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PROOF OF THE INVARIANCE OF THE BETHE-ANSATZ SOLUTIONS UNDER COMPLEX CONJUGATION

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1. The Bethe-ansatz method [1-5] allows us to reduce an exact treatment of certain models in one-dimensional mathematical physics to an analysis of the appropriate system of coupled algebraic equations. The simplest (but typical) example of the Bethe ansatz equations (BAE) is

$$\left(\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}\right)^N = -\prod_{k=1}^{\ell} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}, \quad j = 1, \dots, \ell;\ \ell \le \frac{N}{2}. \tag{1}$$

A solution set $\{\lambda_j\}$ is assumed to consist of distinct complex numbers (otherwise, it is not relevant to Bethe's ansatz).

If $\{\lambda_j\}$ is a solution of (1), then, obviously, its complex conjugate $\{\bar{\lambda}_j\}$ is also a solution. The aim of this paper is to prove a less trivial statement: $\{\bar{\lambda}_j\} = \{\lambda_j\}$. The full proof of this significant property of the BAE solutions appears to be missing in the literature. Nevertheless, the equality $\{\bar{\lambda}_j\} = \{\lambda_j\}$ has the status of a highly plausible hypothesis and serves as a basis in any further treatment of BAE. For the special class of solutions (when $\{\lambda_j\}$ must contain a sea of real roots) the coincidence of $\{\lambda_j\}$ and $\{\bar{\lambda}_j\}$ has been shown in [6].

2. In the present paper we prove $\{\bar{\lambda}_j\} = \{\lambda_j\}$ for all solutions of eqs.(1) and some other BAE. When doing this we rely very heavily on the prehistory of BAE in the context of algebraic Bethe's ansatz [2,3]. Therefore, we need here some relations from the quantum inverse scattering formalism.

The underlying physical model in the case of eqs.(1) is known as the one-dimensional isotropic Heisenberg magnet or else XXX model of spin 1/2. Its Hamiltonian

$$H = -\frac{1}{2} \sum_{n=1}^{N} \left(\sum_{a=1}^{3} \sigma_n^a \sigma_{n+1}^a - 1\right), \quad \sigma_{N+1}^a = \sigma_1^a$$

describes N interacting spins 1/2 on a periodic chain. The Pauli matrices σ_n^a act in the quantum space $V_n = \mathbf{C}^2$ associated with the site n. We introduce the 4x4 matrix

$$L_n(\lambda) = \lambda + \frac{i}{2} \sum_{a=1}^{3} \sigma_0^a \otimes \sigma_n^a$$

which acts in $V_0 \otimes V_n$ ($V_0 \simeq \mathbf{C}^2$ is an auxiliary space), and the transfer matrix $\tau(\lambda) = \mathrm{tr}\,(L_N(\lambda)\dots L_1(\lambda))$ acting in $W = \prod_{n=1}^{N} \otimes V_n$. Each solution $\{\lambda_j\}$ of (1) defines a vector $|\{\lambda_j\}\rangle$ in W which is an eigenvector of $\tau(\lambda)$ for all complex λ,

$$\tau(\lambda)|\{\lambda_j\}\rangle = \Lambda(\lambda, \{\lambda_j\})|\{\lambda_j\}\rangle \qquad (2)$$

with the eigenvalue

$$\Lambda(\lambda, \{\lambda_j\}) = (\lambda - \frac{i}{2})^N \prod_{j=1}^{\ell} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} + (\lambda + \frac{i}{2})^N \prod_{j=1}^{\ell} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j}. \qquad (3)$$

Vectors $|\{\lambda_j\}\rangle$ are the eigenvectors of H as well due to

$$H = -i \frac{d}{d\lambda} \ln \tau(\lambda)\Big|_{\lambda = \frac{i}{2}} + N. \qquad (4)$$

The precise definition of $|\{\lambda_j\}\rangle$ and the derivation of eqs.(2)-(4) can be found in refs.[2,3]

