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ON EXACTLY SOLUBLE MODEL IN QUANTUM ELECTRODYNAMICS

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1. INTRODUCTION

The problem of the three-level atom interacting with the electromagnetic field is the object of gained active research for the last ten years. It is central to discussions of two-photon coherence [1,2] resonance Raman scattering and double-resonance processes [3], three-level superradiance [4,5], two-mode laser [6], three-level echoes [7], population dynamics, and spectra of a driven three-level system [8-11].

A number of recent papers has been dedicated to a careful consideration of the problem of dynamics of a single three-level atom interacting with two resonant modes of the radiation field. The semiclassical formalism for the treatment of this problem has been discussed [12,13,9,10]. In another series of articles [6,11,15,16,17] the fully quantized theory has been studied. Exact Schrödinger wave functions have been obtained for some special initial states [6,11]. In the work of Li and Bei [15] the explicit expression of the evolution operator has been derived in the interaction picture for the case of exact one-photon resonance. The rigorous examinations of the dynamical behaviour of level populations and photon numbers have been realized in the Heisenberg picture by Bogolubov et al. [16] for the three-level two-photon lambda configuration. On the other hand the exact solution of the nonlinear equation for the energy operator of a few-level atom interacting with a single mode radiation field has been obtained by Buck and Sukumar [20]. In this paper we shall show that the operator equations for the three-level two-photon ladder configuration detuned from one-photon resonance can be solved explicitly. By using the exact solution obtained here we shall examine the dynamical behaviour of photon numbers and level populations for arbitrary initial states of the field.

The remainder of this paper is organized as follows. Section 2: Model Hamiltonian. Section 3: Exact solution of operator equations for level populations and photon numbers. Section 4: Time evolution of photon numbers and level populations in the case of quantum initial states. Section 5: Time evolution of photon numbers and level populations in the case of arbitrary initial field. Section 6: Summary.

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2. MODEL HAMILTONIAN

We consider a three-level atom of ladder configuration (see Fig.1) in which nonzero dipole moments exist only between levels 1 and 3, and 2 and 3. The dipole transition between levels 1 and 2 is thus forbidden. Let the atom be at rest in a lossless cavity and interact with a two-mode radiation field. The energy operator for the atom is

$$\hat{H}_A = \sum_{j=1}^{3} \hbar\Omega_j \hat{R}_{jj}. \qquad (1)$$

Here, the operator $\hat{R}_{jj} = |j\rangle\langle j|$ describes the population of level j and $\hbar\Omega_j$ is the corresponding energy. The atomic eigenstate vectors $|j\rangle$ (j = 1,2,3) form the basis of the state space of the three-level atom

$$\hat{H}_A |j\rangle = \hbar\Omega_j |j\rangle, \qquad \langle i|j\rangle = \delta_{ij}, \quad \sum_{j=1}^{3} |j\rangle\langle j| = 1. \qquad (2)$$

The field Hamiltonian is

$$\hat{H}_F = \sum_{\alpha=1}^{2} \hbar\omega_\alpha \hat{a}^+_\alpha \hat{a}_\alpha. \qquad (3)$$

The operators $\hat{a}_\alpha, \hat{a}^+_\alpha$ describe near-resonant mode α of the radiation field in the cavity. The corresponding frequencies of the modes are ω_α, where $|(\omega_\alpha - |\Omega_3 - \Omega_\alpha|)| \ll \omega_\alpha$. The atom-field interaction is described in the dipole and rotating wave approximations [21] by

$$\hat{H}_{AF} = \hbar g_1(\hat{a}_1 \hat{R}_{31} + \hat{a}^+_1 \hat{R}_{13}) + \hbar g_2(\hat{a}_2 \hat{R}_{23} + \hat{a}^+_2 \hat{R}_{32}). \qquad (4)$$

Here, the operator $\hat{R}_{ij} = |i\rangle\langle j|$ describes the atomic transition from level j to level i $(i \neq j)$. The parameters g_α are the constants of atom-mode coupling. Thus, the total model Hamiltonian of the "atom field" system is

