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STATIC ELASTIC AND THERMOELASTIC FIELD FLUCTUATIONS IN MULTIPHASE COMPOSITES

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1. Introduction

The elastic field in a multiphase composite exhibits spatial fluctuations owing to the random spatial variation of the material properties. In a previous work dealing with the characterization and evaluation of these fluctuations /1/, hereafter referred to as I, we have derived some exact relations between the square means of the field in each phase and the analytic properties of the effective material parameters. Since these relations do not determine the square means completely, a modified effective-medium approximation for calculating them in composites with aggregate topology has been presented. In the present paper the procedure given in I is extended to the evaluation of the static thermoelastic field fluctuations. Moreover, we give another representation of our previous results, which is more convenient for applications.

The basic equation of linear thermoelastostatics in the absence of body forces is given by

$$\sigma_{ik,k} = 0 . \tag{1.1}$$

where the stress tensor σ_{ik} is related to the strain tensor $\varepsilon_{ik} = (u_{i,k} + u_{k,i})/2$, ($\underline{u}$ denotes the displacement field) by the local constitutive law

$$\sigma_{ik}(\underline{r}) = C^{ra}_{ik\,lm}(\underline{r}) \left[\varepsilon_{lm}(\underline{r}) - \hat{\alpha}^{ra}_{lm}(\underline{r})\, \vartheta \right] \tag{1.2}$$

with the elastic moduli tensor C^{ra} and the tensor of thermal expansion $\hat{\alpha}^{ra}$. The temperature difference ϑ is supposed to be homogeneous. Furthermore, the considered medium is assumed to possess an aggregate topology in contrast with a matrix-inclusion one. It consists of a random arrangement of homogeneous regions (grains) ν, and the material parameters take the form

$$C^{ra}(\underline{r}) = \sum_{\nu} C^{\nu}\,\Theta_{\nu}(\underline{r}) \ , \qquad \sum_{\nu} \Theta_{\nu}(\underline{r}) = 1 \ , \tag{1.3}$$

(and analogously for $\hat{\alpha}^{ra}$), where c^{ν} means the material parameter of grain ν . The step function Θ_{ν} is equal to one inside the grain ν and zero otherwise.

In the case $\vartheta = 0$ the mean fields $\langle\sigma\rangle$ and $\langle\varepsilon\rangle$ defined as ensemble or volume averages are connected by the effective constitutive law $\langle\sigma\rangle = c\langle\varepsilon\rangle$. With this definition of the effective elastic moduli tensor c , the effective thermal expansion tensor $\hat{\alpha}$ can be defined by

$$\langle\sigma\rangle = c(\langle\varepsilon\rangle - \hat{\alpha}\vartheta). \tag{1.4}$$

Some works concerning the evaluation of the effective thermal expansion coefficient have been reviewed in /2/. A recent paper devoted to this problem in the case of anisotropic phases has been published by Hashin /3/. However, this is not the subject of the present paper.

Another task of practical importance, which has received growing attention in the last years, is the calculation of the mean values of the field in special phases of the composite as well as its fluctuations. (For some references the reader is referred to I). The mean strain in the phase A can be defined by

$$\langle\varepsilon\rangle_A = \langle\Theta_A(\underline{r})\,\varepsilon(\underline{r})\rangle / \vartheta_A \tag{1.5}$$

$$\Theta_A(r) = \sum_{\nu\in A} \Theta_\nu(\underline{r}) , \qquad \vartheta_A = \langle\Theta_A\rangle , \tag{1.6}$$

where the sum is over all grains ν occupied by the phase A and ϑ_A denotes its volume fraction. The field fluctuations in the phase A are characterized by the products

$$q_A = \langle\varepsilon\otimes\varepsilon\rangle_A - \langle\varepsilon\rangle_A \otimes \langle\varepsilon\rangle_A ,$$

$$\langle\varepsilon\otimes\varepsilon\rangle_A = \langle\Theta_A(\underline{r})\,\varepsilon(\underline{r})\otimes\varepsilon(r)\rangle/\vartheta_A \tag{1.7}$$

which represent tensors of fourth rank. A general scheme for calculating the moments of the random elastic field in the case of a composite with matrix-inclusion topology has been presented by Kanaun /4/. This procedure is similar to that proposed in the following. Another approach to the evaluation of the fluctuations based on information theory has recently been given by Pompe and Kreher /5/.

In the present work the mean fields (1.5) and the fluctuations (1.7) are calculated within a self-consistent single-grain approximation including thermal expansion. The procedure is briefly outlined in Section 2. The thermoelastic field fluctuations are calculated in Section 3. Explicit results are obtained for phases with isotropic material properties by supposing approximately spherical grains and

homogeneous mean fields $\langle\varepsilon\rangle$ and $\langle\sigma\rangle$. Section 4 is devoted to the derivation of some rigorous relations between the fluctuations and the effective thermal expansion coefficient. They are satisfied by our self-consistent approximation. Finally, a numerical analysis of the obtained results for a two-phase composite is given in Section 5.

