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ON THE KINETICS OF DICKE-TYPE SYSTEMS

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1. INTRODUCTION

The experimental development of quantum electronics and nonlinear optics leads to a considerable increase of the attention to the theoretical investigation of two- and multilevel systems interacting with the resonant or quasiresonant electromagnetic field. The two- or multilevel emitters which form such a system can be of a diverse microscopic nature, e.g., molecular, atomic, or nuclear. But the general properties of macroscopic systems are mainly determined by the conditions of resonance between the field and some transition in the emitter, by the correlation conditions for the emitters, and not by a concrete physical nature of the emitters [1]. Under these circumstances one can consider some general model problems describing the interaction between an abstract two- or multilevel system of emitters and the field.

The simplest model problem of the type mentioned above goes back to the famous paper of Dicke [2] and is usually formulated in the following manner [1,8]. N two-level emitters in the volume $V_c = \prod_{\alpha} L_{\alpha}$, $\alpha = x, y, z$ form the "Matter" (M-subsystem), e.g., for a crystal radiating system with a simple cubic lattice $V_c = Na^3$, where a is a lattice parameter. Usually it is supposed that the wave length of the electromagnetic radiation $\lambda \gg a$, i.e., the optical or infrared radiation occurs. The physical effects connected with the condition were considered in a number of papers (see [4,8] and references given therein).

The two-level transitions in an f-th emitter are described by the operators $r_f^{\pm}$:

$$r_f^{\pm}|\pm>_f = 0; \quad r_f^{\pm}|\mp>_f = |\pm>; \quad r_f^{\pm} = \frac{1}{2}(\sigma_{fx} \pm i\sigma_{fy}).$$

Here $\sigma_{f\alpha}$ $(\alpha = x, y, z)$ are Pauli operators. The state $|+>_f$ corresponds to the excitation of the f-th emitter.

The electromagnetic field is quantized in a cubic box of a volume $V \gg V_c$.

Ignoring the electrostatic dipole interaction one can represent a Hamiltonian of the system under consideration in the following form:

$$H = H_F + H_M + H_{FM}. \qquad (1)$$

Here H_F is the free field energy $H_F = \sum_{\vec k} \hbar\omega_{\vec k} b_{\vec k}^{+} b_{\vec k}$, $\vec k$ is the wave-vector. The operator H_M describes free emitters (the "Matter")

in the dipole representation [1]

$$H_M = \sum_{f=1}^{N} \hbar\Omega_f r_{fz}; \quad r_{fz}|\pm>_f = \frac{1}{2}|\pm>_f; \quad r_{fz} = \frac{1}{2}\sigma_{fz},$$

where Ω_p is the transition frequency of an f-th emitter. The interaction between the field and matter is described by the operator

$$H_{FM} = \hbar N^{-1/2} \sum_f \sum_{\vec k} g_{\vec k} \{ b_{\vec k} e^{i\vec k \vec x_f}(r_f^{+} + \mu r_f^{-}) + b_{\vec k}^{+} e^{-i\vec k \vec x_f}(r_f^{-} + \mu r_f^{+}) \}.$$

Here $\vec x_f$ is a radius-vector of the f-th emitter; μ,- a real parameter, and

$$g_{\vec k} = \sqrt{\frac{2\pi\rho}{\hbar\omega_k}}\, \Omega\, (\vec d^{\,2} - (\hat k \vec d)^2)^{1/2},$$

where $\rho = N/V$ is the density of emitters in the system, $\vec d$ is the matrix element of the dipole moment operator, and $\hat k$ is the unit vector $\hat k = \vec k / |\vec k|$.

It should be noted that some effects due to the direct electrostatic dipolar coupling were examined in papers [5,6].

The equilibrium properties of the Hamiltonian (1) were rigorously studied in [3,7,8]. It has been established that the Hamiltonian (1) describes an equilibrium phase transition into a state with the spontaneous polarization of the ferroelectric type. For such a state the mean occupation number of a resonant mode is proportional to the specific polarization of matter in the second degree [8].

But the radiation process is nonequilibrium and thus it ought to be described in the framework of the kinetic theory. The derivation of an exact kinetic equation for the Dicke type system (1) is, however, confronted with considerable mathematical difficulties. Therefore such a derivation is usually accompanied by supplementary physical assumptions and simplifications.

As usual, the semiphenomenological Markoffian Master Equation is used to describe the spontaneous radiation process in a system with the Hamiltonian (1) [9]. The Non-Markoffian effects were considered in [10] within the Zwanzing method [11].

