объединенный
институт
ядерных
исследований
дубна

E17-85-873

R.Gielerak

UNIQUENESS FOR SOME CLASS OF (NONFERROMAGNETIC) LATTICE SYSTEMS OF CONTINUOUS SPINS

Submitted to Acta Physica Polonica

1985

1. Introduction

In recent years, a great attention has been focused on statistical mechanics of systems of unbounded spins /1-6/. The aim of this note is to contribute to the problem of uniqueness of the corresponding equilibrium Gibbs measures. A variant of the Dobrushin uniqueness theorem has been formulated for some class of unbounded-spin system in /5/. In the case of ferromagnetic local specifications several uniqueness criteria have been proved in /1,6/. In this note we consider systems that correspond to the trigonometric perturbation of the Gaussian measures. The uniqueness criterion we prove in this note does not depend on the validity of the FKG inequalities and is valid for nonferromagnetic interactions. Our result may be formulated as follows: differentiability and independence of the tempered boundary conditions of the infinite volume free energy field for positive trigonometric coupling uniqueness of the tempered, translationally invariant equilibrium Gibbs measure. Independence of the free energy of tempered boundary conditions can be proved actually for a class of superstable two-body interactions /1/. Further assuming the FKG inequalities to hold we prove uniqueness of the tempered Gibbs equilibrium measures without the assumption of translational invariance. The main ingredient of the proof is a collection of some correlation inequalities of the Ginibre type.

2. Preliminary Definitions

On the lattice $\mathbb{Z}^\nu$ we consider a random field $\mathbb{Z}^\nu \ni x \to s_x \in R^1$. The configuration space of the system is thus $\Omega = \{R^{\mathbb{Z}^\nu} \ni \underline{s}\}$. The restriction of the spin configuration $\underline{s} \in \Omega$ to the given $\Lambda \subset \mathbb{Z}^\nu$ will be denoted by s_Λ. From the denumerability of $\mathbb{Z}^\nu$ it follows that Ω is a polish space for the product topology. We denote by $\mathcal{B}$ the product Borel σ-algebra of Ω and for $\Lambda \subset \mathbb{Z}^\nu$ we denote by $\mathcal{B}(\Lambda)$ the corresponding sub-σ-algebras of $\mathcal{B}$. The configuration space Ω has the following decomposition property: $\Omega = \Omega_\Lambda \otimes \Omega_{\Lambda^c}$ for any $\Lambda \subset \mathbb{Z}^\nu$ which implies also $\mathcal{B} = \mathcal{B}(\Lambda) \otimes \mathcal{B}(\Lambda^c)$. As usual, in the case of noncompact configuration space we have to consider some distinguished Borel subsets of Ω of some reasonable spin configurations in order to exclude some pathological situations. For this let us consider two Borel subsets of Ω :

$$s'(\mathbb{Z}^\nu)=\{\underset{\sim}{s} \mid \underset{N>0}{\exists} : \sup_{x\in\mathbb{Z}^\nu} d(0,x)^{-N}|s_x|<\infty\},$$

where $d(0,x)$ is the standard Euclidean distance on $\mathbb{Z}^\nu$, and with

$$R_N=\{\underset{\sim}{s}\in\Omega \mid \sum_{|x|<j} s_x^2 \le N\,(2j+1)^d,\ \forall j>0\}$$

we define the set

$$R=\bigcup_{N=1}^{\infty} R_N .$$

The set $s'(\mathbb{Z}^\nu)$ naturally arises whenever we have to deal with the positive-definite two-body interactions and the set R in the case of superstable interactions. It is easy to see that $R\subset s'(\mathbb{Z}^\nu)$. Let us consider also the set of fastly decreasing configurations:

$$s(\mathbb{Z}^\nu)=\{\underset{\sim}{s}\in\Omega \mid \underset{N\ge 0}{\forall} \ \sup_{x\in\mathbb{Z}^\nu}(1+|x|)^N|s_x|<\infty\}.$$

It is easy to see that the collection of the norms $\{\|\cdot\|_N\}_{N=1,2,\dots}$ where $\|\underset{\sim}{s}\|_N=\sup_x(1+|x|)^N|s_x|$ defines a nuclear topology in the space $s(\mathbb{Z}^\nu)$ and then a rigging

$$s(\mathbb{Z}^\nu)\subset \ell_2(\mathbb{Z}^\nu)\subset s'(\mathbb{Z}^\nu)$$

is the Hilbert nuclear rigging. In particular, $s'(\mathbb{Z}^\nu)$ is the nuclear space in the topology of the weak dual of the space $\{s(\mathbb{Z}^\nu),\|\cdot\|_N\}$. Therefore, we are in the situation where the Minlos theorem works.

