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Bi-HAMILTONIAN STRUCTURE AND CONSERVED FUNCTIONALS OF LIOUVILLE's EQUATION

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The last decade has shown the exciting prospects of tackling nonlinear field theories (in two dimensions) nonperturbatively by exploiting their complete integrability properties [1,2].

The concept of completely integrable Hamiltonian systems with finitely many degrees of freedom goes back to the last century [3]. Briefly they are nonlinear ordinary differential equations admitting a Hamiltonian description and possessing sufficiently many constants of motion so that they can be integrated by quadratures.

Some qualitative features of these systems remain true in some special classes of infinite-dimensional Hamiltonian systems expressed by nonlinear field equations (Korteweg-de Vries (KdV), sine-Gordon (sG), Nonlinear Schroedinger (NS),...) [1,4].

A relevant progress in the study of these systems, with an infinite dimensional phase manifold M, was the introduction of Lax Representation [5] which played an important role in formulating the inverse scattering method [6].

Recent papers [7-14] are concerned with the search for tensorial equations able to capture the geometrical nature of Lax Representation. It has been shown how it can be thought of as the vanishing, along the dynamics, of the covariant derivative of a section of an (M-based) associated bundle of the (co-) adjoint type [7-9], or as the vanishing of the Lie derivative of a $\binom{1}{1}$ tensor field.

According to the latter point of view, recent papers [10-14] have emphasized the role played by invariant mixed tensor fields on the phase manifold M in characterizing the integrability of a dynamical system.

More precisely it has been proved the following integrability theorem [11,12] which constitutes an extension [12,15] to field theories of the classical complete integrability Liouville's theorem.

A dynamical vector field Λ with finitely (or infinitely) many degrees of freedom which admits an invariant mixed tensor field T satisfying the following general properties:

1. It is invariant under the dynamics,
2. Its Nijenhuis tensor vanishes,
3. It has(a double degenerate continuous spectrum and finitely many discrete) eigenvalues with bidimensional invariant spaces,

completely separates into one which has one degree of freedom. For those degrees whose corresponding eigenvalues are nowhere stationary the dynamics are integrable and hamiltonian.

The relevance of such a tensor field lies, shortly, in the fact that it allows to construct, in a very simple way, a sequence of conserved constants of motion and to define through its eigenvectors, a privileged system of coordinates (action-angle coordinates) in which the dynamics is trivially integrable.

In spite of its success in accounting for the integrability of solitonic evolution equations (and also, by removing property 3, of dissipative dynamics /16/), tensor field never appeared in the physical literature of relativistic string, Yang-Mills fields (in two dimensions), or, more specifically, on the Liouville's evolution equation (LE) in light-cone coordinates:

$$\sigma_{xt} = e^{\sigma}, \quad \sigma : (x,t) \in R^2 \to \sigma(x,t) \in R. \qquad (1)$$

The purpose of the present note is to fill up this gap.

A tensor field T satisfying properties 1.,2.,3., which allows to relate Liouville and KdV equations shall be constructed. As a consequence, the construction of an infinite "complete" set of conserved functionals /17/ is accomplished in a straightforward way.

LE is among the most celebrated of the "completely integrable" theoreies that have attracted a great deal of attention in the literature /18,19/ . It was first written and solved by Liouville, in the nineteenth centure during a study of vortices /20/, arises in many branches of physics /21/ and mathematics and, recently, was important for particle physicists in reformulations of the dual string model /22/ and as a two dimensional mode for graviry /23/.

Consider, for instance, on SL(2,C) Yang-Mills field in two dimensions:

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2] \qquad (2)$$

with $A_i : R^2 \to \mathcal{G}_{SL(2,C)}$ $i = 1,2$, $\mathcal{G}_{SL(2,C)}$ denoting the Lie algebra of SL(2,C).

The zero field strength (curvature) condition, i.e. $F_{12} = 0$, implies

$$A_1 = g^{-1} \partial_1 g, \quad g : R^2 \to SL(2,C). \qquad (3)$$

Both the zero-curvature condition and the requirement that g satisfies the constraints /24,17/:

$$g^{-1}\partial_1 g = U^{+} H + \frac{1}{\sqrt{2}} f^{+} E^{+}, \qquad (4a)$$

$$g^{-1}\partial_2 g = U^{-} H + \frac{1}{\sqrt{2}} f^{-} E^{-}, \qquad (4b)$$

where H, E^{+}, E^{-} are the generators in a Cartan-Weyl basis

$$[H, E^{\pm}] = 2E^{\pm}, \quad [E^{+}, E^{-}] = H, \qquad (5)$$

give:

$$\partial_1 \ln f^{-} = 2U^{+}, \qquad (6a)$$

$$\partial_2 \ln f^{+} = -2U^{+}, \qquad (6b)$$

$$\partial_1\partial_2 \ln f^{+}f^{-} = f^{+}f^{-}. \qquad (7)$$

Identifying x^1, x^2 with light-cone coordinates (x,t) and $\sigma = \ln f^{+}f^{-}$, equation (7) becomes LE.