Recently, a new approach to the kinetics of the matter+boson field system has been developed in papers [12,13] in connection with the polaron problem. An appreciable analogy in the mathematical formulation of the Dicke-type model and of the polaron problem suggests an idea that the method mentioned above can be applied for the derivation of the exact kinetic equation for the system (1) [14].

In this paper we shall consider in detail the derivation of the exact kinetic equation for the systems (1). Some results for spontaneous - radiation processes will be obtained here on the basis of the mentioned equation.

2. THE HIERARCY FOR THE M-SUBSYSTEM

Nonequilibrium characteristics of the system (1) can be described with the aid of statistical operator D_t which obeys the Liouville equation

$$i\hbar\frac{\partial}{\partial t} D_t = [H, D_t]. \tag{2}$$

We shall consider initial conditions of the special form

$$D_{t_0} = \rho_M \otimes D_F,\quad D_F = \exp(-\beta H_F)/\mathrm{Tr}_{(F)}\exp(-\beta H_F),\quad \mathrm{Tr}_{(M)}\rho_M = 1. \tag{3}$$

Here t_0 is the initial point, ρ_M is the equilibrium statistical operator for an M-subsystem, and D_F the same for the field-subsystem. These conditions (3) describe the situation when interaction between the initially equilibrium subsystems started at a moment $t = t_0$. It is easily seen that $\mathrm{Tr}_{(F,M)} D_t = 1$. The equations of motions for the boson operators are

$$i\hbar\frac{\partial}{\partial t} b^+_{\vec k}(t) = -\hbar\omega_{\vec k} b^+_{\vec k}(t) - \hbar A_{\vec k}(t),\quad i\hbar\frac{\partial}{\partial t} b_{\vec k}(t) = \hbar\omega_{\vec k} b_{\vec k}(t) + \hbar A^+_{\vec k}(t),$$

where

$$A_{\vec k}(t) \equiv N^{-1/2}\sum_f g_{\vec k}\, e^{i\vec k\vec x_f}[r^+_f(t) + \mu r^-_f(t)].$$

Their formal solutions are

$$b^+_{\vec k}(t) = e^{i\omega_{\vec k}(t-t_0)} b^+_{\vec k}(t_0) + i\int_{t_0}^{t} e^{i\omega_{\vec k}(t-\tau)} A_{\vec k}(\tau)d\tau,\quad b_{\vec k}(t) = (b^+_{\vec k}(t))^+. \tag{4}$$

Let $O_t(M)$ be an operator of an M-subsystem in the Heisenberg representation. It means that $O(M)$ acts on the eigenfunctions of the Hamiltonian (1) only as on functions of the variables of the M-subsystem. The suitable equation of motion is

$$i\hbar\frac{\partial}{\partial t} O_t(M) = [O_t(M), H]. \tag{5}$$

Its right-hand side contains the terms

$$\hbar N^{-1/2}\sum_{\vec k}\sum_f g_{\vec k} b_{\vec k}(t)\, e^{i\vec k\vec x_f}\,[O_t(M), (r^+_{ft} + \mu r^-_{ft})],$$
$$\hbar N^{-1/2}\sum_{\vec k}\sum_f g_{\vec k} b^+_{\vec k}(t)\, e^{-i\vec k\vec x_f}\,[O_t(M), (r^-_{ft} + \mu r^+_{ft})].$$

Substituting here the formal expressions (4) instead of $b^+_{\vec k}(t)$, $b_{\vec k}(t)$ one can obtain

$$\hbar\sum_{\vec k} b^+_{\vec k}(t_0)\, e^{i\omega_{\vec k}(t-t_0)}[O_t(M), A^+_{\vec k}(t)] + i\hbar\sum_{\vec k}\int_{t_0}^{t} e^{i\omega_{\vec k}(t-\tau)} A_{\vec k}(\tau)[O_t(M), A^+_{\vec k}(t)]d\tau \tag{6}$$

and a conjugate expression follows immediately. Now we shall perform the following action in (5)

$$i\hbar\,\mathrm{Tr}_{(F,M)}\left(\frac{\partial}{\partial t} O_t(M)\right)D_{t_0} = \mathrm{Tr}_{(F,M)}\{[O_t(M), H]\}D_{t_0}.$$