To obtain[2] the Hermitian conjugate $\tau^+(\lambda)$ of the transfer matrix, we must conjugate $L_n(\lambda)$ only in the quantum space V_n. It means complex conjugation of $L_n(\lambda)$ as a whole and taking a transpose of σ_n^a. In the explicit notation

$$L_n^+(\lambda) = \bar{\lambda} - \frac{i}{2}(\sigma_0^1 \otimes \sigma_n^1 - \sigma_0^2 \otimes \sigma_n^2 + \sigma_0^3 \otimes \sigma_n^3)$$

it is easy to observe that

$$L_n^+(\lambda) = \sigma_0^2 L_n(\bar{\lambda})\, \sigma_0^2 \qquad (5)$$

whence it follows that $\tau^+(\lambda) = \tau(\bar{\lambda})$ and

$$\bar{\Lambda}(\lambda, \{\lambda_j\}) = \Lambda(\bar{\lambda}, \{\lambda_j\}) \qquad (6)$$

for $\{\lambda_j\}$ obeying (1) and λ arbitrary.

3. Eqs.(3) and (6) suffice for proving $\{\bar{\lambda}_j\} = \{\lambda_j\}$. Substituting (3) into (6), replacing then $\bar{\lambda}$ by λ and getting rid of denominators give

$$(\lambda - \frac{i}{2})^N [\prod_{j=1}^{\ell} (\lambda - \bar{\lambda}_j + i)(\lambda - \lambda_j) - \prod_{j=1}^{\ell} (\lambda - \lambda_j + i)(\lambda - \bar{\lambda}_j)] =$$

$$= (\lambda + \frac{i}{2})^N [\prod_{j=1}^{\ell} (\lambda - \lambda_j - i)(\lambda - \bar{\lambda}_j) - \prod_{j=1}^{\ell} (\lambda - \bar{\lambda}_j - i)(\lambda - \lambda_j)] \qquad (7)$$

or

$$(\lambda - \frac{i}{2})^N f(\lambda) = (\lambda + \frac{i}{2})^N f(\lambda - i), \qquad (8)$$

where $f(\lambda)$ is $[\dots]$ in the l.h.s. of (7).

From the condition $\ell \le N/2$ in (1) it follows that $f(\lambda)$ is a polynomial in λ with the degree less than N. Therefore, $f(\lambda)$ cannot contain a factor $(\lambda + i/2)^N$ as it is prescribed by (8). We conclude that $f(\lambda) = 0$, i.e., for any λ

$$\prod_{j=1}^{\ell} (\lambda - \bar{\lambda}_j + i)(\lambda - \lambda_j) = \prod_{j=1}^{\ell} (\lambda - \lambda_j + i)(\lambda - \bar{\lambda}_j). \qquad (9)$$

The remainder of the proof follows ref.[6]. If the set $\{\lambda_j\}$ includes real or complex conjugate roots, then the corresponding factors in (9) cancel. The other members of $\{\lambda_j\}$ form a subset $\{\lambda_m\} \subset \{\lambda_j\}$ such that $\{\lambda_m\} \cap \{\bar{\lambda}_m\} = \emptyset$. Let $m = 1, \dots, \ell'$, $0 \le \ell' \le \ell$. Then

$$\prod_{m=1}^{\ell'} (\lambda - \bar{\lambda}_m + i)(\lambda - \lambda_m) = \prod_{m=1}^{\ell'} (\lambda - \lambda_m + i)(\lambda - \bar{\lambda}_m). \qquad (10)$$

Now choose $\lambda_p \in \{\lambda_m\}$ such that $\mathrm{Im}\,\lambda_p \ge \mathrm{Im}\,\lambda_m$, $m = 1, \dots, \ell'$. The factor $(\lambda - \lambda_p)$ in the l.h.s. of (10) must have its counterpart in the r.h.s. But our assumptions prevent the occurrence of such a factor in the r.h.s. of (10). The only way out is $\ell' = 0$. We have shown that $\{\bar{\lambda}_j\} = \{\lambda_j\}$.