Then to every $g: R^2 \to SL(2,C)$ satisfying eqs.(4), a solution of LE is associated, and vice versa.

Equation (1) will be here considered as a dynamical equation of the functional space M of field functions $\sigma(x,t)$, regarded as functions of the space coordinates only, defined on the whole real axis and satisfying the boundary conditions /18/

$$\lim_{x \to \pm\infty} \sigma_x = \pm a, \quad a \in R^{+}; \qquad \lim_{x \to \pm\infty} \sigma_t = \pm b, \qquad b \in (R - \{0\}). \qquad (8)$$

LE can be written in the form

$$\sigma_t = P \frac{\delta H_1}{\delta \sigma}. \qquad (9)$$

where P and $H_1[\sigma]$ are defined as follows:

$$P\phi = \frac{1}{18}\Big(\int_{-\infty}^{x} \phi\, dx - \int_{x}^{\infty} \phi\, dx \Big), \qquad H_1[\sigma] = 9 \int_{-\infty}^{+\infty} e^{\sigma} dx, \qquad (10)$$

and $\frac{\delta H}{\delta \sigma}$ denotes the gradient of a functional $H[\sigma]$ with respect to the L_2 scalar product (,).

The corresponding symplectic form

$$\Omega_1(\delta_1\sigma, \delta_2\sigma) = (\delta_1\sigma, 9D\,\delta_2\sigma), \qquad (11)$$

where $D \equiv \partial/\partial x$ and $\delta_1\sigma$ and $\delta_2\sigma$ belong to the tangent space TM, is easily verified to be closed.

It follows that LE is a Hamiltonian system. In terms of Poisson brackets $\{ , \}$, defined in the usual way:

$$\{L, L'\} \equiv \Omega_1\Big(P\frac{\delta L'}{\delta\sigma}, P\frac{\delta L}{\delta\sigma}\Big) = \Big(\frac{\delta L}{\delta\sigma}, P\frac{\delta L'}{\delta\sigma}\Big), \qquad (12)$$

where $L[\sigma]$, $L'[\sigma]$ are any pair of functionals (Frechet differentiable) defined on M, Liouville's equation assumes the form:

$$\dot{\sigma} = \{\sigma, H_1\}. \qquad (13)$$

On the other hand LE can also be written in the form

$$\dot{\sigma} = E_L \frac{\delta H_2}{\delta \sigma}, \qquad (14)$$

where E_L and $H_2[\sigma]$ are defined by

$$E_L\phi \equiv \frac{1}{9}[D - \sigma_x P \sigma_x + a^2 D^{-1}], \qquad H_2 = -\frac{9}{a^2}\int_{-\infty}^{\infty} e^{\sigma} dx, \qquad (15)$$

and $D^{-1} \equiv 9P$.

The operator E_L is an antysymmetric operator with respect to the inner product used to define the gradient. It is easy to verify that the 2-form

$$\Omega_2(\delta_1\sigma,\delta_2\sigma)\equiv(\delta_1\sigma, E_L^{-1}\delta_2\sigma), \qquad (16)$$

is (weakly) not degenerate and closed, i.e. a (weakly) symplectic form, and, then, LE is an infinite dimensional hamiltonian system also with respect to Ω_2.

This fact allows to construct an infinite sequence of gradients of conserved and in involution functionals by means of the operator:

$$\check{T}_L \equiv 9DE_L = D^2 - D\sigma_x P\sigma_x + a^2. \qquad (17)$$

The operator E_L generates a sequence of conserved functionals $L_n[\sigma]$ in the sense that $\frac{\delta L_n}{\delta\sigma} = \check{T}_L^{n-1}\frac{\delta L_1}{\delta\sigma}$ $\forall n\in N$ is the sequence of their gradients, where:

$$L_1[\sigma] = -\frac{9}{2}\int_{-\infty}^{\infty}(\sigma_x^2 - a^2)\,dx. \qquad (18)$$

Involutivity and Invariance of L_n.

The involutivity of L_n is straightforward verified, because for every $n<m$:

$$\{L_n, L_m\} = \left(\frac{\delta L_n}{\delta\sigma}, P\frac{\delta L_m}{\delta\sigma}\right) = \left(\frac{\delta L_n}{\delta\sigma}, E_L\frac{\delta L_{m-1}}{\delta\sigma}\right) = -\left(E_L\frac{\delta L_n}{\delta\sigma}, \frac{\delta L_{m-1}}{\delta\sigma}\right) =$$

$$= -\left(P\frac{\delta L_{n+1}}{\delta\sigma}, \frac{\delta L_{m-1}}{\delta\sigma}\right) = -\{L_{n+1}, L_{m-1}\}.$$

By iteration it results that $\exists k\in N, n\le k<m$:

$$\{L_n, L_m\} = \left(\frac{\delta L_k}{\delta\sigma}, B\frac{\delta L_k}{\delta\sigma}\right),$$

with B equal to one of the operators P or E_L.