Then using the expression (6) and the evident equalities

$$\mathrm{Tr}_{(F,M)}\{A(t)D_{t_0}\} = \mathrm{Tr}_{(F,M)}\{AD_t\},\quad [b_{\vec k}(t), O_t(M)] = [b^+_{\vec k}(t), O_t(M)] = 0$$

together with the Lemma of Bogolubov [12]

$$\mathrm{Tr}_{(F)}\{b_{\vec k}(t_0)\mathfrak{U}D(F)\} = (1 + N_{\vec k})\,\mathrm{Tr}_{(F)}\{[b_{\vec k}(t_0), \mathfrak{U}]D(F)\},$$
$$\mathrm{Tr}_{(F)}\{b^+_{\vec k}(t_0)\mathfrak{U}D(F)\} = -N_{\vec k}\,\mathrm{Tr}_{(F)}\{[b^+_{\vec k}(t_0), \mathfrak{U}]D(F)\},$$
$$N_{\vec k} \equiv e^{-\beta\hbar\omega_{\vec k}/2}\,/\,2\,\mathrm{sh}(\beta\hbar\omega_{\vec k}/2),$$

one can obtain the following equation

$$\mathrm{Tr}_{(M)}\left\{O(M)\frac{\partial\rho_{Mt}}{\partial t} - i\left[\sum_f \Omega_f r_{fz}, O(M)\right]\rho_{Mt}\right\} =$$
$$= \sum_{\vec k}\int_{t_0}^{t} d\tau\,\mathrm{Tr}_{(F,M)}\, e^{-i\omega_k(t-\tau)}\{N_k A^+_{\vec k}(\tau)[O_t(M), A_{\vec k}(t)] +$$
$$+ (1 + N_k)[A_{\vec k}(t), O_t(M)]A^+_{\vec k}(\tau)\}D_{t_0} + \tag{7}$$
$$+ \sum_{\vec k}\int_{t_0}^{t} d\tau\,\mathrm{Tr}_{(F,M)}\, e^{i\omega_k(t-\tau)}\{(1 + N_k)A_{\vec k}(\tau)[O_t(M), A^+_{\vec k}(t)] +$$
$$+ N_k[A^+_{\vec k}(t), O_t(M)]A_{\vec k}(\tau)\}D_{t_0}.$$

There is some hierarchy because the right-hand side contains the averages of a higher order that the left-hand side. This hierarchy is exact. It can be rewritten in terms of the collective variables [9, 14]. Let the vectors $\vec\nu$ correspond to the modes in the volume V_c, and let their number be equal to N: $\nu_\alpha = 2\pi n_\alpha/L_\alpha$; $\alpha = x, y, z$. Here n_α is an integer. Then the collective operators of the emitters are

$$R^\pm_{\vec\nu} = \sum_f r^\pm_f e^{\pm i\vec\nu\vec x_f},\quad R_{\vec\nu z} = \sum_f r_{fz} e^{i\vec\nu\vec x_f};\quad \Omega_\nu = N^{-1}\sum_f \Omega_f e^{-i\vec\nu\vec x_f}.$$

It is obvious that

$$N^{-1}\sum_f e^{i(\vec\nu-\vec\nu')\vec x_f} = \delta_{\nu\nu'},\quad N^{-1}\sum_{\vec\nu} e^{i\vec\nu(\vec x_f - \vec x_{f'})} = \delta_{ff'}.$$

Then from (7) one can obtain

$$\underset{(M)}{\mathrm{Tr}}\,\{ O(M)\frac{\partial \rho_{Mt}}{\partial t} - i[\sum_{\vec\nu}\Omega_\nu R_{\nu z},\, O(M)]\rho_{Mt}\} =$$

$$= \sum_{\vec k,\vec\nu,\vec\nu'}\int_{t_0}^{t} d\tau \underset{(F,M)}{\mathrm{Tr}}\; e^{-i\omega_k(t-\tau)}\{N_k U^{+}_{\vec\nu\vec k}(\tau)[O_t(M),\, U_{\vec\nu'\vec k}(t)] +$$

$$+ (1+N_k)[U_{\vec\nu\vec k}(t),\, O_t(M)]U^{+}_{\vec\nu'\vec k}(\tau)\}D_{t_0} + \qquad (8)$$

$$+ \sum_{\vec k,\vec\nu,\vec\nu'}\int_{t_0}^{t} d\tau \underset{(F,M)}{\mathrm{Tr}}\; e^{i\omega_k(t-\tau)}\{(1+N_k)U_{\vec\nu\vec k}(\tau)[O_t(M),\, U^{+}_{\vec\nu'\vec k}(t)] +$$

$$+ N_k[U^{+}_{\vec\nu\vec k}(t),\, O_t(M)]U_{\vec\nu'\vec k}(\tau)\}D_{t_0}.$$

Here

$$U_{\vec\nu\vec k}(t) \equiv N^{-1/2} g_k\{R^{+}_{\vec\nu}(t)\phi(\vec k-\vec\nu) + \mu R^{-}_{\vec\nu}(t)\phi(\vec k+\vec\nu)\},$$

$$\phi(\vec\Delta) \equiv N^{-1}\sum_f e^{i\vec\Delta\vec x_f}.$$

This is a collective form of our exact hierarchy.

