell-k.$$

The case $k\geq\ell$ is examined analogously. Formulae (4) and (5) at $f\equiv 1$, $k=0$, and $\ell=1$ transform into formulae of the conventional Jaynes-Cummings model. Expressions obtained in /2/ for various averages of the energy operator of an atomic system $S_3(t)=e^{iHt}S_3e^{-iHt}$ follow from formulae (4) and (5) if one puts $f(x)=\sqrt{x}$, $k=0$, $\ell=1$. Note also that in /3/ squares are found for normal modes of oscillations of an atomic system $W_n^2 = (\mathcal{E}_n^{(+)} - \mathcal{E}_n^{(-)})^2$ within model (1) at $f=1$,

$k=0$, and arbitrary ℓ. Composing the expression $W_n^2 = (\mathcal{E}_n^{(+)} - \mathcal{E}_n^{(-)})^2$ by formulae (4) and (5) and setting $f=1$, $k=0$, we arrive at the result of ref. /3/.

2. Now, consider a generalization of the Jaynes-Cummings model to the case of nonlinear interaction of a two-level system with a two-mode radiation field a, b:

$$H = \omega_1 a^+a + \omega_2 b^+b + \omega_0 S_3 + \lambda a^\ell (b^+)^k S_+ + \lambda (a^+)^\ell b^k S_- \qquad (6)$$

It is essential that this model possesses two independent integrals of motion, $\hat N = a^+a + \frac{\ell}{2}S_3$ and $\hat M = b^+b - \frac{k}{2}S_3$. A complete system of eigenvectors of operators $H, \hat N$ and $\hat M$ should correspond to all possible sets of combined values of "charges" $\hat N$ and $\hat M$:

$$\left\{N^{(-)} = n - \frac{\ell}{2},\ M^{(-)} = m + \frac{k}{2};\ n,m\geq 0\right\}, \quad \left\{N^{(+)} = n+\frac{\ell}{2},\ M^{(+)} = m - \frac{k}{2};\ n,m\geq 0\right\}$$

where $\hat N|\Phi_{nm}^{(\pm)}\rangle = N^{(\pm)}|\Phi_{nm}^{(\pm)}\rangle$, $\hat M|\Phi^{(\pm)}_{nm}\rangle = M^{(\pm)}|\Phi^{(\pm)}_{nm}\rangle$.

Therefore we look for solutions to the Schrödinger equation (3) with Hamiltonian (6) in the form of a union of two system of vectors

$$\left\{|\Psi_{nm}^{(1)}(\alpha_1)\rangle = \exp(\alpha_1 a^\ell b^{+k} S_+)|\Phi^{(-)}_{nm}\rangle\right\}, \quad \left\{|\Psi^{(2)}_{nm}\rangle = \exp(\alpha_2 a^{+\ell} b^k S_-)|\Phi^{(+)}_{nm}\rangle\right\} \qquad (7)$$

the first of which corresponds to the set $\{N^{(-)}, M^{(-)}\}$, while the second to $\{N^{(+)}, M^{(+)}\}$. The Schrödinger equation for the vector $|\Psi^{(1)}\rangle$ is transformed into the corresponding equations for $|\Psi^{(2)}\rangle$ by the unitary transformation $a\leftrightarrow b$, $S_3\to -S_3$, $S_+\leftrightarrow S_-$ with subsequent change $k\leftrightarrow\ell$, $\omega_1\leftrightarrow\omega_2$, $\omega_0\to-\omega_0$.