Then

$$\{L_n, L_m\} = 0. \qquad (19)$$

The invariance of L_n is likewise easily verified; by observing that $\{H_1, L_1\}=0$ and

$$\{H_1, L_n\} = \left(\frac{\delta H_1}{\delta\sigma}, P\frac{\delta L_n}{\delta\sigma}\right) = \left(\frac{\delta H_1}{\delta\sigma}, E_L\frac{\delta L_{n-1}}{\delta\sigma}\right) = -\left(E_L\frac{\delta H_1}{\delta\sigma}, \frac{\delta L_{n-1}}{\delta\sigma}\right) =$$

$$= \frac{1}{a^2}\left(E_L\frac{\delta H_2}{\delta\sigma}, \frac{\delta L_{n-1}}{\delta\sigma}\right) = \frac{1}{a^2}\left(P\frac{\delta H_1}{\delta\sigma}, \frac{\delta L_{n-1}}{\delta\sigma}\right) = -\frac{1}{a^2}\{H_1, L_{n-1}\}.$$

By iteration

$$\{H_1, L_n\} = 0. \qquad (20)$$

Invariance of $\check{T}_L$

Involutivity of two arbitrary functionals $L[\sigma], L'[\sigma]$

$$\{L, L'\} = 0 \qquad (21)$$

in terms of their gradients G, G' reads

$$(G, PG') = 0 \qquad (22)$$

and by taking the derivative with respect to σ, leads to

$$(G_\sigma X, PG') + (G, PG'_\sigma X) = 0, \qquad (23)$$

where X is a tangent vector and

$$G_\sigma X \equiv \frac{d}{d\epsilon}G[u+\epsilon X]\Big|_{\epsilon=0}. \qquad (24)$$

Operator G_σ is symmetric with respect to (,) by Schwartz theorem. Because of the symmetry of G_σ eq. (24) leads to

$$G_\sigma PG' = G'_\sigma PG. \qquad (25)$$

In particular if L' is the Hamiltonian ($L'\equiv H_1$, $G' = 9e^\sigma$) eq.(18) implies

$$\dot{G} = -A^+ G, \qquad (26)$$

where A^+ is the adjoint of the operator $A = 9Pe^\sigma$ which appears in the linearized LE equation and the dot denotes the time-derivative along integral curves, solutions of (1).

Time derivative, along solutions of (1) of the sequence $G_{n+1} = \check{T}_L G_n$ gives

$$\dot{G}_{n+1} = \dot{\check{T}}_L G_n + \check{T}_L\dot{G}_n. \qquad (27)$$

By using for $\dot{G}_{n+1}$ and $\dot{G}_n$ eq.(19), it becomes

$$-A^+G_{n+1} = \dot{\check{T}}_L G_n - \check{T}_L A^+ G_n, \qquad (28)$$

and then

$$\dot{\check{T}}_L G_n = [\check{T}_L, A^+]G_n. \qquad (29)$$

As a matter of fact operator $\check{T}_L$ satisfies the relation

$$\dot{\check{T}}_L = [\check{T}_L, A^+], \qquad (30)$$

whose geometrical meaning is obvious once the mixed tensor field

$$T_L(\alpha, X) \equiv \langle\alpha, \hat{T}_L X\rangle = \langle\check{T}_L\alpha, X\rangle \qquad (31)$$

is introduced, where α denotes a differential form, X a vector field and $\langle\,,\rangle$ the usual contraction operation between forms and vectors.

So T_L is the endomorphism of the module of differential forms naturally associated with the $\binom{1}{1}$ tensor field contraction of $\binom{2}{0}$ skew-symmetric tensor field Ω_1^{-1} with the skew-symmetric $\binom{0}{2}$ tensor field Ω_2

$$T_L \equiv C(\Omega_1^{-1}\otimes\Omega_2). \qquad (32)$$

Equation (30) is just the transcription in a specific frame of the invariance of T_L under the dynamics $\Delta_L \equiv 9Pe^{\sigma}$ of LE, expressed by

$$\mathcal{L}_{\Delta_L} T_L = 0, \tag{33}$$

where $\mathcal{L}_\Delta$ denotes the Lie derivative with respect to Δ. In terms of $\hat{T}_L \equiv D^2 - \sigma_x P\sigma_x D + a^2$ the adjoint of T_L with respect to L_2 scalar product, it becomes:

