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# RENORMALIZATION GROUP INVARIANTS FROM A LARGE-n LIMIT MECHANICAL ANALOGY

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It is well established [1] that differential formulations of the Wilson renormalization group (RG) are far more convenient and effective to apply than the finite recursion relation approaches. This is especially the case where one manipulates over large domains of the variable involved and the RG nonlinear properties are to be investigated. With this in mind, we have recently presented [2] a differential formulation of the dynamic RG (DRG) in the large- $n$ limit, for the time-dependent generalization of the $n$-vector classical model with purely relaxational dynamics, also including the static RG (SRG)[3] as a particular case.

Some peculiar aspects of the RG differential equations allowed to introduce [2] an interesting mechanical analogy which gives the possibility to introduce an alternative geometrical picture for both DRG and SRG in the large- $n$ limit. As has already been pointed out in ref.[2], this may be particularly relevant for obtaining a more intuitive insight into the structure of the DRG, whose general features are at present less understood [4] than those of the static one [5].

Since the proposed mechanical analogy constitutes a new point of view in exploring the RG properties in the large- $n$ limit, further more detailed investigations in such direction are necessary.

In this short note we present some preliminary aspects of the dual mechanical problem which may have some relevance in realizing the above specified program. A more detailed study of the problem, including direct physical insights into the original RG formulation, will be presented elsewhere.

We find it convenient to start with a summary of the considerations leading to the mechanical analogy for the RG in the large- $n$ limit.

We consider the $n$-vector $d$-dimensional classical model described by the effective Hamiltonian:

$$\mathcal{H}\{\Phi\} = \int d^dx\,[(\nabla\vec{\Phi})^2 + U(\Phi^2)], \qquad (1)$$

where $\vec{\Phi} \equiv \{\Phi_a ; a = 1,\dots,n\}$ is the $n$-component order parameter with a wave vector cut off assumed to be equal to unity and [4]:

$$U(\Phi^2) = \sum_{m=1}^{\infty} \frac{u_{2m,2}}{m} (\Phi^2)^m, \quad u_{2m,2} = O(n^{1-m}). \qquad (2)$$

The time evolution of the time-dependent order parameter in the purely relaxational case is governed by the Langevin equation:

$$\frac{\partial \Phi_\alpha}{\partial \tau} = - L \frac{\delta \mathcal{H}}{\delta \Phi_\alpha} + \zeta_\alpha , \qquad (3)$$

where $\zeta(\vec{x}, \tau) \equiv \{\zeta_\alpha(\vec{x}, \tau);\ \alpha = 1, \dots, n\}$ is a Gaussian white noise source and $L = \Gamma_0 (i\nabla)^c$ $(c = 0,2)$. Here $\Gamma_0$ is a real constant, conveniently assumed equal to unity, and $c = 0$ $(c = 2)$ corresponds to a nonconserved (conserved) order parameter. By using the finite recursion relations of the DRG in the large- n limit as given in ref.[4] or starting from the general form of the exact RG differential equations [1], one obtains [2,3,6] the large- n limit DRG differential equations

$$\frac{\partial t_i}{\partial \ell} + (d-2)(\Psi^2 - F)\frac{\partial t_i}{\partial \Psi^2} + (d+c)(\theta - G)\frac{\partial t_i}{\partial \theta} = a_i t_i , \qquad (4)$$

$$(i = 1,2)$$

where $\ell$ is a parameter describing the progress of the RG averaging, $t_i(\ell, \Psi^2, \theta)(i=1,2)$ are global functions [2] of $\ell$ and the new fields $\Psi^2 = \Phi^2/N_c$ , $\theta = \frac{d+c}{d-2}\frac{\phi}{N_c}$ (with $N_c = \frac{1}{2} n K_d/(d-2)$ , $K_d = \pi^{-d/2} 2^{1-d} \Gamma(\frac{d}{2})$) and $\phi = \phi(\vec{x}, \tau)$ is an additional field generated by the construction of the "initial large- n action" within a path functional representation [4,7] of the stochastic process (3). In eqs. (4):

$$F(t_1, t_2) = [(1 + t_1^2) - 2t_2]^{-1/2} ,$$

$$G(t_1, t_2) = 1 - (1 + t_1) F(t_1, t_2), \qquad (5)$$

and $a_1 = 2$, $a_2 = 4 + c$.