2. Self-Consistent Approximation

Analogously to the case of electrostatics extensively treated in I, one can transform (1.1) into an equation for the displacement field $\underline{u}$

$$L^{ra}_{il} u_l := -\partial_k c^{ra}_{iklm} \partial_m u_l = -\partial_k c^{ra}_{iklm} \hat{\alpha}^{ra}_{lm} \vartheta =: q^{ra}_i . \tag{2.1}$$

The differential operator L^{ra} and the source term q^{ra} depend on the random material parameters c^{ra} and $\hat{\alpha}^{ra}$. Thus, in contrast to I.(3.1), we are dealing here with a linear differential equation that contains a random source term. The mean field $\langle\underline{u}\rangle$ is governed by the effective equation

$$L_{il}\langle u_l\rangle := -\partial_k c_{iklm}\partial_m\langle u_l\rangle = -\partial_k c_{iklm}\hat{\alpha}_{lm}\vartheta =: q_i \tag{2.2}$$

with the corresponding effective material properties c and $\hat{\alpha}$.

Choosing the still unknown effective medium as the homogeneous reference medium, we can perform the following decompositions:

$$L^{ra} = L + \sum_\nu \delta L^\nu , \qquad \delta L^\nu_{il} = -\partial_k \delta c^\nu_{iklm}\Theta_\nu(\underline{r})\partial_m \tag{2.3}$$

$$\delta c^\nu = c^\nu - c$$

$$\underline{q}^{ra} = \underline{q} + \sum_\nu \delta\underline{q}^\nu , \qquad \delta q^\nu_i = -\partial_k \delta(c_{iklm}\hat{\alpha}_{lm})^\nu \Theta_\nu(\underline{r})\vartheta \tag{2.4}$$

$$\delta(c\hat{\alpha})^\nu = c^\nu\hat{\alpha}^\nu - c\hat{\alpha}$$

$$\underline{u} = \langle\underline{u}\rangle + \sum_\nu \underline{u}^\nu \tag{2.5}$$

and the basic equation (2.1) may be split up into an equivalent set of equations

$$L\langle\underline{u}\rangle = \underline{q} , \tag{2.6}$$

$$(L + \delta L^\nu)\underline{u}^\nu = -\delta L^\nu(\langle\underline{u}\rangle + \sum_{\nu'\neq\nu} \underline{u}^{\nu'}) + \delta\underline{q}^\nu . \tag{2.7}$$

After some formal transformations the following set of equations for the strain tensor $\varepsilon_{ik} = u_{(i,k)}$ results from (2.5) and (2.7):

$$\varepsilon = \langle \varepsilon \rangle + \sum_{\nu} \varepsilon^{\nu} , \tag{2.8}$$

$$\varepsilon^{\nu} = A^{\nu} (\langle \varepsilon \rangle + \sum_{\nu' \neq \nu} \varepsilon^{\nu'} - \psi^{\nu}) . \tag{2.9}$$

where

$$A^{\nu} = \Gamma \delta c^{\nu} \Theta_{\nu} (1 - \Gamma \delta c^{\nu} \Theta_{\nu})^{-1} , \quad \Gamma_{iklm} = \partial_{(k} G_{i)(l} \partial_{m)} , \tag{2.10}$$

$$\psi^{\nu} = (\delta c^{\nu})^{-1} \delta (c \hat{\alpha})^{\nu} \vartheta \tag{2.11}$$

and G denotes the inverse operator L^{-1}.

Starting from the rigorous system of equations (2.8), (2.9) the first and second moments of the field in each phase of the composite have been calculated in I for the case $\vartheta = 0$ within a self-consistent single-grain approximation. According to (2.9) the results of I may easily be extended to the case $\vartheta \neq 0$ by the replacement

$$A^{\nu} \langle \varepsilon \rangle \longrightarrow A^{\nu} (\langle \varepsilon \rangle - \psi^{\nu}) . \tag{2.12}$$

For isotropic materials and spherical grain shapes we get inside the grain ν

$$(\Theta_{\nu} A^{\nu} \langle \varepsilon \rangle)_{ik} =: (\hat{A}^{\nu} \langle \varepsilon \rangle)_{ik} = - a_{\nu} \delta_{ik} \mathrm{Tr} \langle \varepsilon \rangle - 2 b_{\nu} e_{ik} \tag{2.13}$$

with

$$a_{\nu} = (\varkappa_{\nu} - \varkappa)/(3\varkappa_{\nu} + 4\mu) ,$$

$$b_{\nu} = 3(\mu_{\nu} - \mu)/[6\mu_{\nu} + \mu(9\varkappa + 8\mu)/(\varkappa + 2\mu)] , \tag{2.14}$$

$$e_{ik} = \varepsilon_{ik} - \delta_{ik} \mathrm{Tr}\, \varepsilon / 3 , \tag{2.15}$$

where μ_{ν}, $\varkappa_{\nu}$, μ and $\varkappa$ are the shear and bulk moduli of the grains and the effective medium, respectively. Furthermore, the thermal expansion coefficients simplify to scalars

$$\hat{\alpha}^{\nu}_{lm} = \alpha_{\nu} \delta_{lm} , \quad \hat{\alpha}_{lm} = \alpha \delta_{lm} , \tag{2.16}$$

$$\psi^{\nu}_{lm} = \delta_{lm} (\varkappa_{\nu} \alpha_{\nu} - \varkappa \alpha) \vartheta / (\varkappa_{\nu} - \varkappa)$$

and we finally obtain

$$(\hat{A}^{\nu} (\langle \varepsilon \rangle - \psi^{\nu}))_{ik} = - a_{\nu} \delta_{ik} [\mathrm{Tr} \langle \varepsilon \rangle - 3 (\varkappa_{\nu} \alpha_{\nu} - \varkappa \alpha) \vartheta / (\varkappa_{\nu} - \varkappa)] - 2 b_{\nu} e_{ik} . \tag{2.17}$$

