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AXIAL TOROID MOMENTS IN ELECTRODYNAMICS AND PHYSICS OF CONDENSED MATTER

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I. Introduction

In the course of studying the electromagnetic properties of systems with distributed charges and currents there arises the problem of the choice of macroscopic characteristics that provide an adequate description for the interactions of these systems with external fields and currents. An appropriate mathematical tool for solving such a problem is the formalism of multipole expansion of the microscopic charge density $\rho(\vec{r},t)$ and current density $\vec{j}(\vec{r},t)$. In the framework of classical electrodynamics this formalism is well expounded, e.g., in ref. /1/. There it has also been shown that together with two known families of charged and magnetic multipole moments the third family, that of toroid multipole moments also appears. A formal reason for the latter is the splitting of a vorticity field (in the given case, of a transversal component of the current density $\vec{j}_{\perp}(\vec{r},t)$, i.e., of the function for which $\operatorname{div}\vec{j}_{\perp}=0$). in the two components

$$\vec{j}(\vec{r},t) = \operatorname{rot}\left[\vec{r}\,\psi(\vec{r},t)\right] + \operatorname{rot}\operatorname{rot}\left[\vec{r}\,\chi(\vec{r},t)\right], \tag{1}$$

where $\psi(\vec{r},t)$ and $\chi(\vec{r},t)$ are, respectively, pseudoscalar and scalar functions. We shall call this sum by the Neumann–Debye representation /2,3,4/. The expansion of $\psi(\vec{r},t)$ over radial spherical harmonics generates a family of magnetic multipole moments among which generating is the magnetic dipole moment $\vec{M}$ since with it one may construct all higher moments of that family $\hat{M}_{ik} = r_i \hat{M}_k + r_k \hat{M}_i$ etc. An analogous expansion of χ generates a family of toroid multipole moments /1,2/. A toroid family is generated by the toroid dipole moment $\vec{T}$. Besides, the expansion of the charge density $\rho(\vec{r},t)$ generates a family of charge multipole moments, among which a vector characteristic is the charge dipole moment $\vec{P}$. The above vector quantities $\vec{P}, \vec{M}, \vec{T}$ differ in symmetry properties (the behaviour under the operation of coordinate $\hat{I}$ and time $\hat{R}$ inversions) from each other and are transformed according to table 1.

It is seen, however, that for constructing the complete vector basis of multipole representation of the group of space-time inversions

Table 1

	$\hat I$	$\hat R$
$\vec P$	−	+
$\vec M$	+	−
$\vec T$	−	−
$\vec G$	+	+

$\hat R \otimes \hat T$ the set of vectors $\vec P, \vec M, \vec T$ is not sufficient and, in principle, it is necessary to introduce one more axial vector $\vec G$ (see, e.g., /5/), the symmetry properties of which are presented in the lowest line of Table 1. In the classical electrodynamics by Maxwell and Lorentz, in which

$$\vec j_e(\vec r,t) = \sum e_i \dot{\vec r}\, \delta(\vec r - \vec r_i(t))$$

there is no place for a vector with such properties. Nevertheless, as is shown in this paper, there exists a number of physical applications of the mathematical method of multipole expansions, where the appearance of the generating dipole moment $\vec G$ and the family **of its multipoles** turns out to be inevitable.

At a multipole analysis of a magnetic charge system (sect.2) the vector $\vec G$ is the toroid moment of the magnetic current. It is analogous (dual) to the toroid moment $\vec T$ of the electric current. However, owing to pseudoscalar properties of the density $\rho_g(\vec r,t)$ of magnetic charges $\vec G$ is an axial vector, while $\vec T$ is a polar vector.

In the electrodynamics of condensed matter the quantity $\vec G$ may be introduced for describing systems with distributed charge dipole moments (sect. 3). In this case $\vec G$ is an analog of the induced toroid moment $\vec T_{ind}$ in media with distributed magnetic dipole moments /1,6,7,8/ *).

Proceeding from the symmetry properties and the analogy with toroid multipoles in the Maxwell–Lorentz electrodynamics we shall call the vector $\vec G$ the axial toroid moment, and the vector $\vec T$ the polar toroid moment.

The introduction of spin necessitates the classification of the toroid moments $\vec T$ and $\vec G$ with respect to the inversion transformation $\hat R_\sigma$ in the spin space. In case it is necessary we shall supply the even (singlet) vectors with index (s) , and odd (triplet) ones with index (t) .

*) Note, the term "induced" is actual characteristic of dipoles, arising due to the polarizability properties (of media), and therefore in the corresponding Hamiltonian there appears the additional degree of an "inducing" field. But a circular chain of dipoles is considered in this context as a primary object and the term "induced" as applied to it, e.g., in the book /8/, is confused.

In the description of phase transitions in crystals vectors $\vec T$ and $\vec G$ (or their higher multipoles) can naturally be taken to serve as order parameters transforming according to certain irreducible representations of the magnetic space group of a highly symmetric phase. Let us discuss what is the peculiarity of the toroid types of ordering, and for which systems it makes sense to introduce these concepts.

From the point of view of a formal symmetry a lot of the known order parameters are transformed analogously to $\vec T$ and $\vec G$. So, for instance, for describing the spin magnets multipole order, parameters (spin densities) have been introduced, some of which are analogous to T_t in transformation properties. The simplest is the case of a two-sublattice antiferromagnet.

Analogous to $\vec G_s$, the vector ω transforms, which is dual to the antisymmetric part of the distortion tensor /11/ , as well as the director $\vec n$ characterizing the orientational ordering in liquid crystals /12/ . In the theory of "spin nematics" /13/ some order parameters are introduced, analogous in symmetry to $\vec G_{s,t}$ and $\vec T_{s,t}$. However, the description of these systems in terms of toroid distributions is not physically transparent thought it may happen to be constructive for description of interactions with an electromagnetic field.

Separation of toroid moments in a special group of order parameters is much more justified in case of systems with itinerant electrons. In particular, the polar toroid moment describes the orbital antiferromagnetic ordering with formation of the current-density wave /14/ . The vector $\vec T_t$ can be used for describing a number of itinerant antiferro-magnets with spin-density waves.

The axial toroid moment $\vec G_s$ describes a specific charge ordering like the itinerant antiferroelectricity (see below). The vector $\vec G_t$ characterizes the orientational ordering in itinerant magnets and is connected with the spin-current density wave (SCDW) /15/ .

The scheme of multipole expansions allows a rather constructuve description for the general macroscopic properties of a number of systems with intricate distributions of charges and currents (in particular, the response to an electromagnetic field), however, an important problem is also the study of concrete quantum-mechanical models realizing different types of multipole structures. An illuminating example is the theory of polar toroid ordering in crystals constructed in refs. /14/.

In this paper, besides a general phenomenological consideration of the axial toroid ordering (sect.4), microscopic model of the phase transition with formation of such an ordering is proposed (sect.5). To be more precise, we consider two band semimetals or semiconductors with the charge-density wave (CDW) and allowed interband transition in the orbital moment. Earlier /15/ another microscopic model with formation of the (SCDW) was considered, therefore we do not concentrate on its specific features (a general outline is given in sect.4).

These models clearly illustrate the physical meaning of $\vec{G}$ and allow a better understanding of the nature of formation of $\vec{G}$ in crystals.