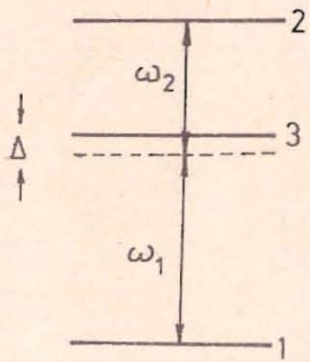

Fig.1. A ladder-configuration three-level atom interacting with a two-mode near-resonant radiation field.

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF} = \sum_{j=1}^{3} \hbar\Omega_j \hat{R}_{jj} + \sum_{\alpha=1}^{2} \hbar\omega_\alpha \hat{a}^+_\alpha \hat{a}_\alpha +$$
$$+ \hbar g_1(\hat{a}_1 \hat{R}_{31} + \hat{a}^+_1 \hat{R}_{13}) + \hbar g_2(\hat{a}_2 \hat{R}_{23} + \hat{a}^+_2 \hat{R}_{32}). \qquad (5)$$

Note that the operators $\hat{R}_{ij} = |i\rangle\langle j|$ (i, j = 1,2,3) are the generators of the group SU(3) and obey the following relations:

$$\hat{R}_{ij} \hat{R}_{k\ell} = \hat{R}_{i\ell} \delta_{kj}, \qquad (6)$$

$$\sum_{i=1}^{3} R_{ii} = 1. \qquad (7)$$

By using (6) the commutation rules

$$[\hat{R}_{ij}, \hat{R}_{k\ell}] = \hat{R}_{i\ell}\delta_{kj} - \hat{R}_{kj}\delta_{i\ell} \qquad (8)$$

are quickly established [4]. The commutation relations of the photon operators $\hat{a}_\alpha$, $\hat{a}^+_\alpha$ (α = 1,2) are

$$[\hat{a}_\alpha, \hat{a}^+_{\alpha'}] = \delta_{\alpha\alpha'}, \quad [\hat{a}_\alpha, \hat{a}_{\alpha'}] = 0, \quad [\hat{a}^+_\alpha, \hat{a}^+_{\alpha'}] = 0. \qquad (9)$$

Assuming that there is exact two-photon resonance, the detuning parameter Δ can be defined as

$$\Delta = (\Omega_3 - \Omega_1) - \omega_1 = \omega_2 - (\Omega_2 - \Omega_3). \qquad (10)$$

3. EXACT SOLUTION OF OPERATOR EQUATIONS FOR LEVEL POPULATIONS AND PHOTON NUMBERS

Starting from the Hamiltonian (5) and the commutators (8) and (9) we write down the Heisenberg equations for various operators in the usual way, i.e., $\dot{\hat{C}} = (i/\hbar)[H, \hat{C}]$. It is convenient to define the subsidiary operators

$$\hat{A}_1 = i(\hat{a}_1 \hat{R}_{31} - \hat{a}^+_1 \hat{R}_{13}), \qquad \hat{A}_2 = i(\hat{a}_2 \hat{R}_{23} - \hat{a}^+_2 \hat{R}_{32}). \qquad (11)$$

Then, the Heisenberg equations for the level-population operators $\hat{R}_{\alpha\alpha}$ and the photon-number operators $\hat{N}_\alpha = \hat{a}^+_\alpha \hat{a}_\alpha$ (α = 1,2) are quickly established

$$\dot{\hat{R}}_{11}(t) = \dot{\hat{N}}_1(t) = g_1 \hat{A}_1(t), \qquad \dot{\hat{R}}_{22}(t) = -\dot{\hat{N}}_2(t) = -g_2 \hat{A}_2(t). \qquad (12)$$

It follows that

$$\hat{N}_1(t) - \hat{R}_{11}(t) = \mathrm{const} \equiv \hat{M}_1, \quad \hat{N}_2(t) + \hat{R}_{22}(t) = \mathrm{const} \equiv \hat{M}_2 + 1, \tag{13}$$

where $\hat{M}_\alpha$ are time-independent operators.