$$\dot{\hat{T}}_L = [A, \hat{T}_L]. \tag{34}$$

Nijenhuis Condition

Operator $\check{T}_L$, as it has been shown, maps gradients of (conserved) functionals in gradients of (conserved) functionals. This very peculiar property is assured by the following condition fulfilled by $\hat{T}_L$ [10,25]:

$$(\hat{T}_L)_\sigma (X, \hat{T}Y) - (\hat{T}_L)_\sigma (Y, \hat{T}X) = \hat{T}_L[(\hat{T}_L)_\sigma (X, Y) - (\hat{T}_L)_\sigma (Y, X)], \tag{35}$$

where

$$(\hat{T}_L)_\sigma (X, Y) \equiv \frac{d}{d\epsilon} \hat{T}_L[\sigma + \epsilon Y] X |_{\epsilon = 0} \tag{36}$$

is the derivative of operator $\hat{T}_L$ and X, Y are any pair of tangent vectors.

Equation (35) whose geometrical reading [11,12] is easily verified to be

$$(\mathcal{L}_{\hat{T}_L X} T_L)^{\wedge} = \hat{T}_L (\mathcal{L}_X T_L)^{\wedge}. \tag{37}$$

jointly with Δ invariance expressed by (37), it assures that the sequence $\frac{\delta L_{n+1}}{\delta\sigma} = \check{T}_L^n \frac{\delta L_1}{\delta\sigma}$ is a sequence of gradients of conserved functionals, or, what it is all the same, that vector fields $(\Delta_L)_n$ of the sequence

$$(\Delta_L)_{n+1} = \hat{T}_L^n (\Delta_L)_1, \quad n \in N; \quad ((\Delta_L)_1 = \sigma_x, (\Delta_L)_2 = \sigma_{3x} - \frac{1}{2}\sigma_x^3 + \frac{a^2}{2}\sigma_x), \tag{38}$$

constitute an abelian (Lie) algebra of symmetries for LE.

Relation with KdV

For the Korteweg- de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \tag{39}$$

similar results hold [2,24]. It is an infinite dimensional bi-Hamiltonian system for which a mixed tensor field T_K:

$$\hat{T}_K \equiv D^2 + \frac{2}{3} u + \frac{u_x}{3} D^{-1}, \tag{40}$$

satisfying properties 1.,2.,3. exists. The conserved functionals $K_n[u]$ are determined by the sequence

$$\frac{\delta K_{n+1}}{\delta u} = \check{T}_K \frac{\delta K_n}{\delta u}, \quad K_1[u] = 3 \int_{-\infty}^{\infty} u dx, \tag{41}$$

and the symmetries by the sequence:

$$(\Delta_K)_{n+1} = \hat{T}_K^n (\Delta_K)_1, \quad (\Delta_K)_1 = -u_x. \tag{42}$$

The likeness of T_K and T_L suggests the existence of an operator $F: C^\infty \to C^\infty$ such that the map

$$u = F[\sigma], \tag{43}$$

transforms conserved functionals $K_n[u]$ of KdV into conserved functionals $L_n[\sigma]$ of LE, i.e. such that:

$$L_n[\sigma] = K_n[F[\sigma]]. \tag{44}$$

Following the same procedure of ref. [2], it results

$$F[\sigma] = -\frac{3}{2}(\sigma_x^2 + \sigma_{xx} - a^2), \tag{45}$$

and then

$$\check{T}_L F_\sigma^+ = F_\sigma^+ \tilde{\check{T}}_K, \tag{46}$$

where $F_\sigma = -3(\sigma_x D + D^2)$ is the weak-derivative of the operator $F[\sigma]$, $F_\sigma^+ = 3(\sigma_x D + \sigma_{xx} - D^2)$ its adjoint and $\tilde{\check{T}}_K \equiv (\check{T}_K)_{u=F[\sigma]}$. As a consequence

$$\frac{\delta L_n}{\delta\sigma} = F_\sigma^+ \frac{\delta K_n}{\delta F}, \quad (\Delta_L)_n = F_\sigma (\tilde{\Delta}_K)_n, \tag{47}$$

where

$$\frac{\delta K_n}{\delta F} \equiv \frac{\delta K_n}{\delta u} |_{u=F[\sigma]}, \quad (\tilde{\Delta}_K)_n \equiv (\Delta_K)_n |_{u=F[\sigma]}.$$

So the map $F: C^\infty \to C^\infty$, transforms the Hamiltonian flow of KdV onto a one-parameter symmetry group of LE.

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Построено инвариантное составное тензорное поле с нулевым тензором Ниенхьюза, дающее сохраняющиеся функционалы уравнения Лиувилля.

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Bi-Hamiltonian Structure and Conserved Functionals of Liouville's Equation

An invariant mixed tensor field, with vanishing Nijenhuis tensor, which accounts for conserved functionals of Liouville's equation, is constructed.

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