Let A be a symmetric, <u>strictly</u> positive operator on the Hilbert space $\ell_2(\mathbb{Z}^\nu)$ and such that the bilinear form $(\underset{\sim}{s},A\underset{\sim}{s})$ being restricted to $s(\mathbb{Z}^\nu)\otimes s(\mathbb{Z}^\nu)$ is then continuous (in the topology of s). From the Minlos theorem then follows that the functional

$$\Gamma_A(\underset{\sim}{\alpha})=\exp-\tfrac{1}{2}(\underset{\sim}{\alpha},A\underset{\sim}{\alpha}) \tag{2.1}$$

defined on the space $s(\mathbb{Z}^\nu)$ is the characteristic functional of some Gaussian measure $\mu_A^0(d\underset{\sim}{s})$ supported on the space $s'(\mathbb{Z}^\nu)$, i.e.,

$$\Gamma_A(\underset{\sim}{\alpha})=\int_{s'(\mathbb{Z}^\nu)} e^{i(\underset{\sim}{\alpha},\underset{\sim}{s})}\mu_A^0(d\underset{\sim}{s}), \tag{2.2}$$

where $(\underset{\sim}{\alpha},\underset{\sim}{s})=\sum_x \alpha_x s_x$ is the canonical pairing of the dual pair $\{s(\mathbb{Z}^\nu),s'(\mathbb{Z}^\nu)\}$.

Now we come back to the definition of our systems. For each finite $\Lambda\subset\mathbb{Z}^\nu$ and for each spin configuration t_{Λ^c} in Λ^c we define a Gibbsian probability distribution of $\underset{\sim}{s}_\Lambda$ conditioned by t_{Λ^c}

via the following formulas: ($s_{\Lambda^c}=t_{\Lambda^c}$)

$$E_\Lambda(d\underset{\sim}{s}_\Lambda \mid t_{\Lambda^c})=(Z_\Lambda(t_{\Lambda^c}))^{-1}\exp\Big(-\sum_{\substack{x\in\Lambda\\ y\in\mathbb{Z}^\nu}} J(x-y)s_x s_y\Big)$$
$$\exp\Big(-\frac{m^2}{2}\sum_{x\in\Lambda}s_x^2\Big)\prod_{x\in\Lambda}\exp(z\cos\alpha s_x)\prod_{x\in\Lambda}ds_x, \tag{2.3}$$

$$Z_\Lambda(t_{\Lambda^c})=\int_{R^{|\Lambda|}}\prod_{x\in\Lambda}ds_x\prod_{x\in\Lambda}\exp(z\cos\alpha s_x)\prod_{x\in\Lambda}\exp-\frac{m^2}{2}s_x^2$$
$$\cdot\exp\Big(-\sum_{\substack{x\in\Lambda\\ y\in\mathbb{Z}^\nu}}J(x-y)s_x s_y\Big), \tag{2.4}$$

where J is a matrix inverse to the operator A on the space $\ell_2(\mathbb{Z}^\nu)$. Then J is strictly positive, symmetric operator on the space $\ell_2(\mathbb{Z}^\nu)$. Parameter α will be chosen arbitrary in the set $[0,2\pi)$. The trigonometric coupling constant z will be assumed throughout this note to be <u>non-negative</u>.

Let $\mathcal{G}(z)$ be the set of probability measures on the $\{\Omega,\mathcal{B}\}$ with the following property:

$$\mu\in\mathcal{G}(z)\Longleftrightarrow$$

for each finite $\Lambda\subset\mathbb{Z}^\nu$

(DLR) $$\mu\circ E_{\Lambda^c}=\mu . \tag{2.5}$$

Then the set $\mathcal{G}(z)$ is called the set of the Gibbs equilibrium measures corresponding to the local specification (2.3). Its subset $\mathcal{G}^T(z)$ consisting of measures supported on the set R is called the set of tempered Gibbs measures.