2. Multipole expansion of a system of magnetic charges

In ref. /1/ it is noticed that in a dual-invariant scheme of electrodynamic there appears total symmetry between forms of multipole sources and types of fields (specifically, of radiation)[1]. Note that in the electromagnetic theory invariant under $\hat{R}$ and $\hat{I}$ inversions the current of electric charges should be a vector, while that of magnetic charges a pseudovector if we adopt a conventional picture of space-time properties of fields $\vec{E}$ and $\vec{H}$ (see., e.g., /16/). Obvious (see ref. /17/), the formulation of an electromagnetic theory with pointlike magnetic charges turns our to be difficult (however, see , /18/), since, in particular, in the ratio $\operatorname{div}\vec{H} = g\,\delta(\vec{r})$ either the charge g is not a number, or it is a number, and parity is not conserved in the theory. In a macroscopic formulation of the theory this difficulty does not arise since in the equation $\operatorname{div}\vec{H} = \rho_g(\vec{r},t)$ the function $\rho_g(\vec{r},t)$ may be always considered to be odd, and in $\operatorname{rot}\vec{E} = -\dot{\vec{H}} - \vec{j}_g(\vec{r},t)$ the function $\vec{j}_g(\vec{r},t)$ to be an axial vector. With this difference of magnetic from electric charges, the problem we are interested in of the multipole expansion of densities $\rho_g(\vec{r},t)$ and $\vec{j}_g(\vec{r},t)$ is readily solved. One should, e.g., in the corresponding formulae of ref. /1/, to make the change: $\rho_e \to \rho_g$ and $\vec{j}_e \to \vec{j}_g$.

Let us write the corresponding formulae denoting the quantities having the same form in the electric and magnetic worlds by the double symbol g/e . In particular, the expansion of $\rho_{g/e}(\vec{r},t)$ is written in the form

[1] Dual-invariant should be not only field equations but also the equations of motion of charged particles. The latter is achieved by introducing into theory of particles having both electric and magnetic charges /17/ . When e/g = const for all types of particles a scheme like that may by a dual transformation be reduced to a conventional one-charge electrodynamics.

$$\rho_{g/e}(\vec{r},t) = (2\pi)^{-3}\int_0^\infty \sum_{\ell,m} (-ik)^\ell \frac{\sqrt{4\pi(2\ell+1)}}{(2\ell+1)!!} F_{\ell m k}(\vec{r})\, Q^{g/e}_{\ell m}(k^2,t)\, k^2 dk,$$

$$F_{\ell m k} = f_\ell(kr)\, Y_{\ell m}(\hat{\vec{r}}), \quad \hat{\vec{r}} \equiv \vec{r}/|\vec{r}|, \tag{2}$$

where charge multipole distributions $Q_{\ell m}(k^2,t)$ /1/ are given by

$$Q^{g/e}_{\ell m}(k^2,t) = \frac{(2\ell+1)!!}{(-ik)^\ell\sqrt{4\pi(2\ell+1)}} \int \rho_{g/e}(\vec{r},t)\, F^*_{\ell m k}(\vec{r})\, d^3r. \tag{3}$$

Hence there follow definitions of the charge multipole moments

$$Q^{g/e}_{\ell m}(0,t) = \sqrt{\frac{4\pi}{2\ell+1}} \int r^\ell\, Y^*_{\ell m}(\hat{\vec{r}})\, \rho_{g/e}(\vec{r},t)\, d^3r \tag{4}$$

and their 2n-power radii

$$\overline{r^{2n}_{\ell m}}(t) = \sqrt{\frac{4\pi}{2\ell+1}} \int r^{\ell+2n}\, Y^*_{\ell m}(\hat{\vec{r}})\, \rho_{g/e}(\vec{r},t)\, d^3r, \tag{5}$$

that complete the multipole parametrization of the initial function $\rho_{g/e}(\vec{r},t)$.

The multipole expansion of the current density of magnetic (electric) charges with a separated toroid part is as follows /1/:

$$\vec{j}_{g/e}(\vec{r},t) = (2\pi)^{-3}\int \sum_{\ell,m} (-ik)^{\ell-1} \frac{\sqrt{4\pi(2\ell+1)(\ell+1)/\ell}}{(2\ell+1)!!} \times$$

$$\times \Big\{ k\, \vec{F}^{(0)}_{\ell m k}(\vec{r})\, M^{g/e}_{\ell m}(k^2,t) +$$

$$+ \vec{F}^{(+)}_{\ell m k}(\vec{r}) \Big[\dot{Q}^{g/e}_{\ell m}(0,t) + k^2\, T^{g/e}_{\ell m}(k^2,t) \Big] + \tag{6}$$

$$+ \sqrt{\frac{\ell}{\ell+1}}\, \vec{F}^{(-)}_{\ell m k}(\vec{r})\, \dot{Q}^{g/e}_{\ell m}(k^2,t) \Big\} k^2 dk,$$

where 2):

$$\vec{\mathcal{F}}^{(\lambda)}_{\ell m k}(-\vec{r}) = (-1)^{\ell+\lambda}\,\vec{\mathcal{F}}^{(\lambda)}_{\ell m k}(\vec{r})\,, \quad \lambda = 0, \pm 1\,, \tag{7}$$

the magnetic multipole distributions

$$M^{g/e}_{\ell m}(k^2,t) = \frac{-i\,(2\ell+1)!!}{(-ik)^{\ell}\sqrt{4\pi(2\ell+1)(\ell+1)/\ell}} \times \int \vec{\mathcal{F}}^{*(0)}_{\ell m k}(\vec{r})\,\vec{j}_{g/e}(\vec{r},t)\,d^3r\,, \tag{8}$$

the magnetic multipole moments

$$M^{g/e}_{\ell m}(0,t) = \sqrt{\frac{4\pi}{(2\ell+1)(\ell+1)^2}}\int r^{\ell}\,\vec{Y}^{*}_{\ell\ell m}(\hat{r})\,\vec{j}_{g/e}(\vec{r},t)\,d^3r\,, \tag{9}$$

the toroid multipole distributions

$$T^{g/e}_{\ell m}(k^2,t) = \frac{(2\ell+1)!!}{(-ik)^{\ell+1}\sqrt{4\pi(2\ell+1)(\ell+1)/\ell}} \times \int\Big\{\vec{\mathcal{F}}^{(+1)*}_{\ell m k}(\vec{r}) - \frac{4\pi(-ikr)^{\ell-1}}{\sqrt{2\ell+1}}\,\frac{\sqrt{\ell+1}}{(2\ell-1)!!}\,\vec{Y}^{*}_{\ell\ell-1m}(\hat{r})\Big\}\,\vec{j}_{g/e}(\vec{r},t)\,d^3r\,, \tag{10}$$

the toroid multipole moments

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2) The basis of expansion (6) is introduced as follows $\vec{\mathcal{F}}^{(-)}_{\ell m k} = -\frac{i}{k}\vec{\nabla}F_{\ell m k}$, $\vec{\mathcal{F}}^{(0)}_{\ell m k} = \frac{i}{\sqrt{\ell(\ell+1)}}\,\mathrm{rot}\{\vec{r}F_{\ell m k}\}$, $\vec{\mathcal{F}}^{(+)}_{\ell m k} = \frac{i}{k\sqrt{\ell(\ell+1)}}\,\mathrm{rot\,rot}\{\vec{r}F_{\ell m k}\}$