As an immediate consequence, one obtains that the eigenvalues i_k of the local integrals of motion[7,8] I_k are real:

$$I_k = i(-)^{k+1} \frac{d^k}{d\lambda^k} \ln \tau(\lambda)\Big|_{\lambda = \frac{i}{2}} - N i^{-(k-1)}(k-1)!,$$

$$I_k |\{\lambda_j\}\rangle = i_k |\{\lambda_j\}\rangle, \quad i_k = -i \sum_{j=1}^{\ell} \frac{d^k}{d\lambda_j^k} \ln \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} = \bar{i}_k. \qquad (11)$$

4. The XXX model admits an integrable generalization to arbitrary spin s [9,10]. The L-matrix takes the form

$$L_n(\lambda) = \lambda + i \sum_{a=1}^{3} \sigma_0^a \otimes S_n^a,$$

where S_n^a are generators in the spin-s representation of SU(2). The BAE look now like

$$\left(\frac{\lambda_j - is}{\lambda_j + is}\right)^N = - \prod_{k=1}^{\ell} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}, \quad j = 1, \dots, \ell;\ \ell \le Ns \qquad (12)$$

and the eigenvalues are given by

$$\Lambda(\lambda, \{\lambda_j\}) = (\lambda - is)^N \prod_{j=1}^{\ell} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} + (\lambda + is)^N \prod_{j=1}^{\ell} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j}. \quad (13)$$

In complete analogy with the $s = 1/2$ case, $L_n^+(\lambda) = \sigma_0^2 L_n(\bar\lambda)\sigma_0^2$, $\tau^+(\lambda) = \tau(\bar\lambda)$, and the key relation (6) is left unchanged.

Let us prove the equality $\{\bar\lambda_j\} = \{\lambda_j\}$ for system (12). Instead of (8) we have

$$(\lambda - is)^N f(\lambda) = (\lambda + is)^N f(\lambda - i) \quad (14)$$

with the same $f(\lambda)$ (however, the solutions $\{\lambda_j\}$ for $s > 1/2$ are, of course, different). From (14) we find that $f(\lambda) = (\lambda + is)^N \phi(\lambda)$, $\phi(\lambda)$ is a polynomial. Hence $f(\lambda - i) = (\lambda + i(s-1))^N \phi(\lambda - i)$. At the same time, $f(\lambda - i)$ must equal $(\lambda - is)^N \phi(\lambda)$ by (14). Therefore

$$(\lambda - is)^N \phi(\lambda) = (\lambda + i(s-1))^N \phi(\lambda - i).$$

We are to repeat this procedure: $\phi(\lambda) = (\lambda + i(s-1))^N \chi(\lambda)$ and so on. Finally,

$$f(\lambda) = (\lambda + is)^N (\lambda + i(s-1))^N \ldots (\lambda - i(s-1))^N \omega(\lambda), \quad (15)$$

where $\omega(\lambda)$ is a polynomial obeying $\omega(\lambda) = \omega(\lambda - i)$ so that $\omega(\lambda) = \text{const}$. Remember now that the degree of the polynomial $f(\lambda)$ (see eq.(7)) must be strictly less than 2ℓ and hence less than $2Ns$. This is consistent with (15) iff $\omega(\lambda) = 0$ and $f(\lambda) = 0$. As is shown above, $f(\lambda) = 0$ entails $\{\bar\lambda_j\} = \{\lambda_j\}$. We have proved that all solutions $\{\lambda_j\}$ of the BAE for XXX model of arbitrary spin s obey $\{\bar\lambda_j\} = \{\lambda_j\}$.

5. The XXZ model

$$H = -\frac{1}{2} \sum_{n=1}^{N} (\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \Delta \sigma_n^3 \sigma_{n+1}^3)$$

will be one more illustration of our approach. Let us consider the case $\Delta \geq 1$ and set $\Delta = \operatorname{ch}\gamma$, $\gamma \geq 0$. The essential formulas of Bethe's ansatz for this model are given in [2]. We choose to deal with arbitrary spin s from the very beginning. The corresponding generalization of the XXZ model has been proposed in ref. [11] (see also [12]).