Throughout this note we will assume that J is translational-invariant and that $J(x-y)$ does satisfy the superstability condition

$$\underset{k>0,\,\epsilon>0}{\exists}\qquad |J(x-y)|\le K|x-y|^{-d-\epsilon} \quad \text{as } |x-y|\uparrow\infty \tag{2.6}$$

From the last assumption it follows that our local specification belongs to the class for which the superstability estimates [2] are valid.

3. Correlation Inequalities of Ginibre

Let us denote by $\mu_\Lambda^{m,0}$ the Gaussian measure on the space $R^{|\Lambda|}$ with mean zero and the covariance given by $A_\Lambda^m(x,y) = (J_\Lambda + \frac{1}{2}m^2\Delta)^{-1}$. Then the probability distribution of s_Λ conditioned by $t_{\Lambda^c}=0$ can be written as:

$$E_{\Lambda^c}^{\Lambda'}(ds_\Lambda|0)_{|\Lambda'=\Lambda} = (Z_\Lambda^{(\Lambda')}(z))^{-1}\prod_{x\in\Lambda} e^{z\cos\alpha s_x}\mu_{\Lambda'}^{m,0}(ds_\Lambda)_{|\Lambda'=\Lambda} . \quad (3.1)$$

Using duplicate variable trick and invariance of the Gaussian integration with respect to the orthogonal transformations one can easily prove the following collection of the correlation inequalities (see also /7/).

Proposition 3.1.

For any $z\geq 0$, any $\Lambda\subset Z^\nu, \Lambda'\subset Z^\nu$ with $|\Lambda|<\infty$ the following correlation inequalities hold:

$$E_{\Lambda'^c}^{\Lambda}\Big(\prod_{i=1}^n \cos\alpha_i s_{x_i}|0\Big) \geq 0 \quad (3.2)$$

$$E^\Lambda_{\Lambda'^c}\Big(e^{i(\alpha,s)}; \prod_{j=1}^m \cos\beta_j s_{x_j}|0\Big)^T \geq 0 \quad (3.3)$$

$$E^\Lambda_{\Lambda'^c}\Big(e^{t(\alpha,s)}; \prod_{j=1}^m \cos\beta_j s_{x_j}|0\Big)^T \leq 0 \quad (3.4)$$

$$E^\Lambda_{\Lambda'^c}\big((\alpha,s)^2|0\big) \leq \langle \alpha,\alpha\rangle_{A^m_{\Lambda'}} \leq \sum_{x,y}\alpha_x A^m(x,y)\alpha_y \quad (3.5)$$

$$E^\Lambda_{\Lambda'^c}\Big(\prod_{i=1}^n\cos\alpha_i s_{x_i}; \prod_{j=1}^m \cos\beta_j s_{x_j}|0\Big)^T\geq 0 \quad (3.6)$$

$$Z_{\Lambda_1\cup\Lambda_2}(z)\geq Z_{\Lambda_1}(z)\cdot Z_{\Lambda_2}(z). \quad (3.7)$$

Other correlation inequalities hold for the arbitrary boundary condition. Let us denote by $E_{\Lambda^c}(-|0,s'_{\Lambda^c})$ the expectation value with over respect to the tensor product measure $E_{\Lambda^c}(ds_\Lambda|0)\otimes E_{\Lambda^c}(ds'_\Lambda|s'_{\Lambda^c})$

Proposition 3.2.

For any $z\geq 0$, $\Lambda\subset Z^\nu$ the following correlation inequalities hold

$$0\leq E_{\Lambda^c}\Big(\Big(\prod_{i=1}^n\cos(\alpha_i s_{x_i}+\theta_i) - \prod_{i=1}^n \cos\alpha s'_{x_i}\Big) \exp\Big(\pm\delta\sum_{x\in Z^\nu}\alpha_x\cos\alpha s_x\cos\alpha s'_x\Big)|0,s'_{\Lambda^c}\Big) \quad (3.8)$$

for any $\delta\in R^1$ and $\theta_i\in[0,2\pi)$.

$$\Big|E_{\Lambda^c}\Big(\prod_{i=1}^n \cos(\alpha s_{x_i}+\theta_i)|s(\Lambda^c)\Big)\Big| \leq E_{\Lambda^c}\Big(\prod_{i=1}^n \cos\alpha s_x|0\Big) \quad (3.9)$$

Let us note the following consequence of the inequality (3.8). By $\lim_{\Lambda\uparrow Z^\nu}$ we mean some well defined kind of the convergence to be described below.

Proposition 3.3.