As has been noted in I an analysis of the self-consistency condition $\langle \sum_{\nu} u^{\nu} \rangle = 0$ (cf. eq.(2.5)) for $\vartheta = 0$ leads to the condition $\sum_A v_A \hat{A}^A \langle \varepsilon \rangle = 0$ that yields the two equations

$$\langle b_A \rangle = \sum_A v_A b_A = 0 , \quad \langle a_A \rangle = \sum_A v_A a_A = 0 \tag{2.18}$$

determining the effective elastic moduli $\varkappa$ and μ. The sums run over all phases A of the composite and v_A denotes the phase volume fraction. (Here and in the following the quantities with the phase index A are obtained from those with the grain index ν by replacing ν by A). For $\vartheta \neq 0$, according to the substitution (2.12), we get the additional condition

$$\sum_A v_A (\hat{A}^A \psi^A)_{ik} = - 3 \delta_{ik} \sum_A v_A (\varkappa_A \alpha_A - \varkappa \alpha) \vartheta / (3\varkappa_A + 4\mu) = 0. \tag{2.19}$$

By the use of (2.18) equation (2.19) yields the effective thermal expansion coefficient α given by Budiansky[6]

$$\alpha = (3 + 4\mu/\varkappa) \sum_A v_A \alpha_A \varkappa_A / (3\varkappa_A + 4\mu) . \tag{2.20}$$

After this short rederivation of the effective-medium approximations of the effective material parameters let us now discuss the mean fields and the fluctuations of the field in the phases defined by (1.5) and (1.7), respectively. According to the substitution (2.12), instead of the former result $\langle \varepsilon \rangle_A = (1 + \hat{A}^A) \langle \varepsilon \rangle$, (cf. I. (4.4)), the mean strain in the phase A is now given by

$$\langle \varepsilon_{ik} \rangle_A = (1 - 2b_A) \langle e_{ik} \rangle + (1 - 3a_A) \delta_{ik} \mathrm{Tr} \langle \varepsilon \rangle / 3 + 3 \delta_{ik} (\varkappa_A \alpha_A - \varkappa \alpha) \vartheta / (3\varkappa_A + 4\mu) . \tag{2.21}$$

By the use of the constitutive laws (1.2) and (1.4) the strains may be replaced by the more relevant stresses. Then, we obtain

$$\langle \sigma_{ik} \rangle_A = (\mu_A/\mu)(1 - 2b_A) \langle s_{ik} \rangle + (\varkappa_A/\varkappa)(1 - 3a_A) \delta_{ik} \mathrm{Tr} \langle \sigma \rangle / 3 + 12 \delta_{ik} \mu \varkappa_A (\alpha - \alpha_A) \vartheta / (3\varkappa_A + 4\mu) , \tag{2.22}$$

where the deviator S_{ik} of σ_{ik} is defined analogously to (2.15).

In the case $\vartheta = 0$ the fluctuations q_A (1.7) of the strain in the phase A were given by I.(4.18)

$$q_A = G^A(1-H)^{-1}H \langle\varepsilon\rangle \otimes \langle\varepsilon\rangle \tag{2.23}$$

with

$$H = \langle \sum_\nu (1-\Theta_\nu(\underline{r}))\, A^\nu(\underline{r}) \otimes A^\nu(\underline{r})\rangle, \tag{2.24}$$

$$G^A = (1+\hat{A}^A) \otimes (1+\hat{A}^A). \tag{2.25}$$

For $\vartheta \neq 0$, according to (2.12), we have to replace

$$H\langle\varepsilon\rangle\otimes\langle\varepsilon\rangle \longrightarrow \langle \sum_\nu (1-\Theta_\nu)\left[A^\nu(\langle\varepsilon\rangle-\psi^\nu)\otimes A^\nu(\langle\varepsilon\rangle-\psi^\nu)\right]\rangle. \tag{2.26}$$

By splitting up the mean field $\langle\varepsilon\rangle$ into

$$\langle\varepsilon_{ik}\rangle = \bar{\varepsilon}_{ik} + \delta_{ik}\alpha\vartheta, \qquad \bar{\varepsilon}_{ik} := C^{-1}_{iklm}\langle\sigma_{lm}\rangle \tag{2.27}$$

and inserting (2.26) and (2.27) into (2.23) one gets finally

$$q^A = G^A(1-H)^{-1}\left\{ H\mathcal{E}^m + H^{mt}\mathcal{E}^{mt} + H^t\mathcal{E}^t \right\} \tag{2.28}$$

with

$$\begin{aligned}
\mathcal{E}^m_{iklm} &= \bar{\varepsilon}_{ik}\bar{\varepsilon}_{lm}, \qquad \mathcal{E}^t_{iklm} = \delta_{ik}\delta_{lm}\vartheta^2,\\
\mathcal{E}^{mt}_{iklm} &= (\delta_{ik}\bar{\varepsilon}_{lm} + \bar{\varepsilon}_{ik}\delta_{lm})\vartheta,\\
H^{mt} &= \langle \sum_\nu (1-\Theta_\nu) A^\nu \otimes A^\nu \omega_\nu\rangle,\\
H^t &= \langle \sum_\nu (1-\Theta_\nu) A^\nu \otimes A^\nu \omega_\nu^2\rangle,\\
\omega_\nu &= \varkappa_\nu(\alpha-\alpha_\nu)/(\varkappa_\nu-\varkappa).
\end{aligned} \tag{2.29}$$

Equation (2.28) represents a general expression for the fluctuations of the strain field in the case of a simultaneous mechanical load and thermal expansion. Let us now turn to a detailed analysis of this equation.