By using the relations (6) the Heisenberg equations for $\hat{A}_\alpha$ are found to be

$$\begin{aligned} \dot{\hat{A}}_1(t) &= -\Delta \hat{C}_1(t) + 2g_1(\hat{M}_1 + 1)\,[1 - 2\hat{R}_{11}(t) - \hat{R}_{22}(t)] - g_2 \hat{B}(t), \\ \dot{\hat{A}}_2(t) &= \Delta \hat{C}_2(t) - 2g_2(\hat{M}_2 + 1)\,[1 - 2\hat{R}_{22}(t) - \hat{R}_{11}(t)] + g_1 \hat{B}(t), \end{aligned} \tag{14}$$

where

$$\hat{B} \equiv \hat{a}_1 \hat{a}_2 \hat{R}_{21} + \hat{a}_1^+ \hat{a}_2^+ \hat{R}_{12}, \quad \hat{C}_1 \equiv \hat{a}_1 \hat{R}_{31} + \hat{a}_1^+ \hat{R}_{13}, \quad \hat{C}_2 \equiv \hat{a}_2 \hat{R}_{23} + \hat{a}_2^+ \hat{R}_{32}. \tag{15}$$

The operators $\hat{B}$ and $\hat{C}_\alpha$ obey the following equations of motion:

$$\begin{aligned} \dot{\hat{B}}(t) &= -g_1(\hat{M}_1 + 1)\,\hat{A}_2(t) + g_2(\hat{M}_2 + 1)\,\hat{A}_1(t), \\ \dot{\hat{C}}_1(t) &= \Delta \hat{A}_1(t) + g_2 \hat{D}(t), \qquad \dot{\hat{C}}_2(t) = -\Delta \hat{A}_2(t) - g_1 \hat{D}(t), \end{aligned} \tag{16}$$

where

$$\hat{D} \equiv i(\hat{a}_1 \hat{a}_2 \hat{R}_{21} - \hat{a}_1^+ \hat{a}_2^+ \hat{R}_{12}). \tag{17}$$

Finally, the Heisenberg equation for the operator $\hat{D}$ is found to be

$$\dot{\hat{D}}(t) = g_1(\hat{M}_1 + 1)\,\hat{C}_2(t) - g_2(\hat{M}_2 + 1)\,\hat{C}_1(t). \tag{18}$$

The equations (12), (14), (16) and (18) form a closed linear system which has two following integrals of motion:

$$\begin{aligned} -g_1 g_2 \hat{B}(t) + g_1^2(\hat{M}_1 + 1)\,\hat{R}_{22}(t) + g_2^2(\hat{M}_2 + 1)\,\hat{R}_{11}(t) &= \mathrm{const} \equiv \hat{K}, \\ g_1 \hat{C}_1(t) + g_2 \hat{C}_2(t) - \Delta\{\hat{R}_{11}(t) + \hat{R}_{22}(t)\} &= \mathrm{const} \equiv \hat{Q}. \end{aligned} \tag{19}$$

Here, $\hat{K}$ and $\hat{Q}$ are time-independent operators. It is easy to establish that the operators $\hat{M}_\alpha$, $\hat{K}$ and $\hat{Q}$ commute with $\hat{H}$ and each other. Taking into account (19) one can obtain from (12), (14), (16) and (18) the operator equations