Let us assume that

$$\lim_{\Lambda\uparrow Z^\nu} E_{\Lambda^c}(\cos\alpha s_x|t_{\Lambda^c}) = \lim_{\Lambda\uparrow Z^\nu} E_{\Lambda^c}(\cos\alpha s_x|0))>0. \quad (3.10)$$

Then for any $n>1$ we have

$$\lim_{\Lambda\uparrow Z^\nu} E_{\Lambda^c}\Big(\prod_{i=1}^n\cos\alpha s_{x_i}|t_{\Lambda^c}\Big) = \lim_{\Lambda\uparrow Z^\nu} E_{\Lambda^c}\Big(\prod_{i=1}^n\cos\alpha s_{x_i}|0\Big) \quad (3.11)$$

(assuming the corresponding limits do exist).

Proof:

Expanding the right-hand side of the inequality (3.8) in the powers of δ we have:

$$\begin{aligned} 0 \leq\ & E_{\Lambda^c}\Big(\prod_{i=1}^n\cos\alpha s_{x_i}|0\Big) - E_{\Lambda^c}\Big(\prod_{i=1}^n \cos\alpha s_{x_i}|t_{\Lambda^c}\Big) \\ & \pm\delta\sum_{x\in Z^\nu}\alpha_x\Big[E_{\Lambda^c}\Big(\cos\alpha s_x\prod_{i=1}^n\cos\alpha s_{x_i}|0\Big)E_{\Lambda^c}(\cos\alpha s_x|t_{\Lambda^c}) \\ & - E_{\Lambda^c}\Big(\cos\alpha s_x\prod_{i=1}^n\cos\alpha s_{x_i}|t_{\Lambda^c}\Big)\cdot E_{\Lambda^c}(\cos\alpha s_x|0)\Big] \\ & + O_\Lambda(\delta^2). \end{aligned}$$

Taking the thermodynamic limit $\lim_{\Lambda\uparrow Z^\nu}$ with $\alpha_x=\delta_{x,x_0}$ and using the inductive hypothesis (3.10) and (3.11) we obtain:

$$\begin{aligned} 0 \leq \pm\delta\Big[&\lim_{\Lambda\uparrow Z^\nu} E_{\Lambda^c}\Big(\cos\alpha s_{x_0}\prod_{i=1}^n\cos\alpha s_{x_i}|0\Big) \\ & - \lim_{\Lambda\uparrow Z^\nu} E_{\Lambda^c}\Big(\cos\alpha s_x \prod_{i=1}^n \cos\alpha s_{x_0}|t_{\Lambda^c}\Big)\Big] + O_\Lambda(\delta^2), \end{aligned}$$

where as it follows from the definition

$$\sup_{\Lambda} |O_\Lambda(\delta^2)| \leq C\delta^2$$

for some constant C. Dividing by δ we complete the proof by applying induction principle.

q.e.d.

4. Uniqueness

Let Π_Λ be an orthogonal projector in the space $\ell_2(\mathbb{Z}^\nu)$ onto the subspace $R^\Lambda = \{\underset{\sim}{s} \mid x \notin \Lambda \to \underset{\sim}{s} = 0\}$. Then the operator $\Pi_\Lambda J \Pi_\Lambda$ has a matrix representation $J_\Lambda(x,y) = J(x-y)$ if $x, y \in \Lambda$, and $J_\Lambda(x,y) = 0$ otherwise. The family

$$\mu_\Lambda^{m,0}(ds_\Lambda) = (Z_\Lambda^0)^{-1} \exp\Big(-\frac{1}{2}\sum_{x,y\in\Lambda} s_x s_y J_\Lambda(x-y)\Big) \tag{4.1}$$

$$\exp\Big(-\frac{m^2}{2}\sum_{x\in\Lambda} s_x^2\Big) \prod_{x\in\Lambda} ds_x$$

then defines a system of compatible probability distributions. From the Kolmogorov theorem then it follows that there exists an extension of the space $\ell^2(\mathbb{Z}^\nu)$ where there lives a unique Gaussian measure $\mu^0_{J_m}$. From the superstability estimates /1,2/ we know actually that $\mu^0_{J_m}$ is supported on the space $\mathbb{R} \subset S'(\mathbb{Z}^\nu)$. Let us denote by A_m the inverse of the operator $J + \frac{1}{2} m^2 \mathbb{1}$. From the Minlos theorem it follows that the characteristic functional of the measure $\mu^0_{J_m}$ is then given by the formula

$$\int_{S'(\mathbb{Z}^\nu)} e^{i(\underset{\sim}{\alpha},\underset{\sim}{s})} \mu^0_{J_m}(d\underset{\sim}{s}) = e^{-\frac{1}{2}(\underset{\sim}{\alpha}, A_m \underset{\sim}{\alpha})} \tag{4.2}$$

and that A_m is a continuous operator on the space $S(\mathbb{Z}^\nu)$.