We shall example below some special consequences from the equations (7) and (8).

3. THE SPONTANEOUS RADIATION

Let us consider the special initial condition

$$D_F = |0\rangle\langle 0|, \qquad (9)$$

i.e., at a time t_0 the radiation is absent in the system. So, the rise of the field for $t > t_0$ can be interpreted as a spontaneous radiation. Bearing in mind the condition (9) we obtain from the equation (9) that

$$\underset{(M)}{\mathrm{Tr}}\,\{O(M)\frac{\partial\rho_{Mt}}{\partial t} - i[\sum_{\vec\nu}\Omega_{\vec\nu}R_{\vec\nu z},\, O(M)]\rho_{Mt}\} =$$

$$= \sum_{\vec k,\vec\nu,\vec\nu'}\int_{t_0}^{t} d\tau \underset{(F,M)}{\mathrm{Tr}}\,\{e^{-i\omega_k(t-\tau)}[U_{\vec\nu\vec k}(t),\, O_t(M)]U^{+}_{\vec\nu'\vec k}(\tau) + \qquad (10)$$

$$+ e^{i\omega_k(t-\tau)}U_{\vec\nu\vec k}(\tau)[O_t(M),\, U^{+}_{\vec\nu'\vec k}(t)]\}D_{t_0}.$$

Here $D_{t_0} = \rho_M \otimes (|0\rangle\langle 0|)$.

For simplicity we shall consider below the special case of inhomogeneous Lorentz broadening /1/. In other words, we shall propose that the frequency distribution for the emitters is

symmetric with a central frequency Ω. Such a distribution is independent of the positions of emitters $\vec x_f$. Then the average with respect to the above distribution is $\overline{\exp(i\Omega_f t)} = e^{i\Omega t}\, e^{-|t|/2T}$, where the parameter T characterises the width of the distribution. Below we shall ignore high-frequency oscillating terms of the type of R^+R^+ and R^-R^- in the equation (10).

The right-hand side of (10) contains the variables that depend on different times τ and t. But if the coupling between the field and the matter is small, from the definition we obtain

$$R^{\pm}_{\vec\nu}(\tau) \equiv \sum_f r^{\pm}_f(\tau)e^{\pm i\vec\nu\vec x_f} \cong \sum_f r^{\pm}_f(t)\, e^{\mp i\Omega_f(t-\tau)}\, e^{\pm i\vec\nu\vec x_f} =$$

$$= R^{\pm}_{\vec\nu}(t)\, e^{\mp i\Omega(t-\tau)}\, e^{-(t-\tau)/2T}.$$

So, the integration in the right-hand side of (10) can be performed. For simplicity we put here $\mu = 1$. It should be also noted that the function $\phi(\vec k-\vec\nu)$ has a sharp maximum at $\vec k = \vec\nu$. Therefore, the function

$$N^{-1}\sum_{\vec k} g^2_{\vec k}\phi^*(\vec k-\vec\nu)\phi(\vec k-\vec\nu')\, e^{-i\omega_k(t-\tau)}$$

also has a sharp maximum at $\vec\nu = \vec\nu'$. Thus, we obtain from (10)

$$\underset{(M)}{\mathrm{Tr}}\,\{O(M)\frac{\partial\rho_{Mt}}{\partial t} - i\sum_{\vec\nu}\Omega_\nu[R_{\vec\nu z},\, O(M)]\rho_{Mt}\} =$$

$$= \frac{1}{2}\sum_{\vec\nu}\underset{(F,M)}{\mathrm{Tr}}\,\{\Gamma^-_{\vec\nu}[R^+_{\vec\nu t},\, O_t(M)]R^-_{\vec\nu t} + \Gamma^+_{\vec\nu}[R^-_{\vec\nu t},\, O_t(M)]R^+_{\vec\nu t} + \qquad (11)$$

$$+ \Gamma^{-*}_{\vec\nu}R^+_{\vec\nu t}[O_t(M),\, R^-_{\vec\nu t}] + \Gamma^{+*}_{\vec\nu}R^-_{\vec\nu t}[O_t(M),\, R^+_{\vec\nu t}]\}D_{t_0}.$$