3. Thermoelastic Field Fluctuations

The field fluctuations q^A (2.23) for the case $\vartheta = 0$ have been presented in I by means of a representation based on a special set of orthonormal base tensors. In the following we give an alternative representation of the obtained final result which is probably more convenient for direct applications.

The fluctuations q^A are fourth rank tensors which, in the isotropic case, have to be built up from the unit tensor I of second rank ($I_{ik} = \delta_{ik}$) and the tensor $\bar{\varepsilon}$ up to quadratic order. Taking into account the symmetries of the tensor q^A with respect to an interchange of the indices we can construct the following tensors:

$$\begin{aligned}
E_0 &= I \otimes I,\\
E_1 &= \bar{e}\otimes I + I\otimes\bar{e}, \qquad \bar{e} = \bar{\varepsilon} - I\,\mathrm{Tr}\,\bar{\varepsilon}/3,\\
E_2 &= \bar{e}^2\otimes I + I\otimes\bar{e}^2, \qquad (\bar{e}^2)_{ik} = \bar{e}_{il}\bar{e}_{lk},\\
E_3 &= \bar{e}\otimes\bar{e},
\end{aligned} \tag{3.1}$$

as well as the corresponding symmetrized tensors $\tilde{E}_n$ ($n = 0,1,2,3$)

$$(\tilde{E}_n)_{iklm} = (E_n)_{iklm} + (E_n)_{ilkm} + (E_n)_{imlk}. \tag{3.2}$$

With these tensors the quantities $\mathcal{E}$ in (2.29) may be written as

$$\begin{aligned}
\mathcal{E}^m &= (\mathrm{Tr}\,\bar{\varepsilon}/3)^2 E_0 + (\mathrm{Tr}\,\bar{\varepsilon}/3)E_1 + E_3,\\
\mathcal{E}^{mt} &= 2(\mathrm{Tr}\,\bar{\varepsilon}/3)\vartheta E_0 + \vartheta E_1,\\
\mathcal{E}^t &= \vartheta^2 E_0.
\end{aligned} \tag{3.3}$$

If the result of I for the contribution of $\mathcal{E}^m$ to q_A is rewritten in our new representation (3.1), (3.2), one obtains

$$\begin{aligned}
q^m_A &= G^A(1-H)^{-1}H\mathcal{E}^m\\
&= \left[F_{11}(\mathrm{Tr}\,\bar{\varepsilon}/3)^2 + (F_{12}/\sqrt{5})\,\mathrm{Tr}\,\bar{e}^2/3\right]E_0\\
&+ \left[(F_{21}/2\sqrt{5})(\mathrm{Tr}\,\bar{\varepsilon}/3)^2 + (F_{22}/10)\,\mathrm{Tr}\,\bar{e}^2/3\right](3\tilde{E}_0 - 5E_0)\\
&+ (F_{33}/21)\left[2\,\mathrm{Tr}\,\bar{e}^2\,\tilde{E}_0/5 + 7\tilde{E}_3 - 2\tilde{E}_2\right]\\
&+ F_{44}(\mathrm{Tr}\,\bar{\varepsilon}/3)E_1 + (4F_{45}/\sqrt{14})\left[(\mathrm{Tr}\,\bar{e}^2/3)E_0 - E_2/2\right]
\end{aligned} \tag{3.4}$$

$$- (F_{54}/\sqrt{14})(\mathrm{Tr}\,\bar{\varepsilon}/3)(3\tilde{E}_1 - 7E_1)$$

$$- (F_{55}/7)[2(\mathrm{Tr}\,\bar{e}^2/3)(3\tilde{E}_0 - 7E_0) - (3\tilde{E}_2 - 7E_2)].$$

For abbreviation the phase index A at the quantities F_{ik} has been omitted. The expressions for F_{ik} are listed in the appendix.

The contributions of ξ^{mt} and ξ^{t} to q_A may easily be obtained from (3.4) by means of suitable replacements. From (2.28) we get $H\xi^m \longrightarrow H^{mt}\xi^{mt}$ and the comparison of ξ^m and ξ^{mt} in (3.3) yields the following substitutions which have to be performed in (3.4): $(\mathrm{Tr}\,\bar{\varepsilon}/3)^2 \longrightarrow 2\xi_1\vartheta\,\mathrm{Tr}\,\bar{\varepsilon}/3$ and $\mathrm{Tr}\,\bar{\varepsilon}/3 \longrightarrow \xi_1'\vartheta$. The quadratic terms in $\bar{e}$ have to be omitted. Thus, we obtain

$$\begin{aligned} q_A^{mt} &= G^A(1-H)^{-1}H^{mt}\xi^{mt} \\ &= [2F_{11}\xi_1(\mathrm{Tr}\,\bar{\varepsilon}/3)E_0 + (F_{21}\xi_1/\sqrt{5})(\mathrm{Tr}\,\bar{\varepsilon}/3)(3\tilde{E}_0 - 5E_0) \\ &\qquad + F_{44}\xi_1' E_1 - (F_{54}\xi_1'/\sqrt{14})(3\tilde{E}_1 - 7E_1)]\,\vartheta \end{aligned} \tag{3.5}$$

with

$$\xi_1 = \langle a_A^2\omega_A\rangle/\langle a_A^2\rangle, \quad \xi_1' = \langle a_A b_A \omega_A\rangle/\langle a_A b_A\rangle. \tag{3.6}$$