and spherical vectors are introduced according to /19/ :

$$\{\vec{Y}_{\ell\ell' m}(\hat{r})\}^{\mu} = \sum_{m'}\langle \ell' m' 1\mu|\ell m\rangle\,Y_{\ell' m'}(\hat{r})\,,$$

$$\int \vec{\mathcal{F}}^{(\lambda)*}_{\ell m k}\,\vec{\mathcal{F}}^{(\lambda')}_{\ell' m' k'}\,d^3r = \delta_{\ell\ell'}\,\delta_{mm'}\,\delta_{\lambda\lambda'}\,\frac{(2\pi)^3}{k^2}\,\delta(k-k')\,,$$

$$\sum_{\ell,m,k,\lambda}\Big[\vec{\mathcal{F}}^{(\lambda)}_{\ell m k}(\vec{r})\Big]^{*}_{i}\,\Big[\vec{\mathcal{F}}^{(\lambda)}_{\ell m k}(\vec{\xi})\Big]_{j} = (2\pi)^3\delta_{ij}\,\delta(\vec{r}-\vec{\xi})\,.$$

$$T^{g/e}_{\ell m}(0,t) = \frac{-\sqrt{4\pi\ell}}{2(2\ell+1)}\int r^{\ell+1}\Big\{\vec{Y}^{*}_{\ell\ell-1m} + \frac{2\sqrt{\ell/\ell+1}}{2\ell+3}\,\vec{Y}^{*}_{\ell\ell+1m}\Big\}\,\vec{j}_{g/e}\,d^3r \tag{11}$$

the longitudinal charge multipole distributions

$$\dot{Q}^{g/e}_{\ell m}(k^2,t) = \frac{\sqrt{4\pi(2\ell+1)}}{(2\ell+1)!!}\int \vec{\mathcal{F}}^{(-)*}_{\ell m k}\cdot\vec{j}_{g/e}(\vec{r},t)\,d^3r\,, \tag{12}$$

the longitudinal charge multipole moments

$$\dot{Q}^{g/e}_{\ell m}(0,t) = \sqrt{4\pi\ell}\int r^{\ell-1}\,\vec{Y}^{*}_{\ell\ell-1m}\cdot\vec{j}_{g/e}(\vec{r},t)\,d^3r\,. \tag{13}$$

When the conditions of spectral expansion of $\rho_{g/e}(\vec{r},t)$ and $\vec{j}_{g/e}(\vec{r},t)$ (the simplest case is a harmonic dependence) are fulfilled, the latter definitions reduce to formulae (3), (4), and any case expressions (12), (13) are functional-dependent on $Q_{\ell m}(k^2,t)$ because of the 4-current conservation. The question of how the "longitudinal" moments $\dot{Q}_{\ell m}(0,t)$ get into the expansion of the current transversal part $(\mathrm{div}\,\vec{j} = 0)$ (see formula (6)) is much more complicated and is considered in ref. /1/ (and in references therein).

Based on the properties of vector spherical functions $\vec{\mathcal{F}}_{\ell m k}$ under $\hat{I}$ - inversions, it is not difficult to establish that distributions $M^{g}_{\ell m}(k^2,t)$ now create fields of the $E\ell$ - type (in particular, they yield $E\ell$ -radiation, electric multipoles) whereas charge (Coulomb) $Q^{g}_{\ell m}(k^2,t)$ and toroid $T^{g}_{\ell m}(k^2,t)$ distributions produce fields of the $M\ell$ -type (specifically, $Q^{g}_{\ell m}(0,t)$ and $T^{g}_{\ell m}(k^2,t)$ are responsible for the emission of magnetic multipoles).

3. Multipole representations of dipole media

Since free magnetic charges (and their currents) are not as yet detected (the history of the problem may be found, e.g., in /20/), it might seem that the results of multipole expansion in the dual electrodynamics presented in section 2 are also of a pure scholastic nature. We shall, however, show that the dual symmetry $M^{e}_{\ell m} \leftrightarrow Q^{g}_{\ell m}$, $M^{g}_{\ell m} \leftrightarrow Q^{e}_{\ell m}$ has a rather profound meaning, and that the formalism expounded in sect. 2 is useful for describing some dipole structures in the electrodynamics of condensed matter.

We shall describe media as a family of elementary magnetic dipoles $\{\mu_i\}$ $(i = 1, 2, 3, \dots N)$. The magnetic current density $\vec{j}_{\mu}(\vec{r},t)$ is given by the known formula /21/ :

$$\vec{j}_{\vec{\mu}}(\vec{r},t)=\sum_i \vec{\mu}_i \times \vec{\nabla}\,\delta(\vec{r}-\vec{r}_i(t)) \rightarrow \mathrm{rot}\,\vec{M}_{\perp}(\vec{r},t), \qquad (14)$$

where the symbol $\longrightarrow$ denotes the transition to a continuous distribution of the magnetic dipole moment density (magnetization) $\vec{M}_{\perp}$. Together with (14) it is convenient to introduce a formal quantity $\rho_{\vec{\mu}}(\vec{r},t)$, a pseudoscalar distribution density of magnetic charges:

$$\rho_{\vec{\mu}}(\vec{r},t) = -\,\mathrm{div}\,\vec{M}_{\parallel}(\vec{r},t). \qquad (15)$$

Note that the usual Maxwell-Lorentz equations do not contain the quantity $\vec{M}_{\parallel}(\vec{r},t)$ (the longitudinal component of magnetization).

If we replace the density ρ_e by $\rho_{\vec{\mu}}$ in formulae (3)-(5), then there arises almost a complete analogy of multipole expansions of these quantities (the only difference is that one should set $M_{00}=\int \rho_{\vec{\mu}}(\vec{r},t)\,d^3r \equiv 0$. Clearly, the corresponding formulae will give "charge" multipole moments of the system of magnetic dipoles. We note that a completely symmetric scheme of multipole expansions does not arise since $\mathrm{div}\,\vec{j}_{\vec{\mu}}(\vec{r},t) \neq \dot{\rho}_{\vec{\mu}}(\vec{r},t)$ and $M_{00}\equiv 0$.

So, the eddy current $\vec{j}_{\vec{\mu}}$ does not contribute to Q_{em} but its contribution to T_{em} is nonvanishing. It is just this contribution due to which toroidal distributions started to be introduced into electromagnetism (see [6] and references therein). Under the name of induced electric moments [8] there was known the inductive part of the toroid moment

$$T^{\mu}_{lm}(t) = i\sqrt{\frac{4\pi l}{(2l+1)(l+1)}} \int r^{l}\, \vec{Y}_{llm}\cdot \vec{M}_{\perp}(\vec{r},t)\, d^3r . \qquad (16)$$

Its normalization is defined as in ref. [6]. It is seen that (16) differs from M^{e}_{lm} for free currents by the change $\vec{j}^{e} \rightarrow \vec{M}_{\perp}$. So, the elementary dipole by analogy with $\vec{M}_{\perp}$, can be written in the form

$$\vec{T}_{\vec{\mu}} = \frac{1}{2}\sum_i (\vec{r}_i \times \vec{\mu}_i). \qquad (17)$$

A geometrical image in the case of the toroid (polar) dipole is a closed circular chain of elementary magnetic dipoles $\{\vec{\mu}_i\}$ (Fig.1a).