The BAE now read

$$\frac{\sin^N(\lambda_j - i\gamma s)}{\sin^N(\lambda_j + i\gamma s)} = - \prod_{k=1}^{\ell} \frac{\sin(\lambda_j - \lambda_k - i\gamma)}{\sin(\lambda_j - \lambda_k + i\gamma)}, \quad j = 1, \ldots, \ell, \quad (16)$$

the eigenvalues of the transfer matrix are

$$\Lambda(\lambda, \{\lambda_j\}) = \sin^N(\lambda - i\gamma s) \prod_{j=1}^{\ell} \frac{\sin(\lambda - \lambda_j + i\gamma)}{\sin(\lambda - \lambda_j)} +$$

$$+ \sin^N(\lambda + i\gamma s) \prod_{j=1}^{\ell} \frac{\sin(\lambda - \lambda_j - i\gamma)}{\sin(\lambda - \lambda_j)},$$

while the relations (5) and (6) are left intact. We get

$$\sin^N(\lambda - is\gamma) F(\lambda) = \sin^N(\lambda + is\gamma) F(\lambda - i\gamma), \quad (17)$$

where $F(\lambda) = \prod_{j=1}^{\ell} \sin(\lambda - \bar\lambda_j + i\gamma)\sin(\lambda - \lambda_j) - \prod_{j=1}^{\ell} \sin(\lambda - \lambda_j + i\gamma)\sin(\lambda - \bar\lambda_j)$.

Let us show that $F(\lambda) = 0$ for $\ell \leq Ns$. To this end we consider the asymptotics of (17) for $\lambda = -ix$, $x \to +\infty$. One has $\sin^N(\lambda \pm i\gamma s) \sim \exp N(x \mp \gamma s)$ and

$$F(\lambda) = e^{2\ell x} [\prod_{j=1}^{\ell} e^{\gamma - i\lambda_j} e^{-i\bar\lambda_j} - \prod_{j=1}^{\ell} e^{\gamma - i\bar\lambda_j} e^{-i\lambda_j}] + \mathcal{O}(e^{2(\ell-1)x}).$$

The expression in square brackets is equal to zero, so that $F(\lambda) \sim e^{\alpha x + \beta}$, $\alpha < 2\ell$. Comparing the asymptotics of both sides of (17) yields $\alpha = 2Ns$. We see that if $\ell \leq Ns$, then necessarily $F(\lambda) = 0$.

From now on we again follow ref. [6]. Omitting real and complex conjugate roots, we are left with $\{\lambda_m\}$, $m = 1, \ldots, \ell'$; $\{\bar\lambda_m\} \cap \{\lambda_m\} = \emptyset$. For any complex λ

$$\prod_{m=1}^{\ell'} \sin(\lambda - \bar\lambda_m + i\gamma)\sin(\lambda - \lambda_m) = \prod_{m=1}^{\ell'} \sin(\lambda - \lambda_m + i\gamma)\sin(\lambda - \bar\lambda_m).$$

Zeros of both sides must coincide. Upon choosing $\lambda_p \in \{\lambda_m\}$ so as $\operatorname{Im}\lambda_p \geq \operatorname{Im}\lambda_m$ we see that either $\ell' = 0$ or there exists $\lambda_q \in \{\lambda_m\}$ such that $\lambda_p = \bar\lambda_q + 2\pi M$, M integer. By construction [2] of vectors $|\{\lambda_j\}\rangle$ one should identify the solutions of BAE (16) which differ merely by a $2\pi M$ shift. Hence, we have proved $\{\bar\lambda_j\} = \{\lambda_j\}$ in the case of XXZ model with arbitrary spin.

In the other physical sectors ($\Delta < 1$) of this model relations (5) and (6) are slightly modified [2], but the proof follows the same line of argument and eventually gives $\{\bar\lambda_j\} = \{\lambda_j\}$.

In all the models considered the eigenvalues of the local integrals of motion (11) are readily shown to be real.

Based solely on formulas (3) and (6), the method for testing the equality $\{\bar\lambda_j\} = \{\lambda_j\}$ proposed here applies in any models where the BAE can be viewed as consistency conditions within a diagonalization procedure for some transfer matrix.

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Владимиров А.А. Е17-84-830
Доказательство самосопряженности решений уравнений анзаца Бете

Доказана самосопряженность решений систем алгебраических уравнений, возникающих при применении анзаца Бете к интегрируемым XXX- и XXZ -магнетикам произвольного спина.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Vladimirov A.A. E17-84-830
Proof of the Invariance of the Bethe-Ansatz Solutions under Complex Conjugation

A simple proof is given of the property $\{\bar{\lambda}_j\} = \{\lambda_j\}$ for any solution set $\{\lambda_j\}$ of the algebraic equations resulting from the Bethe-ansatz treatment of XXX and XXZ magnets with arbitrary spin.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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