$$\begin{aligned} \ddot{\hat{R}}_{11}(t) &= -(3\hat{\lambda}_1^2 + \hat{\lambda}_0^2)\,\hat{R}_{11}(t) - 3\hat{\lambda}_1^2 \hat{R}_{22}(t) - \Delta g_1 \hat{C}_1(t) + 2\hat{\lambda}_1^2 + \hat{K}, \\ \ddot{\hat{R}}_{22}(t) &= -(3\hat{\lambda}_2^2 + \Delta^2)\,\hat{R}_{11}(t) - (3\hat{\lambda}_2^2 + \hat{\lambda}_0^2 + \Delta^2)\,\hat{R}_{22}(t) + \\ &\quad + \Delta g_1 \hat{C}_1(t) + 2\hat{\lambda}_2^2 + \hat{K} - \Delta \hat{Q}, \\ g_1 \ddot{\hat{C}}_1(t) &= \Delta \ddot{\hat{R}}_{11}(t) + \Delta \hat{\lambda}_1^2\{\hat{R}_{11}(t) + \hat{R}_{22}(t)\} - \hat{\lambda}_0^2 g_1 \hat{C}_1(t) + \hat{\lambda}_1^2 \hat{Q}. \end{aligned} \tag{20}$$

Here we have introduced the notation

$$\hat{\lambda}_\alpha \equiv g_\alpha \sqrt{\hat{M}_\alpha + 1}, \qquad \hat{\lambda}_0 \equiv \sqrt{\hat{\lambda}_1^2 + \hat{\lambda}_2^2}. \tag{21}$$

To solve the system of the second-order differential equations (20) we ought to determine the eigenvalues of the characteristic matrix. This leads to the following equation:

$$\det \begin{pmatrix} \hat{X}^2 - (3\hat{\lambda}_1^2 + \hat{\lambda}_0^2) & -3\hat{\lambda}_1^2 & i\Delta \\ -(3\hat{\lambda}_2^2 + \Delta^2) & \hat{X}^2 - (3\hat{\lambda}_2^2 + \hat{\lambda}_0^2 + \Delta^2) & -i\Delta \\ i\Delta(\hat{\lambda}_1^2 - \hat{X}^2) & i\Delta\hat{\lambda}_1^2 & \hat{X}^2 - \hat{\lambda}_0^2 \end{pmatrix}$$
$$= \hat{X}^6 - 2(3\hat{\lambda}_0^2 + \Delta^2)\,\hat{X}^4 + (3\hat{\lambda}_0^2 + \Delta^2)^2 \hat{X}^2 - \hat{\lambda}_0^4(4\hat{\lambda}_0^2 + \Delta^2) = 0. \tag{22}$$

The solutions of this equation are found to be $\hat{\lambda}_+$, $\hat{\lambda}_-$ and $2\hat{\lambda}$ where

$$\hat{\lambda} \equiv \sqrt{\hat{\lambda}_0^2 + \Delta^2/4}, \quad \hat{\lambda}_+ = \hat{\lambda} + \Delta/2, \quad \hat{\lambda}_- = \hat{\lambda} - \Delta/2. \tag{23}$$

They are the operators of the frequencies of nonlinear optical oscillations in the three-level system [22,10,11]. Now the solution of the system (20) can be presented in the form

$$\begin{aligned} \hat{R}_{11}(t) &= \hat{P}_+(t) + \hat{P}_-(t) + \hat{\lambda}_1^2 \hat{P}(t) + \hat{R}_{11}(0), \\ \hat{R}_{22}(t) &= -\hat{P}_+(t) - \hat{P}_-(t) + \hat{\lambda}_2^2 P(t) + \hat{R}_{22}(0), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \hat{P}_+(t) &= \hat{\mu}_+(\cos\hat{\lambda}_+ t - 1) + \hat{\beta}_+ \sin\hat{\lambda}_+ t, \qquad \hat{P}_-(t) = \hat{\mu}_-(\cos\hat{\lambda}_- t - 1) + \hat{\beta}_- \sin\hat{\lambda}_- t, \\ \hat{P}(t) &= \hat{\mu}\,(\cos 2\hat{\lambda} t - 1) + \hat{\beta} \sin 2\hat{\lambda} t. \end{aligned} \tag{25}$$