Electric dipole media will be described by a set of elementary dipoles $\{\vec{d}_i\}$. In this description, the characteristic space scale of the multipole expansion is large as compared to the characteristic scale of dipoles themselves, therefore the latter may be considered pointlike. The electric polarization of the system $\vec{P}(\vec{r},t)$ is introduced in the usual way:

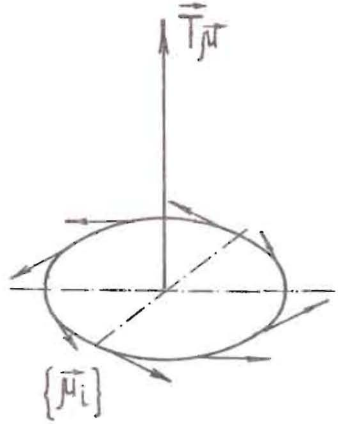

Fig.1a

$$\vec{P}(\vec{r},t) = \sum_i \vec{d}_i\, \delta(\vec{r}-\vec{r}_i(t)). \qquad (18)$$

It is clear that the electric polarization $\vec{P}(\vec{r},t)$ may, due to dual symmetry, imitate multipole moments of magnetic charges. We introduce the axial current "pseudovector" with respect to the time inversion, in the medium of distributed electric dipoles $\{\vec{d}_i\}$:

$$\vec{j}^{(a)}_{\vec{d}}(\vec{r},t) = \sum_i \vec{d}_i \times \vec{\nabla}\,\delta(\vec{r}-\vec{r}_i(t)) \rightarrow \mathrm{rot}\,\vec{P}_{\perp}(\vec{r},t), \qquad (19)$$

where $\vec{P}_{\perp}(\vec{r},t)$ is the transverse part of the density of electric dipole moments (polarization). The longitudinal part of polarization $\vec{P}_{\parallel}(\vec{r},t)$ is described by the scalar density of the distribution of electric charges

$$\rho_{\vec{d}}(\vec{r},t) = \mathrm{div}\,\vec{P}_{\parallel}(\vec{r},t). \qquad (20)$$

We emphasize that the axial current $\vec{j}^{(a)}_{\vec{d}}(\vec{r},t)$ differs in nature from the polar current $\vec{j}_{\vec{d}}(\vec{r},t)$ entering into the Maxwell-Lorentz equations

$$\vec{j}_{\vec{d}}(\vec{r},t) = \sum_i \dot{\vec{d}}_i\, \delta(\vec{r}-\vec{r}_i(t)) \rightarrow \dot{\vec{P}}(\vec{r},t). \qquad (21)$$

Unlike magnetization $\vec{M}_{\perp}$, $\vec{M}_{\parallel}$ both components of polarization ($\vec{P}_{\parallel}$ and $\vec{P}_{\perp}$) enter into the Maxwell-Lorentz equations, and only in the static limit $\vec{P}_{\perp}$ disappears there.

Going back to formula (19) we see that the substitution of the effective current $\vec{j}^{(a)}_{d}$ into the definition of $M^{g}_{\ell m}$ reduces, the definition $M^{g}_{\ell m}$ upon integration by parts, to the conventional definition of the electric part in which $\vec{j}_g$ is replaced by $\vec{P}_\perp$. Here, of course $E^{d}_{00} \equiv 0$. The case with $T^{g}_{\ell m}$ is more intriguing. Substitution of (19) into the definition (11) and transfer of the derivative $\vec{\nabla}$ lead to the formula

$$T^{d}_{\ell m}(t) = i\sqrt{\frac{4\pi\ell}{(2\ell+1)(\ell+1)}}\int r^{\ell}\,\vec{Y}_{\ell\ell m}\cdot\vec{P}_{\perp}(\vec{r},t)\,d^3r \qquad (22)$$

analogous to (16). Hence it immediately follows that the elementary "induced" axial toroid dipole moment is equal to

$$\vec{T}_{d} = \frac{1}{2}\sum_i (\vec{r}_i \times \vec{d}_i) \qquad (23)$$

and its geometric image is a closed chain of electric dipoles $\{\vec{d}_i\}$ (Fig.1b). The last formula demonstrates the simplest possibility of imitation in the dipole representation of symmetry elements absent in a system of pointlike electric charges. (In principle, the completeness of properties under $\hat{R}$ and $\hat{I}$ reflections may also be found in media of elementary higher multipoles of the usual type).

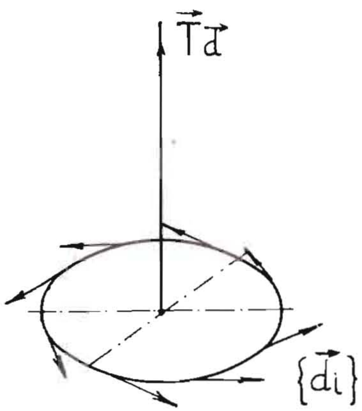

Fig.1b

Let us point to one more possibility for realizing symmetry of $\vec{T}^{g}$ in magnetic media. Recall that in the problem of motion of a particle in a central-symmetric potential correlation arises between the vector of angular momentum $\hat{\vec{L}}$ and moment $\hat{\vec{P}}$. This correlation is described by the Runge-Lentz operator

$$\hat{\vec{\Pi}} = \hat{\vec{L}} \times \hat{\vec{P}}. \qquad (24)$$

This operator already appears in considering the dynamic symmetry of the nonrelativistic Kepler problem, and it it more important for considering the dynamic symmetry of the relativistic Coulomb-problem /22/ or motion of a free relativistic particle obeying the Dirac equation /23/.

The distribution of a vector like $\vec{\Pi}$ being present in a medium (of an orbital or spin origin) also necessitates the introduction of axial toroid moments.

Consider a medium, in which there are distributions of the moment fluxes described, in general, by a second-rank tensor (dyad)

$$\Pi_{ij} = \langle \hat{L}_i \otimes \hat{P}_j \rangle, \quad i,j = x,y,z. \qquad (25)$$

We represent (25) in the form of symmetric $\Pi^{(s)}_{ij}$ and antisymmetric $\Pi^{(as)}_{ij}$ parts. The latter is dual to the polar vector $\vec{\Pi}(\vec{r},t)$ and may be described by analogy with how it has been done above with vector $\vec{P}(\vec{r},t)$ in electrodipole media. Let us introduce the transverse (axial) current

$$\vec{j}^{(a)}_{\Pi}(\vec{r},t) = \mathrm{rot}\,\vec{\Pi}_{\perp}(\vec{r},t) = \mathrm{rot}\langle \hat{\vec{L}} \times \hat{\vec{P}} \rangle. \qquad (26)$$

Substitution of this current into formulae (10), (11), like of $\vec{j}_{d}$, will produce a multipole family $T^{\Pi}_{\ell m}$. In this case the elementary dipole is

$$\vec{T}_{\Pi} = \frac{1}{2}\sum_i \vec{r}_i \times \vec{\Pi}_{\perp}(\vec{r}_i,t). \qquad (27)$$

An ideal geometrical image of that dipole is processing local moments on a circle (Fig.1c).