The angle brackets denote an average with respect to the phases (cf. (2.18)). The additional factors ξ_1, ξ_1' in (3.5) arise from the replacement $H \longrightarrow H^{mt}$.

The contribution of ξ^t is obtained analogously from (3.4) by the replacement $(\mathrm{Tr}\,\bar{\varepsilon}/3)^2 \longrightarrow \xi_2\vartheta^2$ and the omission of all other terms containing $\bar{e}$:

$$\begin{aligned} q_A^{t} &= G^A(1-H)^{-1}H^{t}\xi^{t} \\ &= [F_{11}E_0 + (F_{21}/2\sqrt{5})(3\tilde{E}_0 - 5E_0)]\,\xi_2\vartheta^2. \end{aligned} \tag{3.7}$$

The factor

$$\xi_2 = \langle a_A^2\omega_A^2\rangle/\langle a_A^2\rangle \tag{3.8}$$

is due to the replacement $H \longrightarrow H^t$. For an analysis of equations (3.4), (3.5) and (3.7) the components of the occurring tensors are listed in the Table where the system of principal axes of the strain tensor $\bar{\varepsilon}$ is chosen as coordinate system. All non-indicated components vanish.

Table. Components of the tensors occurring in (3.4) relative to the system of principal axes of the tensor $\bar{e}$ $(\bar{e}_{ik} = e_i\delta_{ik})$.

	iiii	iikk (i≠k)	ikik=ikki (i≠k)
E_0	1	1	0
$3\tilde{E}_0-5E_0$	4	-2	3
$\tilde{\tilde{E}}_0$	3	1	1
$7\tilde{E}_3-2\tilde{\tilde{E}}_2$	$9e_i^2$	$7e_ie_k-2(e_i^2+e_k^2)$	$7e_ie_k-2(e_i^2+e_k^2)$
E_1	$2e_i$	e_i+e_k	0
E_2	$2e_i^2$	$e_i^2+e_k^2$	0
$3\tilde{E}_1-7E_1$	$4e_i$	$-4(e_i+e_k)$	$3(e_i+e_k)$
$3\tilde{\tilde{E}}_0-7E_0$	2	-4	3
$3\tilde{E}_2-7E_2$	$4e_i^2$	$-4(e_i^2+e_k^2)$	$3(e_i^2+e_k^2)$

Finally, instead of the strain fluctuations, let us consider the practically more important fluctuations of the stress. With the constitutive law of phase A, $\sigma = c^A(\varepsilon - \alpha^A\vartheta)$, we get

$$q_A^{\sigma} := \langle\sigma\otimes\sigma\rangle_A - \langle\sigma\rangle_A\otimes\langle\sigma\rangle_A = (c^A\otimes c^A)\,q_A =: C^A q_A . \tag{3.9}$$

Thus, the transition from q_A to q_A^{σ} is performed by simply replacing $F_{ik} \longrightarrow C^A F_{ik}$ in (3.4), (3.5) and (3.7), where

$$C_1^A = (3\varkappa_A)^2, \quad C_4^A = 3\varkappa_A\cdot 2\mu_A, \quad C_i^A = (2\mu_A)^2 \text{ otherwise.} \tag{3.10}$$

Furthermore, the strain $\bar{\varepsilon}$ may be replaced by the mean stress: $\bar{\varepsilon} = c^{-1}\langle\sigma\rangle =: c^{-1}\bar{\sigma}$. This gives

$$\mathrm{Tr}\,\bar{\varepsilon} = \mathrm{Tr}\,\bar{\sigma}/3\varkappa, \quad \bar{e} = \bar{s}/2\mu = (\bar{\sigma} - I\,\mathrm{Tr}\,\bar{\sigma}/3)/2\mu \tag{3.11}$$

which has to be inserted into the definitions (3.1). In this way we may express the stress fluctuations by the mean stress.

4. Exact Relations for the Thermoelastic Fluctuations

In our previous work exact relations between the scalar invariants of the fluctuation tensor q_A^m and the effective elastic moduli $\varkappa$ and μ have been established. These relations are satisfied by our self-consistent single-grain approximation. In an analogous manner we can derive exact relations between the fluctuations q_A^{mt} and the effective thermal expansion coefficient.