Here

$$\Gamma^{\mp}_{\vec\nu} = \gamma^{\mp}_{\vec\nu} - iQ^{\mp}_{\vec\nu}, \quad \gamma^{\mp}_{\vec\nu} = \frac{4T}{(2\pi)^3\rho}\int d^3\vec k\, \frac{g^2_{\vec k}|\phi(\vec k-\vec\nu)|^2}{1+4T^2(\omega_k\mp\Omega)^2},$$

$$Q^{\mp}_{\vec\nu} = \frac{8T^2}{(2\pi)^3\rho}\int d^3\vec k\, g^2_{\vec k}|\phi(\vec k-\vec\nu)|^2\,\frac{\omega_k\mp\Omega}{1+4T^2(\omega_k\mp\Omega)^2}.$$

The parameter $\Omega' = \frac{1}{2}\sum_{\vec\nu}(Q^+_{\vec\nu} - Q^-_{\vec\nu})$ characterises the so-called Bethe part of the Lamb-shift /9/, whereas $\tilde\Omega_{\vec\nu} = Q^+_{\vec\nu} + Q^-_{\vec\nu}$ defines the collective shift of frequences.

It should be noted that in the limit $T\to\infty$ equation (11) develops into the Markoffian equation of paper /9/ obtained in the weak coupling approximation. The equation (11) describes the spontaneous radiation. The same result for a simple one-mode case was obtained in /16/.

4. THE EQUATION FOR $\langle R_z(t)\rangle$

The radiation intensity is defined by the derivative $d/dt\cdot\langle R_z(t)\rangle$. Here $\langle R(t)\rangle \equiv \underset{(F,M)}{\mathrm{Tr}}\, R(t)\, D_{t_0}$. One can obtain $\langle R_z(t)\rangle$ from the hierarchy (8) putting $O_t(M) = R_z(t)$. Then [16]

$$\frac{d}{dt}\langle R_z(t)\rangle = \sum_{\vec\nu} \{\gamma^+_{\vec\nu}\langle R^-_{\vec\nu}(t)\, R^+_{\vec\nu}(t)\rangle - \gamma^-_{\vec\nu}\langle R^+_{\vec\nu}(t)\, R^-_{\vec\nu}(t)\rangle\}.$$

It can be rewritten in the form

$$\frac{d}{dt}\langle R_z(t)\rangle = \sum_{\vec\nu} \{(\gamma^+_{\vec\nu} - \gamma^-_{\vec\nu})\langle R^+_{\vec\nu}(t)\, R^-_{\vec\nu}(t) - 2\gamma^+_{\vec\nu}\langle R_z(t)\rangle\}. \qquad (12)$$

Let us introduce some new operator $S_{\vec\nu}$ by the expression $R^+_{\vec\nu}R^-_{\vec\nu} \equiv 1/2\cdot N + R_z + S_{\vec\nu}$. It is obvious that

$$S_{\vec\nu} \equiv \sum_{\substack{f,f' \\ (f\neq f')}} r^+_f\, r^-_{f'}\, e^{i\vec\nu(\vec x_f - \vec x_{f'})}.$$

Then we obtain instead of (12) that

$$\frac{d}{dt}\langle R_z(t)\rangle = \sum_{\vec\nu} \{\frac{\gamma^+_{\vec\nu} - \gamma^-_{\vec\nu}}{2} N - (\gamma^+_{\vec\nu} + \gamma^-_{\vec\nu})\langle R_z(t)\rangle + (\gamma^+_{\vec\nu} - \gamma^-_{\vec\nu})\langle S_{\vec\nu}(t)\rangle\}. \qquad (13)$$

To solve this equation, the initial state of the M-subsystem should be defined. Here we consider a special case of an initial excitation of the emitters which can be prepared by a quasi-monochromatic impulse with the wave-vector $\vec k_0$ and area equal to π. Then

$$\rho_M = \prod_{f=1}^{N} |\Theta, \psi - \vec k_0 \vec x_f\rangle_f \langle \psi - \vec k_0 \vec x_f, \Theta|_f. \qquad (14)$$

Here Θ is the so-called Bloch angle, the parameter ψ defines the phase of the initial state, and

$$|\Theta, \psi - \vec k_0 \vec x_f\rangle_f = |+\rangle_f\, e^{-\frac{i}{2}(\psi - \vec k_0 \vec x_f)} \sin\frac{\Theta}{2} + |-\rangle_f\, e^{\frac{i}{2}(\psi - \vec k_0 \vec x_f)} \cos\frac{\Theta}{2}.$$