The RG equations (4) constitute a system of "quasi-linear" first order partial equations.

The SRG transformation [3] is simply obtained putting in (4) $\theta = 0$ with $t_2(\ell, \Psi^2, 0) \equiv 0$ and $t_1(\ell, \Psi^2, 0) = t(\ell, \Psi^2) = [\partial U(\ell, \Phi^2)/\partial \ell]_{\Phi^2 = N_c \Psi^2}$. Then, one obtains the single SRG equation:

$$\frac{\partial t}{\partial \ell} + (d-2)[\Psi^2 - \frac{1}{1+t}]\frac{\partial t}{\partial \Psi^2} = 2t . \qquad (6)$$

By inspection of eqs.(4), a very peculiar structure appears: the coefficients of the corresponding derivatives are identical. Thus, they constitute a system of equations with the "same principal part" whose properties are well established [8]. This surprising feature of the DRG transformation (and the static one as a particular case) in the large- n limit gives the possibility of introducing a new approach to the critical dynamic (and static) problem. Firstly, by a known theory [8], the system (4) is equivalent to the single homogeneous linear partial equation:



$$\frac{\partial S}{\partial \ell} + (d-2)(\Psi^2 - F)\frac{\partial S}{\partial \Psi^2} + (d+c)(\theta - G)\frac{\partial S}{\partial \theta} + \sum_{i=1}^{2} a_i t_i \frac{\partial S}{\partial t_i} = 0 \qquad (7)$$

for an unknown function $S(\ell, \Psi^2, \theta, t_1, t_2)$. Then, if we put $q_1 = \Psi^2$, $q_2 = t_1$ , $q_3 = \theta$ , $q_4 = t_2$ , $p_j = \frac{\partial S}{\partial q_j}$ $(j = 1, \dots, 4)$ and introduce the function, not depending explicitly on the parameter $\ell$,

$$H(\{q_i\}, \{p_j\}) = \sum_{j=1}^{4} \alpha_j q_j p_j - [\alpha_1 F(q_2, q_4) p_1 + \alpha_3 G(q_2, q_4) p_3] \qquad (8)$$

with $\alpha_1 = d - 2$ , $\alpha_2 = 2$ , $\alpha_3 = d + c$ , $\alpha_4 = 4 + c$ eq.(7) for $S(\ell, \{q_i\})$ can be rewritten as:

$$\frac{\partial S}{\partial \ell} + H(\{q_j\};\ \{\frac{\partial S}{\partial q_j}\}) = 0. \qquad (9)$$

The corresponding equations of characteristics assume the form:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = - \frac{\partial H}{\partial q_j} , \qquad (j = 1, \dots, 4), \qquad (10)$$

where $\dot{X} = dX/d\ell$, and the integration of the original RG system of quasi-linear partial equations (4) is "equivalent" to the integration of the system of ordinary equations (10).

The results (8)-(10) provide the Key for developing the above-mentioned mechanical analogy of the DRG (and SRG) in the large- n limit. In fact, eq.(9) is a Hamilton-Jacobi equation and eq.(10) is the corresponding canonical system.