To this end the displacement field and strain are split up into

$$\underline{u} = \underline{u}^m + \underline{u}^t, \quad \varepsilon = \varepsilon^m + \varepsilon^t. \tag{4.1}$$

where ε^m is caused by a mechanical load only, i.e. it depends linearly on $\langle\sigma\rangle$ whereas ε^t is proportional to ϑ . According to (2.1) and (2.2) they obey the following equations:

$$\frac{\partial}{\partial \underline{r}} c^{ra}\varepsilon^m = \frac{\partial}{\partial \underline{r}}\sigma = \frac{\partial}{\partial \underline{r}}\langle\sigma\rangle = \frac{\partial}{\partial \underline{r}} c\langle\varepsilon^m\rangle, \quad \langle\varepsilon^m\rangle = \bar\varepsilon, \tag{4.2}$$

$$\frac{\partial}{\partial \underline{r}} c^{ra}\varepsilon^t = \frac{\partial}{\partial \underline{r}} c^{ra}\hat\alpha^{ra}\vartheta, \quad \langle\varepsilon^t\rangle = \hat\alpha\vartheta. \tag{4.3}$$

Now let us carry out a simultaneous variation of c^{ra} and $\hat\alpha^{ra}$ so that $c^{ra}\hat\alpha^{ra}$ remains unchanged. Then (4.3) yields

$$\frac{\partial}{\partial \underline{r}}(\delta c^{ra}\varepsilon^t + c^{ra}\delta\varepsilon^t) = 0. \tag{4.4}$$

Further we consider the scalar expression

$$\begin{aligned}\langle\varepsilon^m\delta c^{ra}\varepsilon^t\rangle &= -\langle \underline{u}^m\frac{\partial}{\partial \underline{r}}(\delta c^{ra}\varepsilon^t)\rangle \\ &= \langle \underline{u}^m\frac{\partial}{\partial \underline{r}}(c^{ra}\delta\varepsilon^t)\rangle = -\langle\varepsilon^m c^{ra}\delta\varepsilon^t\rangle \\ &= \langle\delta\underline{u}^t\frac{\partial}{\partial \underline{r}}(c^{ra}\varepsilon^m)\rangle = \langle\delta\underline{u}^t\frac{\partial}{\partial \underline{r}}c\bar\varepsilon\rangle = -\langle\delta\varepsilon^t\rangle c\bar\varepsilon.\end{aligned} \tag{4.5}$$

Here we have used equations (4.4)and (4.2). The remaining steps are only integrations by parts with respect to the volume integral: $\langle\ldots\rangle = (1/V)\int dV\ldots$. The appearing surface integrals

$$\begin{aligned}&\oint d\underline{S}\,[\underline{u}^m(\delta c^{ra}\varepsilon^t + c^{ra}\delta\varepsilon^t) - \delta\underline{u}^t(c^{ra}\varepsilon^m - c\bar\varepsilon)] \\ &= \oint d\underline{S}\,[\underline{u}^m\delta(c^{ra}\varepsilon^t) - \delta\underline{u}^t(\sigma-\bar\sigma)]\end{aligned} \tag{4.6}$$

may be omitted because they vanish for a suitable choice of the boundary conditions for u^m and u^t (e.g. $d\underline{S}\sigma = d\underline{S}\bar\sigma$ as boundary condition for u^m and $d\underline{S}c^{ra}\varepsilon^t = 0$ for u^t).

Inserting the last equation of (4.3) into (4.5) we obtain

$$\langle\varepsilon^m\delta c^{ra}\varepsilon^t\rangle = -\delta\hat\alpha\,\vartheta\,c\bar\varepsilon \qquad \text{for} \qquad \delta(c^{ra}\hat\alpha^{ra}) = 0. \tag{4.7}$$

For isotropic phases this leads to

$$\begin{aligned}&\sum_A v_A\,[\,\delta\varkappa_A\langle\mathrm{Tr}\,\varepsilon^m\,\mathrm{Tr}\,\varepsilon^t\rangle_A + 2\delta\mu_A\langle e^m_{ik}e^t_{ik}\rangle_A\,] \\ &= -3\varkappa\,\delta\alpha\,\vartheta\,\mathrm{Tr}\,\bar\varepsilon \qquad \text{for} \qquad \delta(\varkappa_A\alpha_A) = 0.\end{aligned} \tag{4.8}$$

If the effective thermal expansion is considered as a function of the variables $\varkappa_A$, μ_A and $\varkappa_A\mu_A$, i.e. $\alpha = \alpha(\varkappa_A, \mu_A, \varkappa_A\mu_A)$, equation (4.8) may be split up into

$$\begin{aligned}\langle(\mathrm{Tr}\,\varepsilon)^2\rangle_A^{mt} &= 2\langle\mathrm{Tr}\,\varepsilon^m\,\mathrm{Tr}\,\varepsilon^t\rangle_A = -\frac{6}{v_A}\left(\frac{\partial\alpha}{\partial\varkappa_A}\right)_{\varkappa_A\alpha_A}\varkappa\,\vartheta\,\mathrm{Tr}\,\bar\varepsilon, \\ \langle\mathrm{Tr}\,e^2\rangle_A^{mt} &= 2\langle e^m_{ik}e^t_{ik}\rangle_A = -\frac{3}{v_A}\frac{\partial\alpha}{\partial\mu_A}\varkappa\,\vartheta\,\mathrm{Tr}\,\bar\varepsilon.\end{aligned} \tag{4.9}$$

The index $\varkappa_A\alpha_A$ at the derivation refers to the condition $\delta(\varkappa_A\alpha_A) = 0$. Equations (4.9) represent rigorous relations between the square means and the effective thermal expansion. They are valid for an arbitrary topology of the multiphase composite.