Now we ought to define the action of the emitter operators on the functions of the initial state:

$$r^-_f|\Theta, \psi - \vec k_0 \vec x_f\rangle_f = |-\rangle_f\, e^{-\frac{i}{2}(\psi - \vec k_0 \vec x_f)} \sin\frac{\Theta}{2},$$

$$r^+_f|\Theta, \psi - \vec k_0 \vec x_f\rangle_f = |+\rangle_f\, e^{\frac{i}{2}(\psi - \vec k_0 \vec x_f)} \cos\frac{\Theta}{2}.$$

Then the average $\langle r^+_f r^-_{f'}\rangle$ is $\langle r^+_f r^-_{f'}\rangle = 1/4\cdot \sin^2\Theta\, e^{-ik_0(\vec x_f - \vec x_{f'})}$; $f \neq f'$. It is also obvious that $\langle R_z\rangle = -\frac{N}{2}\cos\Theta$. Then in compliance with the definition of S_ν we have

$$\langle S_{\vec\nu}\rangle = \sum_{\substack{f,f' \\ (f\neq f')}} \{\frac{1}{4} - N^{-2}\langle R_z\rangle^2\} e^{i(\vec\nu - \vec k_0)(\vec x_f - \vec x_{f'})}.$$

This expression corresponds to the initial time t_0. In the so-called zero approximation [18,16] we obtain

$$r^\pm_f(t) \simeq r^\pm_f\, e^{\pm i\Omega_f t}; \qquad r_{fz}(t) \simeq r_{fz} \quad t > t_0.$$

Taking into account the inhomogeneous Lorentz broadening we now get

$$\langle r^+_f(t)\, r^-_{f'}(t)\rangle = \langle r^+_f r^-_{f'}\rangle\, e^{-|t|/T}; \qquad f \neq f'.$$

Then the last term in the right-hand side of (13) can be transformed in the following way

$$N^{-2}\langle S_{\vec\nu}(t)\rangle = N^{-2}\sum_{\substack{f,f' \\ (f\neq f')}} e^{i(\vec\nu - \vec k_0)(\vec x_f - \vec x_{f'})}\left(\frac{1}{4} - N^{-2}\langle R_z(t)\rangle^2\right)e^{-|t|/T} =$$
$$= [\Lambda(\vec\nu - \vec k_0) - N^{-1}][\frac{1}{4} - N^{-2}\langle R_z(t)\rangle^2]\, e^{-|t|/T}.$$

Here

$$\Lambda(\vec\nu - \vec k_0) = |N^{-1}\sum_f e^{i(\vec\nu - \vec k_0)\vec x_f}|^2.$$

Let us introduce the following notation

$$\sum_{\vec\nu}(\gamma^+_{\vec\nu} + \gamma^-_{\vec\nu}) \equiv \tau^{-1},\ \sum_{\vec\nu}(\gamma^-_{\vec\nu} - \gamma^+_{\vec\nu}) \equiv a\tau^{-1},\ \sum_{\vec\nu}(\gamma^-_{\vec\nu} - \gamma^+_{\vec\nu})[\Lambda(\vec\nu - \vec k_0) - N^{-1}] \equiv \kappa\tau^{-1}.$$

Then the equation (13) takes the form

$$\frac{d}{dt}\langle R_z(t)\rangle = -\frac{1}{\tau}\{\frac{aN}{2} + \langle R_z(t)\rangle + \kappa[\frac{N^2}{4} - \langle R_z(t)\rangle^2]e^{-t/T}\}. \qquad (15)$$

This equation describes the spontaneous radiation in our system with the initial conditions (3), (14). One can see easily that the function $\Lambda(\vec\nu - \vec k_0)$ has the maximum at $\vec\nu = \vec k_0$. So, in the isotropic system of the emitters there is the priority direction for the spontaneous radiation which coincides with the direction of the vector $\vec k_0$.

In the case of $T \to \infty$ we obtain

$$\gamma^{+}_{\vec\nu} = 0, \quad \gamma^{-}_{\vec\nu} = \frac{1}{(2\pi)^2 \rho} \int d^3\vec{k}\, g^2_{\vec k} |\phi(\vec k - \vec\nu)|^2 \delta(\omega_{\vec k} - \Omega).$$

So $a = 1$ and the equation (15) transforms into the following form

$$\frac{d}{dt}\langle R_z(t)\rangle = -\frac{1}{r'}\{\frac{N}{2} + \langle R_z(t)\rangle + \kappa'[\frac{N^2}{4} - \langle R_z(t)\rangle^2]\}, \qquad (16)$$

where

$$\frac{1}{r'} = \sum_{\vec\nu} \gamma^{-}_{\vec\nu}, \quad \kappa' = r' \sum_{\vec\nu} \gamma^{-}_{\vec\nu} [\Lambda(\vec\nu - \vec k_0) - N^{-1}].$$

Equation (16) coincides with the known result of Rehler and Eberly /17/ obtained on the basis of the semiphenomenological approach.