We shall not write solutions for $|\Psi^{(1)}\rangle$ and $|\Psi^{(2)}\rangle$ separately and only list a final result for eigenfunctions and energy levels of Hamiltonian (6):

When $n\geq\ell$, $m\geq 0$,

$$|\Psi^{\pm}_{nm}\rangle = \left(1+\alpha^{(\pm)2}C^2_{nm}\right)^{-1/2}\begin{pmatrix}\alpha^{(\pm)}_{nm}C_{nm}|n-\ell\rangle|m+k\rangle\\ |n\rangle|m\rangle\end{pmatrix} \qquad (8)$$

$$\mathcal{E}^{(\pm)}_{nm} = \omega_1 n - \frac{\omega_1\ell}{2} + \omega_2 m + \frac{\omega_2 k}{2} \pm \sqrt{\Omega^2_{k\ell} + C^2_{nm}\lambda^2}.$$

$$\alpha^{(\pm)}_{nm} = \frac{\Omega_{k\ell} \pm \sqrt{\Omega^2_{k\ell} + \lambda^2 C^2_{nm}}}{\lambda C^2_{nm}}, \quad C^2_{nm} = \frac{n!\,(m+k)!}{(n-\ell)!\,m!}$$

$$\Omega_{k\ell} = \omega_0 - \omega_1 \ell/2 + \omega_2 k/2$$

When $n<\ell$, $m \geq 0$,

$$|\Psi_{nm}\rangle = \begin{pmatrix} 0 \\ |n\rangle|m\rangle \end{pmatrix}, \quad \mathcal{E}_{nm} = \omega_1 n + \omega_2 m - \omega_0. \tag{9}$$

Besides, for $m<k$, $n \geq 0$ there is also the solution

$$|\Psi'_{nm}\rangle = \begin{pmatrix} |n\rangle|m\rangle \\ 0 \end{pmatrix}, \quad \mathcal{E}'_{nm} = \omega_1 n + \omega_2 m + \omega_0 \tag{10}$$

To verify the completeness of the system of functions (8)-(10), we compose the operator $\sum_{\{i\}} |\Psi_i\rangle\langle\Psi_i|$ and obtain the 2x2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in which, for instance:

$$a_{11} = \sum_{n \geq \ell,\, m \geq 0} \frac{\left(\alpha^{(+)}_{nm} C_{nm}\right)^2}{1 + \left(\alpha^{(+)}_{nm} C_{nm}\right)^2} |n-\ell\rangle\langle n-\ell|\,|m+k\rangle\langle m+k| \; +$$

$$\sum_{n \geq \ell,\, m \geq 0} \frac{\left(\alpha^{(-)}_{nm} C_{nm}\right)^2}{1 + \left(\alpha^{(-)}_{nm} C_{nm}\right)^2} |n-\ell\rangle\langle n-\ell|\,|m+k\rangle\langle m+k| \; +$$

$$\sum_{n \geq 0,\; 0 \leq m < k} |n\rangle\langle n|\,|m\rangle\langle m|.$$

Owing to the relation $(\alpha^{(+)}C)^2/1+(\alpha^+C)^2 + (\alpha^{(-)}C)^2/1+(\alpha^-C)^2 = 1$, $a_{11} = \sum_0^{\infty} |n\rangle\langle n| \sum_0^{\infty} |m\rangle\langle m| = 1$. In the same way it may be shown that $a_{22} = 1$ and $a_{12} = a_{21} = 0$. The completeness of the system of functions (4)-(5) is proved analogously.

References

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Received by Publishing Department
on December 26, 1985.

Кочетов Е.А. E17-85-945

Строгие результаты в задаче о взаимодействии двухуровневого атома с излучением

Рассматривается обобщение модели Джейнса - Каммингса, описывающей взаимодействие двухуровневой системы с полем излучения, на случай нелинейного по бозонным переменным взаимодействия. Для соответствующих гамильтонианов найдены точные волновые функции и уровни энергии.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Kochetov E.A. E17-85-945

Exact Results in the Problem of Interaction of a Two-Level Atom with Radiation

The Jaynes-Cummings model describing interaction of a two-level system with a radiation field is generalized to the case of interaction, nonlinear in bosonic variables. Exact wave functions and energy levels are found for the corresponding Hamiltonians.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985