Let us apply them now to our self-consistent approximation. From the definition of q_A (1.7) and equations (2.21) and (2.27) we obtain

$$\begin{aligned}q^{mt}_{Aiikk} &= 2\langle\varepsilon^m_{ii}\varepsilon^t_{kk}\rangle_A - 2\langle\varepsilon^m_{ii}\rangle_A\langle\varepsilon^t_{kk}\rangle_A \\ &= -6\left\{\frac{\varkappa}{v_A}\left(\frac{\partial\alpha}{\partial\varkappa_A}\right)_{\varkappa_A\alpha_A} + \frac{(3\varkappa+4\mu)(3\varkappa_A\alpha_A+4\mu\alpha)}{(3\varkappa_A+4\mu)^2}\right\}\vartheta\,\mathrm{Tr}\,\bar\varepsilon,\end{aligned} \tag{4.10}$$

$$q^{mt}_{Aikik} = 2\langle e^m_{ik}e^t_{ik}\rangle_A + \frac{1}{3}q^{mt}_{Aiikk},$$

$$q^{mt}_{Aikik} - \frac{1}{3}q^{mt}_{Aiikk} = -\frac{3}{v_A}\varkappa\frac{\partial\alpha}{\partial\mu_A}\vartheta\,\mathrm{Tr}\,\bar\varepsilon.$$

The results (2.20), (3.5) together with (2.18) can be shown to satisfy these exact relations.

5. Numerical Examples and Discussion

An analysis of the final result for the fluctuations q_A has been performed numerically for a two-phase composite. According to our general treatment the effective elastic moduli and thermal expansion have been calculated in the effective-medium approximations (2.18), (2.20).

For the case $\langle\sigma\rangle = 0$ the thermal stress fluctuations are compared with the mean thermal stress in each phase in Fig. 1. The curves are similar to those found by Pompe and Kreher [5]. For the special example of moderate heterogeneity ($\varkappa_1/\varkappa_2 = 5$) the mean quadratic deviations of the stress defined as usually by the square roots of the fluctuations

$$S^{\sigma}_{Aiklm} = \left(q^{\sigma}_{Aiklm}\right)^{1/2}$$

are of the order of the mean stress. At the maximum of $|\langle\sigma_{11}\rangle_1| + S^{\sigma}_{1,1111}$ at $\vartheta_1 = \vartheta_c \approx 0.18$ (cf. Fig. 1) we find $S^{\sigma}_{1,1111}/|\langle\sigma_{11}\rangle_1| \approx 0.36$. The dependence of this characteristic ratio on the heterogeneity ratio $\varkappa_1/\varkappa_2$ is plotted in Fig. 2a. It can easily be shown within our approximation that for a two-phase composite this ratio does not depend on the ratio α_1/α_2. In the limit $\varkappa_1/\varkappa_2 \to \infty$ the fluctuations increase up to infinity if the volume fraction ϑ_1 goes to the percolation threshold $\vartheta_c = 0.5$ (cf. Fig. 2b). This quantitative analysis clearly shows that the fluctuations are of great importance for an estimation of thermal stresses in a strongly heterogeneous composite.

The case of a simultaneous mechanical load and thermal expansion is considered in Fig. 3 where the stress fluctuations are compared with its mean value as a function of the macroscopic mean stress $\langle\sigma\rangle$.

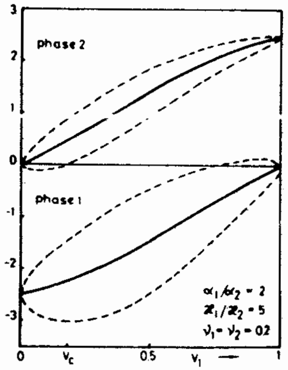

Fig. 1. Mean thermal stresses and their mean quadratic deviations versus volume fraction of phase 1 ($\langle\sigma_{11}\rangle_A / \varkappa_2\alpha_2\vartheta$ - solid lines; $(\langle\sigma_{11}\rangle_A \pm S^{\sigma}_{A1111})/\varkappa_2\alpha_2\vartheta$ - broken lines).

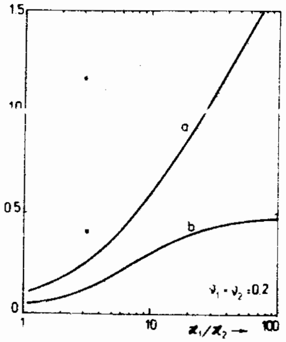

Fig. 2. Ratio of the mean quadratic deviation and the mean value of the thermal stress $S^{\sigma}_{1,1111}/|\langle\sigma_{11}\rangle_1|$ in phase 1 of a two-phase composite at the maximum of $|\langle\sigma_{11}\rangle_1| + S^{\sigma}_{1,1111}$ (cf. $\vartheta_1 = \vartheta_c$ in Fig. 1) versus ratio $\varkappa_1/\varkappa_2$ (a); ϑ_c versus $\varkappa_1/\varkappa_2$ (b).