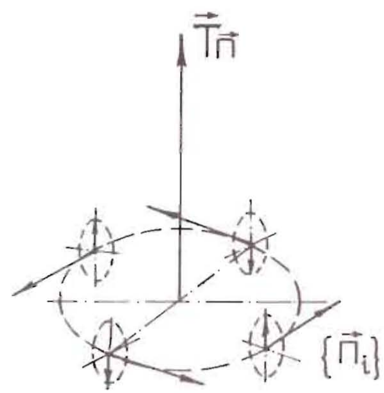

Fig.1c

4. Phenomenological theory of the axial toroid ordering in crystals

Consider a system of itinerant electrons which exhibits a second-order phase transition into a state with the axial vector order parameter $\vec{G}$ even with respect to time inversion. Priori to proceed to particular microscopic models revealing mechanisms of that ordering, we dwell upon some of its phenomenological consequences using only formal symmetry arguments.

Among magnetic classes we may indicate the following 43 classes admitting the existence of the axial vector $\vec{G}$ that is even with respect to time inversion $\hat{R}$:

1) 13 usual crystal classes not containing $\hat{R}$ at all:

$$C_1, C_i, C_s, C_2, C_{2h}, C_4, S_4, C_{4h}, C_6, S_6, C_{3h}, C_{6h}, C_3 . \tag{28}$$

2) the same classes supplemented by the operation $\hat{R}$;

3) 17 actual magnetic classes

$$C_i(C_1), C_2(C_1), C_{2h}(C_i), C_{2h}(C_2), C_{2h}(C_s), C_s(C_1), \tag{29}$$
$$C_4(C_2), S_4(C_2), C_{4h}(C_4), C_{4h}(C_{2h}), C_{4h}(S_4), S_6(C_3),$$
$$C_{3h}(C_3), C_6(C_3), C_{6h}(C_6), C_{6h}(S_6), C_{6h}(C_{3h}).$$

Further we shall consider only systems without nontrivial translations, in which giving a magnetic class is a necessary and sufficient condition for defining a possibility for the vector $\vec{G}$ to exist. Like in the case of the polar vector $\vec{T}$, all the classes listed above are not difficult to obtain by using the tables of irreducible representations of point groups.

Establishment of the axial toroid ordering in crystals may be connected with softening of a certain collective electron mode (in particular, in the model of § 5 this is a CDW oscillation, and in [15] SCDW). We shall call this mode the axial toroid mode considered earlier [24] .

Let us assume that owing to some circumstances the frequency of the axial toroid mode gets anomalously small, and a tendency has arisen for establishing the toroid long-range order.

It is convenient to analyse the general properties of such systems by the effective-Lagrangian method. Assuming that the symmetry group of a highly symmetric phase contains, as a subgroup, one of the above-listed axial magnetic groups, we consider small, low- frequency axial toroid oscillations above the phase-transition point. The effective Lagrangian (with no external field present) has the form

$$\mathcal{L} = K - U, \tag{30}$$

$$K = \frac{1}{2M_{\vec{G}}}(\dot{\vec{G}})^2 + D_{\vec{G}}(\ddot{\vec{G}})^2, \tag{31}$$

$$U = \alpha \vec{G}^2 + \beta \vec{G}^4 + \gamma(\mathrm{rot}\,\vec{G})^2, \tag{32}$$

where $M_{\vec{G}}, D_{\vec{G}}, \alpha, \beta, \gamma > 0$ and the symmetry of the system above the transition point is considered cubic. In the kinetic energy (31) we have retained the term $(\ddot{\vec{G}})^2$, the necessity of which to be taken into account will be explained somewhat later. The interaction with an external electromagnetic field is by symmetry considerations, written in the form

$$\Delta \mathcal{L}_{\vec{A}} = -\frac{\lambda}{c}\dot{\vec{G}}\cdot \mathrm{rot}\,\dot{\vec{A}}, \tag{33}$$

where $\vec{A}(\vec{r},t)$ is the vector-potential, c is the light velocity, λ is a coefficient. Formula (33) may also be written as two equivalent expressions

$$\Delta \mathcal{L}_{\vec{B}} = -\frac{\lambda}{c}\dot{\vec{G}}\cdot\dot{\vec{B}}, \tag{34}$$

$$\Delta \mathcal{L}_{\vec{E}} = \lambda \dot{\vec{G}}\cdot \mathrm{rot}\,\vec{E}, \tag{35}$$

where $\vec{B} = \mathrm{rot}\,\vec{A}$, $\vec{E} = -\frac{1}{c}\dot{\vec{A}}$ and the Maxwell equation is used for the cure of the electric field $\vec{E}$:

$$\mathrm{rot}\,\vec{E} = -\frac{1}{c}\dot{\vec{B}}. \tag{36}$$

From (33) and (34) it is seen that the dynamical magnetic susceptibility $\Delta\chi(\omega)$ at low frequencies ω has the following correction

$$\Delta\chi(\omega) = -\left(\frac{\lambda}{c}\right)^2 \frac{M_{\vec{G}}\,\omega^2}{\omega^2 - \Omega^2_{\vec{G}}}, \tag{37}$$

where $\Omega^2_{\vec{G}} = 2M_{\vec{G}}\,\alpha$ is a normal mode of axial toroid oscillations. Vanishing $\Omega_{\vec{G}}$ corresponds to a second-order phase transition. Note that throughout we are speaking of the transverse axial toroid oscillations, since the longitudinal oscillations do not interact with electric and magnetic fields.

Axial toroid modes could, in principle, react to the current of magnetic charges (if they would exist) as polar toroid modes react to the usual electric current [14] .

Noteworthy is a nontrivial frequency dependence $\Delta\chi(\omega)$ (the numerator is proportional to ω^2 as in the case of polar toroid oscillations where there occurs an analogous to (37) anomaly in the dynamic dielectric susceptibility [24]). Expression (37) is valid only in a low-frequency region when the second term in (31) may be neglected. To find a correct asymptotics for $\omega \gg \Omega_{\vec{G}}$ it is necessary to take this term into account, and then, as it should be, the contribution of $\Delta\chi(\omega)$ vanishes at high frequencies. In the microscopic model (section 5) the second term gets essential for frequencies $\omega \sim E_g$, where E_g is a characteristic one-electron energy of an order of the band gap of a semiconductor.

Interesting effects may occur, in systems in which, together with the axial toroid ordering, there is realized another type of the magnetic long-range ordering. For instance, in the case of antiferromagnets with SDW the appearance of axial toroid order connected with SCDW results in a "weak" ferromagnetism /15/ .

In the case of antiferromagnet containing, in addition to itinerant electrons, localized moments, the axial toroid order also produces in the Lagrangian of the system the term responsible for the "weak" ferromagnetism of local moments:

$$\Delta\mathcal{L}_F = \xi G\left[\vec{L}\times\vec{M}\right], \tag{38}$$

where $\vec{L}$ is the vector of antiferromagnetism, $\vec{M}$ is the average magnetic moment, ξ is a coefficient. Note that in the model /15/ the whole effect is of a pure-exchange origin and has no relativistic smallness, whereas the conventional Dzyaloshinsky-Moriya mechanism of weak ferromagnetism is connected with the spin-orbital or magnetodipole interactions /15/ .