Then, if one looks at the RG parameter $\ell$ as a "time-like" variable, $S(\ell, \{q_j\})$ can be regarded as the "action" of an "equivalent mechanical system" whose Hamiltonian $H(\{q_j\}, \{p_j\})$ does not depend explicitly on "time" $\ell$. Of course, since $\frac{\partial H}{\partial \ell} = \frac{dH}{d\ell} = 0$, the "constant of motion" H is a quantity "invariant" under iteration of the DRG transformation. Note that, if we put $q_3 = q_4 = 0$ and

$$S(\ell, q_1, q_2, 0, 0) \equiv S(\ell, \{q_i\}),$$

$$H(q_1, q_2, 0, 0;\ p_1, p_2, 0, 0) \equiv H^{(s)}(\{q_i\};\ \{p_i\}) = \sum_{i=1}^{2} a_i q_i p_i - \frac{a_1 P_1}{1+q_2} \qquad (11)$$

with $p_i = \frac{\partial S}{\partial q_i}$ $(i=1,2)$, eqs.(7)-(10) reduce just to the corresponding ones for the SRG eq.(6).

Much information [2] about the original statistical mechanics problem may be obtained, at least in principle, starting from the above mechanical analogy and using the Hamilton-Jacobi theory [9]. By postponing this program to a future paper, as a first step in developing the mechanical analogy, here we limit ourselves to show how to use the traditional tools of Hamiltonian mechanics in order to determine the large-$n$ limit RG invariants as constants of motion in the "dual" mechanical problem. In particular we are able:

i) to establish, "a priori" the number of the independent invariants characteristic of the RG in the large-$n$ limit both for the critical statics and for dynamics;

ii) to calculate explicit expressions of invariants depending on the $q'$s and $p'$s;

iii) to derive invariants depending on the $q'$s only and therefore more directly connected with the original many-body problem.

All the points i)-iii) may have a great relevance to give a deeper insight into the internal structure of the DRG (SRG) in the large-$n$ limit and to clarify its eventual hidden symmetries. As concerning iii), we wish to note that the relevance in the present context, of the constants of motion only depending on the $q'$s could suggest the use of Lagrangian formalism. However, it has to be remarked that the linearity in $p'$s of the dual Hamiltonian (8) (and then its degeneracy in the usual mechanical sense) does not allow Legendre transformation in the given natural coordinates.

The point i) immediately follows as a direct consequence of the Hamilton-Jacobi formulation (9)- (10). Indeed, since the Hamilton ordinary differential system (10) is of an order of $\nu$ (equal to eight for critical dynamics and to four for statics), only $\nu-1$ (seven and three respectively) functionally independent RG invariants exist. Furthermore, the integrable Hamiltonian character of the dual problem allows one to use the standard constructive method [9] to exhibit the whole set of constants of motion and therefore the RG invariants in the original classical statistical mechanics problem in the large-$n$ limit.

As concerning the points ii) and iii), we refer, for both clarity and brevity reasons, only to the static mechanical problem with two degrees of freedom Hamiltonian $H^{(S)}$, However, what follows can be extended, in a straightforward way, to large-$n$ limit critical dynamics corresponding to four degrees of freedom Hamiltonian mechanics with seven independent constants of motion.



Just by inspection, the function:

$$\pi_1 = p_1 q_2^{d/2-1} \qquad (12)$$

gives a constant of motion. Both $\pi_1$ and the static Hamiltonian $H^{(S)} = \pi_2$ jointly depend on $q'$s and $p'$s. A further constant of motion only depending on the $q'$s (and therefore a RG invariant depending explicitly only on the original parameters of the RG problem) can now be constructed as conjugate variable of $\pi_1$ in a new canonical coordinate system ($\pi_1, \pi_2, \chi_1, \chi_2$). Here, as usual, the term "canonical" means that it results identically:

$$d \sum_i (p_i dq_i - \pi_i d\chi_i) = 0, \qquad (13)$$

where d denotes the exterior derivative defined by

$$d\beta = \sum_{ij} \left(\frac{\partial \beta_i}{\partial x_j} - \frac{\partial \beta_j}{\partial x^i}\right) dx^j \otimes dx^i = \sum_{ij} \frac{\partial \beta_i}{\partial x^j} dx^j \wedge dx^i \qquad (14)$$

for a differential form $\beta = \sum_i \beta_i dx^i$, the symbols $\otimes$ and $\wedge$ denoting the ordinary and antisymmetric tensorial products, respectively.