5. THE RADIATION INTENSITY

The evolution of the radiation intensity is defined by the expression $I(t) = \hbar\Omega \cdot d/dt \cdot \langle R_z(t)\rangle$. Here $\langle R_z(t)\rangle$ is a solution of the equation (15). We introduce the following notation $X(t) = \langle R_z(t)\rangle$ and put here that $X(t) = Y(t)\, e^{t/T}$. Then from (15) for the variable $Y(t)$ we obtain the following equation

$$\frac{d}{dt}Y(t) = \frac{\kappa}{r}[Y(t) - \frac{r}{2\kappa}(\frac{1}{r} + \frac{1}{T})]^2 - \frac{r}{4\kappa}(\frac{1}{r} + \frac{1}{T})^2 - \frac{Na}{2r} e^{-t/T} - \frac{\kappa N^2}{4r} e^{-2t/T}.$$

By the substitution $\xi = Y - r/2\kappa\,(1/r + 1/T)$ it transforms into the general Riccati equation

$$\frac{d}{dt}\xi(t) = \frac{\kappa}{r}\xi^2 - a - bNe^{-t/T} - cN^2 e^{-2t/T}, \qquad (17)$$

where

$$a = \frac{r}{4\kappa}(\frac{1}{r} + \frac{1}{T})^2, \quad b = \frac{a}{2r}, \quad c = \frac{\kappa}{4r}.$$

This equation is not integrable in quadratures. So, we consider a special case of large T, when $e^{-t/T} \simeq 1 - t/T$. Then from (16) we get

$$\frac{d}{dt}u(t) = u^2(t) + t\frac{D}{T} - E, \quad u = \frac{\kappa}{r}\xi, \quad D = \frac{a\kappa}{2r^2}N + \frac{\kappa^2}{2r^2}N^2,$$

$$E = \frac{1}{4}(\frac{1}{r} + \frac{1}{T})^2 + \frac{2a\kappa N + \kappa^2 N^2}{4r^2}.$$

This is the special form of the Riccati equation. Its standard solution is

$$\langle R_z(t)\rangle = \frac{r}{\kappa} e^{t/T} \{\frac{1}{2}(\frac{1}{r} + \frac{1}{T}) - \frac{P'(t)}{P(t)}\}, \qquad (18)$$

where

$$P(t) = (\frac{T}{D})^{1/3} \sqrt{\frac{t}{T}D - E}\; Z_{1/3}(\frac{2}{3}\frac{T}{D}[\frac{t}{T}D - E]^{3/2}).$$

Here

$$Z_{1/3}(\chi) = C_1 J_{1/3}(\chi) + C_2 N_{1/3}(\chi)$$

and $J_a(\chi)$, $N_a(\chi)$ are the Bessel functions of the first and second order, respectively. The parameters C_1, C_2 can be obtained from the initial conditions. Let us now consider three special cases.

1. $t \ll T$. Then $e^{-t/T} \simeq 1$, and the equation (15) takes the form

$$\frac{d}{dt}X(t) = -L^2 - \frac{X}{r} + \frac{\kappa}{r}X^2, \quad L = \sqrt{\frac{2aN + \kappa N^2}{4r}}.$$

Its solution is

$$X(t) = \frac{1}{2\kappa} - \Delta\, \mathrm{th}\frac{\kappa\Delta}{r}(t - t_m), \qquad (19)$$

where

$$\Delta = \frac{1}{2\kappa}\sqrt{1 + 2a\kappa N + \kappa^2 N^2}.$$

The time t_m corresponds to the maximum of the radiation intensity $I(t)$

$$I(t) = \frac{\hbar\Omega\Delta}{2r_N}\,\mathrm{sech}^2\frac{t - t_m}{2r_N}, \qquad (20)$$

where $r_N = r/2\kappa\Delta$ (see fig.1). For the special case of $T \to \infty$ we have $\Delta \to N/2 + 1/2\kappa$; $r_N \to r/(\kappa N + 1)$ and

$$I(t) = \frac{\hbar\Omega}{4\kappa r}(N\kappa + 1)^2\,\mathrm{sech}^2\frac{t - t_m}{2r_N}, \quad \text{where} \quad t_m = \frac{r}{N\kappa + 1}\ln N\kappa,$$

when $\langle R_z(0)\rangle = N/2$. This particular result can be also obtained on the basis of the Rehler-Eberly equation (16) /17/.