On the space $S'(\mathbb{Z}^\nu)$ let us consider following measure. For $\Lambda \subset \mathbb{Z}^\nu$, $|\Lambda| < \infty$ let us define:

$$\mu_\Lambda(d\underset{\sim}{s}) = Z_\Lambda(z)^{-1} \prod_{x\in\Lambda} e^{z\cos\alpha s_x} \mu^0_{J_m}(d\underset{\sim}{s}), \tag{4.3}$$

$$Z_\Lambda(z) = \int_{S'(\mathbb{Z}^\nu)} \mu^0_{J_m}(d\underset{\sim}{s}) \prod_{x\in\Lambda} e^{z\cos\alpha s_x}. \tag{4.4}$$

The Fourier transform of the measure $\mu_\Lambda(d\underset{\sim}{s})$ can be written as:

$$\Gamma_\Lambda(\underset{\sim}{\alpha}) = \int_{S'(\mathbb{Z}^\nu)} e^{i(\underset{\sim}{\alpha},\underset{\sim}{s})} \mu_\Lambda(d\underset{\sim}{s})$$

$$= \exp -\frac{1}{2}(\underset{\sim}{\alpha}, A_m \underset{\sim}{\alpha})$$

$$\cdot \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{x_1\in\Lambda} \cdots \sum_{x_m\in\Lambda} \int \prod_{i=1}^{m} \frac{1}{2}\big(\delta(\alpha_i - \alpha) + \delta(\alpha_i + \alpha)\big) \tag{4.5}$$

$$\cdot \prod_{i=1}^{m} \Big[e^{-\alpha_i(\underset{\sim}{\alpha} * A_m)(x_i)} - 1\Big] \int_{S'(\mathbb{Z}^\nu)} \mu_\Lambda(d\underset{\sim}{s}) \prod_{i=1}^{m} :e^{i\alpha_i s_{x_i}}.$$

From the correlation inequality (3.3) it follows that $\Gamma_\Lambda(\underset{\sim}{\alpha})$ is monotonically increasing in Λ, and from the inequality (3.4) we have the following estimate for the Laplace transform of the measure μ_Λ:

$$\Big|\int e^{\zeta(\underset{\sim}{\alpha},\underset{\sim}{s})} \mu_\Lambda(d\underset{\sim}{s})\Big| \leq \exp\Big(\frac{|\mathrm{Re}\,\zeta|^2}{2}(\underset{\sim}{\alpha}, A_m \underset{\sim}{\alpha})\Big). \tag{4.6}$$

We conclude that the unique thermodynamic limit $\lim_{\Lambda\uparrow\mathbb{Z}^\nu} \mu_\Lambda = \tilde{\mu}_\infty$ as a weak limit exists whenever Λ tends to $\mathbb{Z}^\nu$ monotonically and by an inclusion. It is not hard to see that

$$\tilde{\mu}_{\infty \restriction \mathcal{B}(\Lambda)}(ds_\Lambda) = (Z_\Lambda)^{-1} \prod_{x\in\Lambda} e^{z\cos\alpha s_x} \mu^0_{J_m \restriction \mathcal{B}(\Lambda)} \tag{4.7}$$

$$= (Z_\Lambda)^{-1} \prod_{x\in\Lambda} e^{z\cos\alpha s_x} \mu_\Lambda^{m,0}(ds_\Lambda).$$

Therefore, we have $\tilde{\mu}_\infty \in \mathcal{G}(z)$ as it fulfills DLR equations (3.5).

Let us define

$$p_\infty(z) = \lim_{\Lambda\uparrow\mathbb{Z}^\nu} \ln (Z_\Lambda)^{-\frac{1}{|\Lambda|}} = \lim_{\Lambda\uparrow\mathbb{Z}^\nu} p_\Lambda(z), \tag{4.8}$$

whenever $\Lambda \uparrow \mathbb{Z}^\nu$ in a certain sense (see below).