In the case of incommensurate structure of the axial toroid moments $\vec{G}$ below the phase-transition point there arises the inhomogeneous spontaneous polarization

$$\vec{P} = -\lambda\,\mathrm{rot}\,\vec{G}, \tag{39}$$

which directly follows from the representation of the term of interaction with electric field $\vec{E}$ by (35). At the phase transition point the static transverse dielectric susceptibility $\Delta\varepsilon_\perp(\vec{q})$:

$$\Delta\varepsilon_\perp(\vec{q}) = \frac{\lambda^2 q^2 4\pi M_G}{\Omega^2_{\vec{G}}(\vec{q})} \tag{40}$$

is divergent. On some wavevector $\vec{q}=\vec{q}_0$ defined from vanishing of the transverse normal modes $\Omega_{\vec{G}}(\vec{q}_0)\to 0, \Delta\varepsilon_\perp(\vec{q}_0)\to\infty$ (for $\vec{q}_0 \neq 0$).

Axial toroid excitations can interact with other collective excitations in crystals. Consider, e.g., a ferromagnet with local moments, in the ground state of which there is no axial toroid ordering. At the same time the collective toroid oscillations are

mixing with usual magnons since the effective Hamiltonian of the system contains terms of the type

$$\Delta\mathcal{V}_{eff}^{(\vec{G})} = -\vec{M}\,\vec{H}^{eff}(\vec{G}), \tag{41}$$

$$\vec{H}^{eff}(\vec{G}) = \lambda_1\dot{\vec{G}}, \tag{42}$$

where λ_1 is a proportionality coefficient, $\vec{M}$ is the magnetic moment. Let us write the Bloch equation, with account of (41) and (42), for small deviations $\vec{m}(\vec{r},t)$ of the magnetic moment $\vec{M}$ from equilibrium value $\vec{M}_0$:

$$\dot{\vec{m}} = \gamma_0\left[\vec{H}_{eff},\vec{M}\right], \tag{43}$$

$$\vec{M} = \vec{M}_0 + \vec{m}, \quad H^{(0)}_{eff} + H_{eff}(\vec{G}) = H_{eff}, \tag{44}$$

$$\vec{H}^{(0)}_{eff} = \alpha_{ik}\frac{\partial^2 m}{\partial x_i\partial x_k} + \vec{H}^{(0)}_{an}. \tag{45}$$

Here $\gamma_0 = \frac{g|e|}{2mc}$, g is the gyromagnetic ratio, $H^{(0)}_{an}$ - the contribution due to magnetic anisotropy which will not explicitly be written here (see ref. /12/). Setting $\hat{\alpha} = \alpha_{ik} n_i n_k$ ($\vec{n}$ is the unit vector directed along the wavevector $\vec{q}$, $\vec{m}\perp\vec{M}_0, g=2$) we get for Fourier components $\vec{m}_\omega$:

$$i\omega\vec{m}_\omega = \gamma_0\left\{\left[\vec{H}^{(0)}_{eff},\vec{M}_0\right]_\omega + i\omega\lambda_1\left[\vec{G}_\omega,\vec{M}_0\right],\right. \tag{46}$$

and for $\vec{G}_\omega$ we make use of the equation resulting from varying the effective Lagrangian (30) with account of (33) and (34):

$$\left(\frac{\omega^2}{2M_{\vec{G}}} - \alpha - \gamma q^2\right)\vec{G}_\omega + \frac{i\omega\lambda_1}{2}\vec{m}_\omega = 0. \tag{47}$$

The dispersion law of magnon-toroid oscillations is given by the equation ($\omega_{\vec{H}}(0)$ is the frequency of a ferromagnetic resonance):

$$\omega = \gamma_0\left[\omega_{\vec{H}}(0) + \hat{\alpha}q^2 + \frac{\lambda_1^2\omega^2}{2D_\omega}\right]M_0, \tag{48}$$

$$D_\omega = \frac{\omega^2}{2M_{\vec G}} - \alpha - \gamma q^2 . \qquad (49)$$

We shall not write down here the general solution to the cubic equation (48) because of very cumbersome expressions involved in the solution. Clearly, the mixing of toroid oscillations and magnons is maximal at quasimomenta $\vec q_0$ given by the approximate relation

$$\gamma_0 M_0 \left[\tilde\alpha q_0^2 + \omega_{\vec H}(0) \right] \approx \Omega_{\vec G}(\vec q_0), \qquad (50)$$
$$\Omega^2_{\vec G}(q_0) = \left(\alpha + \gamma q_0^2 \right) 2M_{\vec G} .$$

The mixing magnons with axial toroid oscillations may also occur in antiferromagnets for the corresponding parity of antiferromagnetic structure.

The microscopic realisation of this intertwinning is possible in the model [15], where the axial toroid mode is taken to be a SCDW oscillation, and the frequency $\Omega_{\vec G}$ can be close to that of the antiferromagnetic resonance. The generalization of formulae (46) - (49) to antiferromagnets is obvious.

Axial toroid oscillations interact with light in a rather peculiar manner; as a result, new branches of polaritons are produced. Indeed, the Lagrangian of the system in an electromagnetic field $\vec A(\vec r, t)$ has the form

$$\mathcal{L} = \mathcal{L}_{\vec G} + \frac{\lambda}{c} \operatorname{rot} \dot{\vec G} \cdot \vec A + \left[\frac{\varepsilon_\infty}{c^2} (\dot{\vec A})^2 - (\operatorname{rot} \vec A)^2 \right] \Big/ 8\pi \qquad (51)$$

Varying (51) with respect to $\vec G$ and $\vec A$ we arrive at the system of equations for the axial toroid moment and the Maxwell equations:

$$- \frac{1}{2M_{\vec G}} \ddot{\vec G} + \gamma \Delta \vec G - \frac{\lambda}{c} \operatorname{rot} \dot{\vec A} = 0, \qquad (52)$$
$$- \frac{\varepsilon_\infty}{c^2} \ddot{\vec A} - \operatorname{rot}\operatorname{rot} \vec A + \frac{4\pi\lambda}{c} \operatorname{rot} \dot{\vec G} = 0 .$$

For normal modes eqs. (51) and (52) yield:

$$\omega^2_{1,2} = \frac{q^2 \tilde c^2 + \varepsilon_\infty \Omega^2_{\vec G} \pm \left[(q^2 \tilde c^2 + \varepsilon_\infty \Omega^2_{\vec G})^2 - 4 \varepsilon_\infty \Omega^2_{\vec G} q^2 c^2 \right]^{1/2}}{2 \varepsilon_\infty}, \qquad (53)$$
$$\tilde c^2 = c^2 + 4\pi M_{\vec G} \lambda^2 .$$

When $q \to 0$, exp. (53) is simplified to

$$\omega_1^2 \to \Omega^2_{\vec G}, \quad \omega_2^2 \to q^2 c^2 / \varepsilon_\infty , \qquad (54)$$

whereas for $q^2 c^2 \gg \varepsilon_\infty \Omega^2_{\vec G}$ we get the asymptotics

$$\omega_1^2 \to \Omega^2_{\vec G} \frac{c^2}{\tilde c^2}, \qquad \omega_2^2 \to q^2 c^2 . \qquad (55)$$

Note that applicability of all the formulae (51)-(55) is limited to the range of low frequencies and small momenta (in the microscopic model, sect.5, $\omega, cq \ll E_g$ where E_g is a band gap of a semiconductor). If $\omega, cq \sim E_g$, then we should keep in Lagrangian the terms with higher-order derivatives of order parameter (analogously to the case of polar toroid oscillations [24]). The result in correct asymptotics for large energies and momenta is:

$$\omega_1^2 \to \Omega^2_{\vec G}, \quad \omega_2^2 = q^2 c^2 / \varepsilon_\infty . \qquad (56)$$

In the system with axial toroid ordering there may occur interesting nonlinear optical effects. In particular, below the transition point there occurs the anomalous contribution to components of the gyration tensor g_{ik}, that is proportional to the electric field $\vec E$:

$$g_{ik} = \gamma_{ikln} G_l E_n . \qquad (57)$$

Note that in the systems with polar toroid ordering there may also occur the anomalous contribution to the tensor g_{ik}, that is, however, proportional to the magnetic field $\vec H$:

(58)

Thus, the anomalous behaviour of electro-optical and magneto-optical characteristics of crystals may help in identifying toroid transitions.