The generating function $S_0 = S_0(\{q_i\}, \{\pi_i\})$ of the canonical transformation $(\{q_i\}, \{p_i\}) \leftrightarrow (\{\pi_i\}, \{\chi_i\})$, which just gives a complete integral of the "time"-independent Hamilton-Jacobi equation for $H^{(S)}$ [2], satisfies the following differential relations:

$$\frac{\partial S_0}{\partial q_i} = p_i(\{\pi_j\}, \{q_j\}) \qquad \frac{\partial S_0}{\partial \pi_i} = \chi_i(\{\pi_j\}; \{q_j\}) \qquad (15)$$

with the condition

$$\det\left(\frac{\partial^2 S_0}{\partial q_i \partial \pi_j}\right) \neq 0. \qquad (16)$$

Then, by eq.(15), it follows:

$$S_0 = \sum_{i=1}^{2} \int p_i(\{\pi_j\}, \{q_j\})dq_i =$$

$$= \frac{\pi_1}{q_2^{d/2-1}} q_1 + \frac{\pi_2}{2} \ln |q_2| + \frac{d-2}{2} \pi_1 \int \frac{dq_2}{q_2^{d/2}(1+q_2)} . \tag{17}$$

Thus, the canonical map (15) is explicitly given by:

$$\pi_1 = p_1 q_2^{d/2-1}, \quad \pi_2 = (d-2) q_1 p_1 + 2 q_2 p_2 - (d-2) \frac{p_1}{1+q_2},$$

$$\chi_1 = \frac{q_1}{q_2^{d/2-1}} + \frac{d-2}{2} \int \frac{dq_2}{q_2^{d/2}(1+q_2)}, \quad \chi_2 = \frac{1}{2} \ln |q_2|, \tag{18}$$

in terms of which the equations of motion read:

$$\dot{\pi}_1 = 0, \quad \dot{\pi}_2 = 0, \quad \dot{\chi}_1 = 0, \quad \dot{\chi}_2 = 1, \tag{19}$$

of course, the corresponding Hamiltonian is $H^{(s)} = \pi_2$. The constant of motion $\chi_1 = \chi_1(q_1, q_2)$ is just the announced large-$n$ limit RG invariant only depending on the q´s.

From our static analysis some relevant aspects emerge:

a) $\pi_1$, $\pi_2$, $\chi_1$ just constitute a maximal set of independent RG invariants. Indeed, $\pi_1$ and $\pi_2$ are independent by inspection and $\chi_1$ is independent by construction;

b) $\chi_1$ is the only invariant not depending on the p´s since the p linearity of $H^{(S)}$ leads to completely separated dynamics for the q´s.

Analogous results can be found for critical dynamics in the large-$n$ limit but this will be subject of a future paper.

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Бусиелло Г. и др. E17-85-925

Инварианты ренормализационной группы из механической аналогии в пределе больших n

Обсуждаются некоторые аспекты аналогии между уравнениями классической механики и уравнениями статической и динамической ренормализационной группы в пределе больших n. Для статического случая определен максимальный набор ренормгрупповых инвариантов. Показано, что существует единственный ренормгрупповой инвариант, явно зависящий от исходных параметров соответствующей статмеханической задачи. Исследован вопрос о глобальном решении статического ренормгруппового уравнения. Метод механической аналогии применим также и в более сложном динамическом случае.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Renormalization Group Invariants from a Large-n Limit Mechanical Analogy

Some relevant aspects of the mechanical analogy recently proposed for the static and dynamic renormalization group (RG) in the large-n limit are pointed out. For the static case, we determine a maximal set of RG invariants and, in particular, we find that there exists a single RG invariant depending explicitly on the original parameters of the statistical mechanics problem. Finally, information about the global solution of the static RG equation is also given. Analogous mechanical methods also apply to the most complex dynamical case.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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