Quantity $p_\infty(z)$ is called the free energy. As a consequence of the inequality (3.7), $p_\infty(z)$ exists and is a convex function of the coupling constant z. From the convexity of $p_\infty(z)$ it follows that p_∞ is almost everywhere differentiable function of z (except for some, at least countable, set of points). The value of $z=z_0$ will be called regular for p_∞ whenever p_∞ is differentiable at the point $z=z_0$.

For any $\underset{\sim}{t} \in \mathbb{R}$ let us define

$$p_\Lambda^{\underset{\sim}{t}}(z) = -\frac{1}{|\Lambda|} \ln Z_\Lambda(z, \underset{\sim}{t}), \tag{4.9}$$

where

$$Z_\Lambda(z, \underset{\sim}{t}) = Z_\Lambda(t_{\Lambda^c}).$$

We note also that $\tilde{\mu}_\infty$ is a measure for which superstability estimates hold. In particular, from this it follows that $\tilde{\mu}_\infty$ is supported on the set $\mathbb{R}$. Using results from paper [1] we have:

<u>Proposition 5.1.</u> (Thm 3.1 in [1])

Whenever $\Lambda \uparrow \mathbb{Z}^\nu$ in the sense of van Hove, then for any $\underset{\sim}{t} \in \mathbb{R}$, $\lim_{\Lambda\uparrow\mathbb{Z}^\nu} p^{\underset{\sim}{t}}_\Lambda(z)$ exists and is equal to $p_\infty(z)$.

From the properties of the convex functions it follows (assuming z_0 is a regular value for p_∞):

$$-\lim_{\Lambda\uparrow\mathbb{Z}^\nu} \frac{d}{dz} p_\Lambda(z)\Big|_{z=z_0} = \lim_{\Lambda\uparrow\mathbb{Z}^\nu} \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \mu_\Lambda^{m,0}(\cos\alpha s_x)\Big|_{z=z_0} \tag{4.10}$$

$$= \tilde{\mu}_\infty(\cos\alpha s_0)\Big|_{z=z_0}$$

(by the translational invariance of $\tilde{\mu}_\infty$)

$$= -\frac{d}{dz}\lim_{\Lambda\uparrow\mathbb{Z}^\nu} p_\Lambda(z)\Big|_{z=z_0} = -\frac{d}{dz}\lim_{\Lambda\uparrow\mathbb{Z}^\nu} p_\Lambda^{\underset{\sim}{t}}(z)\Big|_{z=z_0} \tag{4.11}$$

$$= \lim_{\Lambda\uparrow\mathbb{Z}^\nu} E_{\Lambda^c}\Big(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\cos\alpha s_x \,\Big|\, t_{\Lambda^c}\Big) \equiv \lim_{\Lambda\uparrow\mathbb{Z}^\nu}\mu_\Lambda^{\underset{\sim}{t}}\Big(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\cos\alpha s_x\Big).$$

On the account of the martingale property of the $E_{\Lambda^c}(-|t_{\Lambda^c})$ we have that for every $\underset{\sim}{t}\in\mathbb{R}$ the weak limits $\mu_\infty^{\underset{\sim}{t}}(ds) \equiv \lim_{\Lambda\uparrow\mathbb{Z}^\nu} E_{\Lambda^c}(-|t_{\Lambda^c})$ exist whenever $\Lambda\uparrow\mathbb{Z}^\nu$ monotonically and by inclusion. Assuming that $\mu_\infty^{\underset{\sim}{t}}$ has a translationally invariant first moment we conclude from the chain of equalities written above that

$$\tilde{\mu}_\infty(\cos\alpha s_0) = \mu_\infty^{\underset{\sim}{t}}(\cos\alpha s_0). \tag{4.12}$$

From the inequality (3.6) it follows that $\tilde{\mu}_\infty(\cos\alpha s_0) > 1$. Therefore we are in the position where Proposition 3.2 applies.

<u>Theorem 5.2.</u>

Assume that z_0 is a regular point for $p_\infty(z)$. Then the set $\mathcal{G}_1^{TT}(z)$ of tempered, with translationally invariant first moment, Gibbs measures corresponding to the DLR equations consists exactly of one element $\tilde{\mu}_\infty$.