5. Axial toroid moments in crystals with charge density waves

Consider a two-band model of a semiconductor or semimetal with direct extrema at point $\vec{K}_0$ of the Brillouin zone. Assume that the matrix element of interband dipole transition at point $\vec{K}_0$ equals zero (the wave functions of bands 1 and 2 have the same parity but belong to different irreducible representations of the group of wave vector $\vec{K}_0$), and the matrix element of interband transition over the orbital moment differs from zero. The model with that symmetry has been considered in ref. /26/ where it has been shown that in the case of the electron-hole pairing with an imaginary singlet parameter Δ_{Im} there arises the orbital ferromagnetism of electrons of a filled band. The model Hamiltonian will be written in the $\vec{k}\vec{P}$ approximation in an external electromagnetic field /27/ :

$$\hat{H} = \begin{vmatrix} \frac{1}{2m_1}\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}\right)^2 + \frac{E_g}{2} + e\phi & \eta^{12}_{\alpha\beta}\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}\right)_\alpha\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}\right)_\beta - \hat{\Delta}_{12} \\ \eta^{21}_{\alpha\beta}\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}\right)_\alpha\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}\right)_\beta - \hat{\Delta}_{21} & -\frac{1}{2m_2}\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}\right)^2 - \frac{E_g}{2} + e\phi \end{vmatrix}, \quad (59)$$

where m_1 and m_2 are effective masses of electrons and holes in zones 1 and 2, E_g is a band gap of a semiconductor (for semimetals $E_g < 0$); $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$ are the vector and scalar potentials of the electromagnetic field, $\hat{\Delta}_{ij}(\vec{r},t)$ is the order parameter describing an ordered state below the phase-transition. In the two-band "excitonic insulator" type models $\hat{\Delta}$ has generally the tensor structure

$$\hat{\Delta}_{ij} = \Delta^{s}_{ij}\,\hat{I} + \Delta^{t}_{ij}\,\hat{\vec{\sigma}}, \quad (60)$$

where $\hat{I}$ is the unit matrix, $\hat{\vec{\sigma}}$ is a vector composed of Pauli matrices. In what follows we consider the order parameter $\hat{\Delta}_{ij}$ to be singlet and real, which corresponds to the charge density wave state /28/ . It is assumed that the effective interaction constant is maximal in the case of transition into the state with CDW,

therefore the corresponding transition temperature (or the critical band gap E_g^* in a semiconductor model at $T = 0$) is also maximal, and the state with CDW is energetically the most favourable. The explicit form of effective interaction constant for all possible structures of order parameter $\hat{\Delta}_{ij}$ can be found, e.g., in /28/ .

The tensors $\eta^{ij}_{\alpha\beta}$ in the one-electron part of the Hamiltonian $\hat{H}$ have the form

$$\eta^{12}_{\alpha\beta} = \eta^{*21}_{\alpha\beta} = -\frac{1}{2m^2}\sum_{s\neq 1,2}\left[\frac{1}{E_1 - E_s} + \frac{1}{E_2 - E_s}\right]P^{\alpha}_{1s}P^{\beta}_{s2}, \quad (61)$$

where E_s is the energy of s- band at point $\vec{K}_0$, P_{is} is the matrix element of the momentum between a band i ($i = 1,2$) and a highest band $s \neq 1,2$. Further, the situation is considered when the tensor $\eta^{12}_{\alpha\beta}$ is purely real (this holds, e.g., when Bloch wave functions $\varphi_{nK_0}(\vec{r})$ at point $\vec{K}_0$ can be taken real).

The study of a system with Hamiltonian (59) is carried out by a usual Green functions method, and we shall not dwell upon the calculational technique (see a detailed exposition of the "excitonic insulator" model in /28/). Note only peculiarities due to the reaction to an external electromagnetic field since just these peculiarities allow us to understand the type of ordering arising in system below the phase-transition . To this end we write the effective Lagrangian describing the transition to the state with CDW at $T = 0$ for the model of semiconductor with a small band gap $E_g \simeq E_g^*$, where E_g^* is of an order of the exciton binding energy. Up to the terms of highest order in the parameter $\Delta^{s}_{Re}/E_g \ll 1$, $\left(\Delta^{s}_{Re} = \frac{1}{2}(\Delta^{s}_{12} + \Delta^{s}_{21})\right)$ in a weak and slowly changing transverse field $\vec{A}$, upon cumbersome calculations, we get

$$\mathcal{L}_{eff} = K - \mathcal{V}, \quad (62)$$

$$K = \bar{N}\left(\frac{1}{2M_{\vec{G}}}(\dot{\Delta}^{s}_{Re})^2 + D_{\vec{G}}(\ddot{\Delta}^{s}_{Re})^2\right), \quad (63)$$

$$\mathcal{V} = \bar{N}\left(\alpha(\Delta^{s}_{Re})^2 - \frac{\lambda}{c}(\Delta^{s}_{Re}\,\mathrm{rot}\,\dot{\vec{A}})^2\right), \quad (64)$$

$$M_{\vec{G}} = \gamma E_g^2 . \quad (65)$$

$$D_{\vec{G}} = \frac{5}{256\, E_g^4}\,, \tag{66}$$

$$\alpha = \ln \frac{E_g}{E_g^*}\,, \tag{67}$$

$$\lambda = \frac{e\pi\, \vec{\ell}_{12}}{4 E_g m}\,, \tag{68}$$

$$\vec{\ell}_{12} = -\frac{1}{4m} \sum_{S=1,2} \left(\frac{1}{E_1 - E_S} + \frac{1}{E_2 - E_S} \right) \left(\vec{P}_{1S} \times \vec{P}_{2S} \right) \tag{69}$$

$$\bar{N} = \frac{m^{*3/2}\, E_g^{1/2}}{2\pi^2}\,, \quad m^* = m_1 = m_2\,, \tag{70}$$

where m is electron mass.

Introducing the notation $\vec{G} = \vec{n}_{12} \Delta^3_{Re}$ where $\vec{n}_{12}$ is the unit vector directed along $\vec{\ell}_{12}$ we arrive at an expression analogous to the phenomenological Lagrangian (30) for uniaxial systems.