In conclusion, let us emphasize that the formalism given above is restricted to linear elasticity theory. In many experimental situations, however, the local stresses exceed the limit of linear

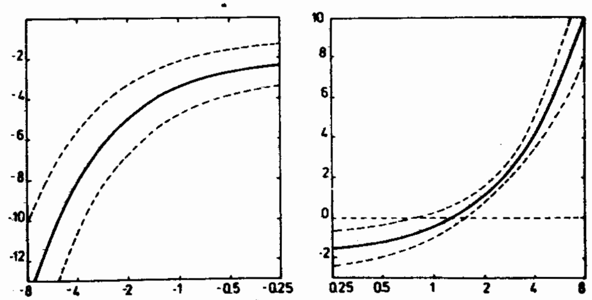

Fig. 3. Mean stress and its mean quadratic deviation in phase 1 of a two-phase composite as a function of the mean load $\langle\sigma_{11}\rangle/\varkappa_2\alpha_2\vartheta$ ($\langle\sigma_{11}\rangle_1/\varkappa_2\alpha_2\vartheta$ - solid lines; $(\langle\sigma_{11}\rangle_1 \pm S^{\sigma}_{1,1111})/\varkappa_2\alpha_2\vartheta$ - broken lines; parameters as in Fig. 1, $\vartheta_1 = 0.3$).

elasticity (compare, e.g., experiments by Hoffmann and Blumenauer /7/. The experimental investigation of the spatially fluctuating local stresses seems to be difficult and has been done mainly by means of X-ray diffraction. Moreover, the measurement of the stress fluctuations requires obviously a high accuracy and statistics of the experiment exceeding that of the measurement of the mean value. These circumstances support the importance of theoretical estimations including more realistic material laws than linear elasticity.

The formalism presented may be extended to the case where, additionally to the thermal expansion, spontaneous internal deformations due to a structural phase transition occur.

Appendix

The coefficients F_{ik}

For completeness we list here the coefficients F_{ik} occurring in (3.4). They have been derived in I.

$$\begin{aligned}
F_{11} &= f_A^2 H_{12} H_{21}/D_{12}, & F_{44} &= f_A g_A (1 - H_{55} - D_{45})/D_{45}, \\
F_{12} &= f_A^2 H_{12}/D_{12}, & F_{45} &= f_A g_A H_{45}/D_{45}, \\
F_{21} &= g_A^2 H_{21}/D_{12}, & F_{54} &= g_A^2 H_{54}/D_{45}, \\
F_{22} &= g_A^2 (1 - D_{12})/D_{12}, & F_{55} &= g_A^2 (1 - H_{44} - D_{45})/D_{45}, \\
F_{33} &= g_A^2 H_{33}/(1 - H_{33}), & &
\end{aligned} \tag{A.1}$$

$$D_{12} = 1 - H_{22} - H_{12}H_{21}, \qquad D_{45} = (1-H_{44})(1-H_{55}) - H_{45}H_{54}, \tag{A.2}$$

$$\begin{aligned} H_{12} &= 50(1-2\nu)^2 h_1/\sqrt{5}, & H_{44} &= 2(1-2\nu)h_3, \\ H_{21} &= 2h_2/\sqrt{5}, & H_{45} &= -10(5-7\nu)(1-2\nu)h_1\sqrt{2/7}, \\ H_{22} &= 2(23-50\nu+35\nu^2)h_1, & H_{54} &= -2(5-7\nu)h_3\sqrt{2/7}, \\ H_{33} &= 4(18-50\nu+35\nu^2)h_1/7, & H_{55} &= -3(11-50\nu+35\nu^2)h_1/7, \end{aligned} \tag{A.3}$$

$$h_1 = \langle b_A^2 \rangle/(4-5\nu)^2, \quad h_2 = \langle (3a_A)^2 \rangle, \quad h_3 = \langle 3a_A b_A \rangle/(4-5\nu), \tag{A.4}$$

$$f_A = 1 - 3a_A, \qquad g_A = 1 - 2b_A. \tag{A.5}$$

The angle brackets in (A.4) are defined in (2.18). The coefficients a_A, b_A are given by (2.14) and ν denotes the Poisson ratio of the effective medium $\nu = (3\varkappa - 2\mu)/2(3\varkappa+\mu)$.

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Бобет М., Динер Г. E17-85-800

Статические упругие и термоупругие флуктуации поля в многофазных композитах

Пространственно флуктуирующие поля напряжения и деформации в случайно гетерогенной термоупругой среде описываются своими средними значениями и квадратными средними. Вычисление этих величин, представленное в прежней работе авторов для случая механической нагрузки, обобщается на случай термического расширения. Выводятся строгие отношения между квадратными средними и аналитическими свойствами коэффициента эффективного термического расширения. Найдены флуктуации поля в приближении эффективной среды, в предположении агрегатной топологии композита. Получены результаты в явной форме для изотропных фаз и шарообразных зерен и представлены в более удобной форме, чем в прежней работе авторов.

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Bobeth M., Diener G. E17-85-800

Static Elastic and Thermoelastic Field Fluctuations in Multiphase Composites

The spatially fluctuating strain and stress fields in a random heterogeneous thermoelastic medium are characterized by their mean values and square means in each phase. In the present paper the calculation of these quantities, which has been presented in a previous work for the case of a mechanical load only, is extended to include thermal expansion. Besides the derivation of some exact relations between the square means and the analytical properties of the effective thermal expansion coefficient, the field fluctuations are calculated within an effective-medium procedure supposing an aggregate topology of the composite. Explicit results obtained for isotropic phases and spherical grain shapes are given in a more convenient representation than in our former work.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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