<u>Proof:</u>

From Proposition 3.2 and the remarks before we see that for every $\underset{\sim}{t}\in\mathbb{R}$

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu} \mu_\Lambda^{\underset{\sim}{t}}\Big(\prod_{i=1}^n \cos\alpha s_{x_i}\Big) = \lim_{\Lambda\uparrow\mathbb{Z}^\nu}\mu_\Lambda\Big(\prod_{i=1}^n\cos\alpha s_{x_i}\Big). \tag{4.13}$$

That the limits on the right-hand side of (4.13) exist follows from the inequality (3.6). From the inequality (3.9) it follows that the sequence $\{\mu_\Lambda^{\underset{\sim}{t}}(\prod_{i=1}^n\cos\alpha s_{x_i})\}$ has a convergent subsequence and is bounded uniformly in Λ. But every accumulation point of $\{\mu_\Lambda^{\underset{\sim}{t}}(\prod_1^n\cos\alpha s_{x_i})\}$ is equal to $\{\tilde{\mu}_\infty(\prod_1^n\cos\alpha s_{x_i})\}$. Therefore the sequence $\{\mu_\infty^{\underset{\sim}{t}}(\prod_1^n\cos\alpha s_{x_i})\}$ is convergent. The moments $\{\mu_\infty^{\underset{\sim}{t}}(\prod_1^n\cos\alpha s_{x_i})\}_n$ do not uniquely determine the measure $\mu_\infty^{\underset{\sim}{t}}$ but only its restriction to the even part of the corresponding set of observables. But from the inequality (3.9) it follows in a standard way that also

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu}\mu_\Lambda^{\underset{\sim}{t}}\Big(\prod_1^n\cos\alpha s_{x_i}\prod_{j=1}^m\sin\alpha s_{y_j}\Big) \tag{4.14}$$

$$= \tilde{\mu}_\infty\Big(\prod_1^n\cos\alpha s_{x_i}\prod_{j=1}^m\sin\alpha s_{y_j}\Big)$$

assuming (4.12) holds.

Similarly as for (4.5), we find the following expression for the Laplace transform of the measure $\mu_\Lambda^{\underset{\sim}{t}}$:

$$\mu_\Lambda^{\underset{\sim}{t}}\big(e^{\zeta(\underset{\sim}{\alpha},\underset{\sim}{s})}\big) = \exp\Big(\frac{\zeta^2}{2}\sum_{x,x'}\alpha_x J_\Lambda^m(x,x')\alpha_{x'}\Big)$$

$$\prod_{x\in\Lambda}\exp\Big(-i\zeta\Big(\sum_{y\notin\Lambda} g_\Lambda^m(\cdot - y)t_y\Big) * J_\Lambda^m\Big)(x)$$

$$\cdot\sum_{n=0}^\infty\frac{z^n}{n!}\sum_{x_1\in\Lambda}\cdots\sum_{x_n\in\Lambda}\int\prod_{i=1}^n\tfrac{1}{2}\big[\delta(\alpha-\alpha_i)+\delta(\alpha+\alpha_i)\big] \tag{4.15}$$

$$\cdot\prod_{i=1}^n\big(e^{i\zeta(J_\Lambda^m * \underset{\sim}{\alpha})(x_i)}-1\big)\,\mu_\Lambda^{\underset{\sim}{t}}\Big(\prod_{i=1}^n :e^{i\alpha_i s_{x_i}}:\Big).$$

Taking into account (4.14) and the simple proved fact that

$$\lim_{\Lambda\uparrow \mathbb{Z}^\nu} \sum_{x\in\Lambda} \Big(\sum_{y\notin\Lambda} \mathcal{J}(x-y)\, t_y \Big) = 0 \tag{4.16}$$

implied by the definition of $\mathbb{R}$ and the assumed decay (2.6), we conclude from the formula (4.15) that

$$\omega\text{-}\lim \mu^{t}_{\Lambda} = \tilde{\mu}_\infty .$$

The full set $\mathcal{G}^T(z)$ can be obtained by taking thermodynamic limits of the suitable convex combinations of z. But the infinite volume free energy still does not depend on such a generalized boundary conditions (see [1]).

q.e.d.

In the case of a ferromagnetic system, i.e. such that $\mathcal{J}(x-y)\geq 0$ for $x\neq y$, we can exclude the assumption of translational invariance.

Corrolary 5.3.

Assume additionally to the hypothesis of Th. 5.2 that $\mathcal{J}(x-y)\geq 0$ for $x\neq y$. Then the set $\mathcal{G}^T(z_0)$ of tempered Gibbs measures corresponding to the DLR equations (3.5) consists exactly of one element.