Thus, in the microsoopic model with Hamiltonian (59) below the CDW phase transition there arises a singlet axial toroid ordering. This allows us not only to illustrate the general phenomenological scheme considered above, but also to consider certain, more specific properties of the system. One of the most interesting is, in our opinion, the problem of influence of collective excitations in the system with Hamiltonian (59) on its optical and magneto-optical properties in a ordered phase. Amplitude excitations in the case of ground state with CDW are, in fact, axial toroid oscillations, whereas phase excitations are "magnons" since at small deviations from equilibrium:

$$\Delta^3_{12}(t) = |\Delta(t)| \exp(i\phi(t)) \approx |\Delta| (1 + i\phi)\,, \tag{71}$$

$$\Delta^3_{Re}(t) = \mathring{\Delta} + \delta |\Delta(t)|\,, \tag{72}$$

$$\Delta^3_{Im}(t) = \mathring{\Delta}\, \phi(t)\,, \tag{73}$$

$$\delta \vec{G}(t) \sim \vec{\ell}_{12}\, \delta |\Delta(t)|\,, \tag{74}$$

$$\delta \vec{M} \sim \vec{\ell}_{12}\, \mathring{\Delta}\, \phi(t)\,, \tag{75}$$

where $\vec{G}(t)$ is the density of the axial toroid moment, $\vec{M}(t)$ the same for the orbital magnetic moment. Amplitude (i.e., axial toroid) oscillations have normal modes $\omega_{\vec{G}} \approx 2\mathring{\Delta}$, and phase oscillations (i.e., orbital magnons) $\omega_M \approx \eta 2\mathring{\Delta}$, where η is a quantity proportional to the difference of effective constants for real and imaginary order parameters (on a phase fixation in "excitonic-insulator" type model see /28/). Both these oscillations give a resonance contribution to the dielectric susceptibility and magnetic susceptibility of the system on the corresponding frequencies.

An interesting situation may occur in the case of incommensurate structure of CDW (soliton lattice). In accordance with conclusions of sect.4, below the transition point in the system there arises a spontaneous inhomogeneous transverse polarization $\vec{P}_\perp(\vec{r}) \sim \operatorname{rot} \vec{G}(\vec{r})$. In a semimetalic model with Hamiltonian (59) (where $\varepsilon_F = -E_g/2$, ε_F is the Fermi-energy), in the region of incommensurate structure of CDW /28/ at $T \lesssim T_{\vec{G}}$, where $T_{\vec{G}}$ is the transition temperature, we have

$$\vec{P}_\perp(\vec{r}) = \xi \operatorname{rot} \left(\vec{\ell}_{12} \Delta^3_{Re} \right), \tag{76}$$

$$\xi = -\frac{e N(0)}{4 \varepsilon_F g_{Re}}\,, \quad N(0) = \frac{m P}{2\pi^2}\,, \tag{77}$$

$$\Delta^3_{Re}(\vec{r}) \cong \mathring{\Delta} \cos \vec{q}_0 \vec{r}\,, \tag{78}$$

and q_0 is the wave vector of superstructure (near the Lifshitz point $q_0 \to 0$). While lowering temperature, there may occur one more transition with the appearance of order parameter $\Delta^3_{Im}(\vec{r})$ in addition to $\Delta^3_{Re}(\vec{r})$ and the spatial distribution of $\Delta^3_{Im}(\vec{r})$ as compared to $\Delta^3_{Re}(\vec{r})$ is shifted by $\pi/2$ (for more detail see, e.g., /29/):

$$\Delta^3_{Im}(\vec{r}) \cong \Delta^3_{Im} \sin \vec{q}_0 \vec{r}\,. \tag{79}$$

As has been mentioned in ref. /26/, the appearance of $\Delta^{1}_{Im}(\vec{r})$ in a system with interband transition allowed with respect to the orbital momentum $(\vec{\ell}_{12} \neq 0)$ signifies the appearance of orbital magnetic ordering. The magnetic-moment density is:

$$\vec{M}(\vec{r}) \sim \vec{\ell}_{12}\, \Delta^{1}_{Im}(\vec{r}). \tag{80}$$

From (79) and (80), it follows that in the system with incommensurate structure of parameters Δ^{1}_{Re} and Δ^{1}_{Im} there arises a peculiar "magnetoelectric" ordering in the range of domain walls, CDW and $\vec{P} \perp \vec{M}$, $\vec{P} \perp \vec{q}_0$, $\vec{M} \perp \vec{q}_0$ if $\vec{q}_0 \perp \vec{\ell}_{12}$. If, however, $\vec{q}_0 \parallel \vec{\ell}_{12}$, then $\vec{P} = 0$, but $\vec{M}$ may be nonzero, and we arrive at the case of orbital long-wave ferromagnetism.

Thus, in the range of domain walls of incommensurate structures with axial toroid moments there may appear various types of electron ordering (ferroelectric, ferromagnetic or magnetoelectric). A detailed analysis of different structure types requires a special consideration and goes beyond the scope of the present paper.

6. Conclusion

Owing to the scheme of multipole expansion being universal one can in principle, also consider more complex structures than discussed above. Of a particular interest would be the study of dipole toroid media specifying by a set of elementary toroid dipoles $\{\vec{t}\}_i$. Despite being seemingly exotic, such a model may happen to be useful for studying phase transformations in a number of molecular crystals.

Generally speaking, using the above scheme one may also "construct" media of higher multipoles (this hierarchy of multipole distributions is considered in /30/).

In particular, these include systems with distributed fluxes of magnetic moments characterizing by the symmetric tensor $\Pi^{(1)}_{ij}(\vec{r},t)$ (see sect.3). However, the microscopic description and discussion of properties of such multipole media seem to us to be somewhat early.

In this paper we have only touched on the problem of longitudinal components of the quantities $\vec{M}_{\parallel}$, $\vec{P}_{\parallel}$ and $\vec{\Pi}_{\parallel}$. Without going into details we only point to a geometrical illustration of elementary distributions of these quantities (Fig.2a,b); a pair of magnetic or electric dipoles ($\vec{M}_{\parallel}$ or $\vec{P}_{\parallel}$) directed towards each other and a pair of spins precessing around a common axis but in opposite directions $(\vec{\Pi}_{\parallel})$.

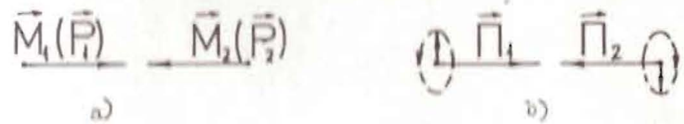

Fig.2

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Аксиальные тороидные моменты
в электродинамике и физике твердого тела

Найдено семейство мультипольных моментов /аксиальных тороидных моментов/, отличающихся по пространственно-временной симметрии от известных в электродинамике Максвелла-Лоренца. Рассмотрена его реализация в системе магнитных зарядов, в электрических дипольных средах, а также в средах с магнитными потоками. В рамках микроскопической модели системы с волной зарядовой плотности исследуется фазовый переход в кристалле с образованием аксиального тороидного момента. Обсуждаются некоторые интересные свойства аксиального тороидного состояния. В частности, мы указываем на существование необычных ветвей поляритонов в кристаллах с волной зарядовой плотности.

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Axial Toroid Moments
in Electrodynamics and Physics of Condensed Matter

A set of multipole moments (axial toroid moments) is found which differs in the properties under the space-time inversions from the ones known in the Maxwell-Lorentz electrodynamics. Its realization is considered in a system of magnetic charges, in electric dipole media, and in magnetic flux media. Within the microscopic model of a wave system, the axial toroid phase transition in a crystal is investigated. Some interesting properties of the axial toroid state are discussed. In particular, the existence of unusual polariton branches in crystals with charge-density wave has been pointed out.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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