The expression (20) describes both the normal radiation and the superradiation. The latter is connected with the maximum of the radiation intensity at $t = t_m$, when $I(t_m) \sim N^2$.

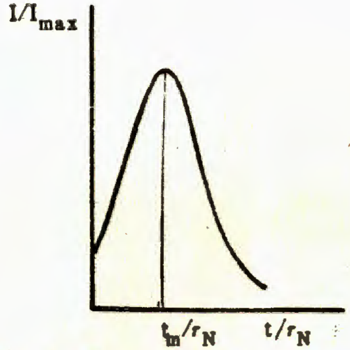

Fig.1. Temporal behaviour of the radiated intensity obtained from eqs. (20) (for $t \ll T$).

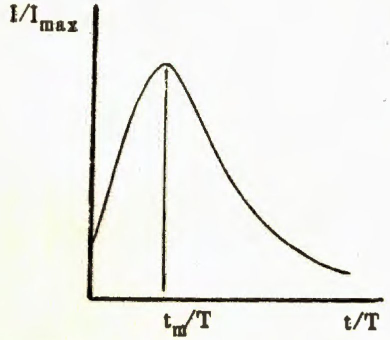

Fig.2. Temporal behaviour of the radiated intensity obtained from eqs. (26) (for $t \sim T$).

2. $t \sim T$. In this case the factor $e^{-t/T}$ in (15) cannot be ignored. But if for so large $t \sim T$ the superradiation takes preference over the normal radiation, then from (15) we get

$$\langle R_z(t)\rangle = -\frac{N}{2}\tanh\left(\frac{\kappa NT}{2\tau}e^{-t/T}-\delta\right).$$

Here the parameter δ is defined by the fuse condition for the expression (20) and

$$I(t) = N^2\hbar\Omega\frac{\kappa}{4\tau}e^{-t/T}\,\mathrm{sech}^2\left(\frac{\kappa NT}{2\tau}e^{-t/T}-\delta\right) \qquad (21)$$

for $t \sim T$. Contrary to the case (20) the radiation intensity (21), has an asymmetric peak of the superradiation (fig.2).

3. $t \gg T$. Then $e^{-t/T} \to 0$ and the equation (15) takes the following form

$$\frac{d}{dt}\langle R_z(t)\rangle = -\frac{N\alpha}{2\tau} - \frac{1}{\tau}\langle R_z(t)\rangle .$$

Its solution is

$$\langle R_z(t)\rangle = -\frac{N\alpha}{2} + N\zeta e^{-t/\tau} ,$$

where the constant ζ is defined by the initial conditions. So, the intensity is

$$I(t) = \frac{\hbar\Omega}{\tau}N\zeta e^{-t/\tau} .$$

It corresponds to the exponential fading of the normal radiance as $t\to\infty$. It is obvious that $\langle R_z(t)\rangle \to -N\alpha/2 \geq -N/2$, when $t\to\infty$. For

the system withought inhomogeneous Lorentz broadening $\alpha = 1$ and $\lim_{t\gg T}\langle R_z(t)\rangle = -N/2$. It means that the whole energy of the initial excitations transforms into the radiation. For finite values of the broadening parameter T: $\alpha<1$ and so $\lim_{t\gg T}\langle R_z(t)\rangle = -N\alpha/2 > -N/2$. Hence some portion of the initial energy is conserved in the system. Such a residual energy permits one to explain the photon echo phenomenon [1].

6. CONCLUSIONS

Thus, it is shown that the kinetics of the Dicke-type system (1) can be rigorously described with the aid of the Bogolubov method [12,13]. This approach leads to the exact hierarchy (7) or (8). The processes of the normal radiation and superradiation can be examined on the basis of this hierarchy. In this way we have obtained here some generalizations for the known semi-phenomenological results connected with the spontaneous radiation. For instance, the expression (18) was established here. This expression gives us the general form of the radiation intensity. The known result of Rehler and Eberly can be obtained from (18) in some particular case.

It should be emphasized that only the simplest consequences from the exact hierarchy (7) or (8) were examined in the present paper. A more detailed investigation is a subject of subsequent pepers.

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О кинетике систем типа Дикке

Получена точная иерархия для модельных проблем типа Дикке. На ее основе исследован процесс спонтанного излучения.

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On the Kinetics of Dicke-Type Systems

An exact hierarchy for the Dicke-type model problems is obtained. On its basis spontaneous-radiation processes are examined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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