Proof:

From the assumption $\mathcal{J}(x-y)\geq 0$ for $x\neq y$ it follows that the local specification (2.3) belongs to the class for which a partial FKG order can be introduced into the set $\mathcal{G}^T(z)$. From the general theory[1,6] it is known that there exists extremal (with respect to FKG order) Gibbs measures. These extremal Gibbs measures are translationally invariant because our local specification is translationally covariant. Assuming two different solutions to exist in $\mathcal{G}^T(z)$ we arrive at the contradiction with Th. 5.2.

q.e.d.

Corrolary 5.4.

Assume that z_0 is a regular point for $p_\infty(z)$ and that $\mathcal{J}$ is a ferromagnetic and of finite range say d. Then the unique tempered Gibbs measure $\tilde{\mu}_\infty$ has a global d-Markov property.
For definition and some results about the global Markov-property for lattice spin systems see [8].

5. Concluding Remarks

Trigonometric perturbations of the Gaussian measures are rather popular models in statistical mechanics and the (Euclidean) Field theory. In particular, when applying the sine-Gordon transformation to the neutral systems of particles interacting via two-body positive-definite potentials (with some regularity properties) we obtain a situation similar to that described in this note [9]. However, the situation obtained in this context is technically much more involved. The work along this line is in progress. Another circle of problems concerns the problem of the global Markov property for the Euclidean fields. Using essentially the strategy of the present note in paper [10], we extend the uniqueness part of the work[11] to cover all regular values of the positive coupling constant for the so-called two-dimensional sine-Gordon models. In the field-theory context, however, the uniqueness result is not sufficient to imply the global Markov property of the corresponding Gibbs field. However, despite the results obtained in this note it seems to be promising that such a proof may be obtained at least for the sine-Gordon-like models.

Further problems that may be attacked by the ideas of the present note are the DLR-equations and Markov property for (generalized) random fields obtained by a trigonometric perturbation of the (Pauli-Villars) regularized free Markov fields. The work in this direction is in progress.

Finally we should remark that the general strategy for the study of uniqueness of solutions of the DLR equations in the above-mentioned problems has been caused by a beautiful study of phase diagrams for the class of abelian (compact) spin systems from[12].

References

1. Lebowitz J.L., Pressutti E. Comm.Math.Phys. 50, 195-218 (1976).
2. Ruelle D. Comm.Math.Phys. 50, 189-194 (1976).
3. Benfatto G., Presutti E., Pulvirenti M. Z.Wahrscheinliohkeitsch Verw. Geb. 41, 305-312 (1978).
4. Cassandro M., Olivieri E., Pellegrinotti A., Presutti E. Z.Wahrcheinlichkeitsch Verw. Geb. 41, 313-334 (1978).
5. Picco P. These de 3 cycle. Marseille (1979).
6. Bellissard J., Høegh-Krohn R., Comm.Math.Phys. 84, 297-327 (1982).
7. Fröhlich J., Park Y.M. Comm.Math.Phys. 59, 235-266 (1978).

8. Albeverio S., Høegh-Krohn R., Olsen G. J.Multivariate Anal. 11, 599-607 (1981).
9. Gielerak R. to appear in Theor. and Math.Phys.
10. Gielerak R. subm. to Journ. Math.Phys.
11. Albeverio S., Høegh Krohn R. Comm.Math.Phys. 68, 95-128 (1979).
12. Fröhlich J., Pfister. Ch-E. Comm.Math.Phys. 89, 303-327 (1983).

Received by Publishing Department
on December 4, 1985.

Гелерак Р. E17-85-873

Единственность для некоторого класса /неферромагнитных/ решеточных систем с непрерывным спином

Рассматривается решеточная система непрерывных спинов, которая соответствует тригонометрическим возмущениям гауссовых сверхустойчивых парных взаимодействий. Показывается единственность предельных трансляционно-инвариантных состояний Гиббса в точках дифференцируемости давления.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Gielerak R. E17-85-873

Uniqueness for Some Class of (Nonferromagnetic) Lattice Systems of Continuous Spins

We consider lattice systems of continuous spins which correspond to the trigonometric perturbation of the Gaussian superstable two-body interactions. Using some correlation inequalities of the Ginibre type we prove uniqueness of the tempered, translationally invariant Gibbs states whenever the pressure is differentiable at the coupling constant.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985