The amplitude operators $\hat{\mu}_\pm$, $\hat{\mu}$, $\hat{\beta}_\pm$ and $\hat{\beta}$ are found to be:

$$\begin{aligned} \hat{\mu} &= \{\hat{\lambda}_1^2 \hat{R}_{11}(0) + \hat{\lambda}_2^2 \hat{R}_{22}(0) - \hat{\lambda}_0^2 \hat{R}_{33}(0) + g_1 g_2 \hat{B}(0) + \\ &\quad + \frac{\Delta}{2}[g_1 \hat{C}_1(0) + g_2 \hat{C}_2(0)]\}/(2\hat{\lambda}_0^2 \hat{\lambda}^2), \\ \hat{\mu}_\pm &= [\hat{R}_{11}(0) - \hat{R}_{22}(0)]\hat{\lambda}_1^2 \hat{\lambda}_2^2/(\hat{\lambda}_0^2 \hat{\lambda} \hat{\lambda}_\pm) + g_1 g_2 \hat{B}(0)\,(\hat{\lambda}_2^2 - \hat{\lambda}_1^2)/(2\hat{\lambda}_0^2 \hat{\lambda} \hat{\lambda}_\pm) \pm \end{aligned}$$

$$\pm[\hat{\lambda}_2^2 g_1 \hat{C}_1(0) - \hat{\lambda}_1^2 g_2 \hat{C}_2(0)]/(2\hat{\lambda}_0^2\hat{\lambda}),$$

$$\hat{\beta} = \{g_1 \hat{A}_1(0) - g_2 \hat{A}_2(0)\}/(2\hat{\lambda}_0^2\hat{\lambda}), \tag{26}$$

$$\hat{\beta}_{\pm} = \{\hat{\lambda}_2^2 g_1 \hat{A}_1(0) + \hat{\lambda}_1^2 g_2 \hat{A}_2(0)\}/(2\hat{\lambda}_0^2\hat{\lambda}) \pm g_1 g_2 \hat{D}(0)/(2\hat{\lambda}\hat{\lambda}_{\pm}).$$

By using the conservation laws (7) and (13) together with eqs. (24) one can obtain

$$\hat{R}_{33}(t) = -\hat{\lambda}_0^2\hat{P}(t) + \hat{R}_{33}(0), \quad \hat{N}_1(t) = \hat{P}_+(t) + \hat{P}_-(t) + \hat{\lambda}_1^2\hat{P}(t) + \hat{N}_1(0),$$
$$\hat{N}_2(t) = \hat{P}_+(t) + \hat{P}_-(t) - \hat{\lambda}_2^2\hat{P}(t) + \hat{N}_2(0). \tag{27}$$

The exact solution (24) of the operator equations (20) and the formulas (27) represent the explicit expressions of time dependence for the level-population and photon-number operators.

From eqs. (25) it is clear that the operators $\hat{\lambda}_+, \hat{\lambda}_-$ and $2\hat{\lambda}$ are the quantum electrodynamic expressions for the two-photon Rabi frequencies /11/. Under the condition of one-photon resonance we have $\Delta = 0$, therefore $\hat{\lambda}_+ = \hat{\lambda}_- = \hat{\lambda}$. In this case there are two branches of the two-photon Rabi frequencies defined by the operators $\hat{\lambda}$ and $2\hat{\lambda}$ /15,16/. It should be noted that the existence of the "soft branch" is a characteristic feature of the three-level system. Such a kind of oscillation frequencies is absent in the two-level system /21,18,19/. Our present results show that the detuning in the case of two-photon resonance leads to the splitting of the "soft branch" to two branches characterized by the frequency operators $\hat{\lambda}_+ = \hat{\lambda} + \Delta/2$, $\hat{\lambda}_- = \hat{\lambda} - \Delta/2$. This conclusion of the full quantised theory is in accord with the results of the semiclassical theory /9,10,13/.

4. TIME EVOLUTION OF PHOTON NUMBERS AND LEVEL POPULATIONS IN THE CASE OF QUANTUM INITIAL STATES!

Let $\hat{\rho}(0)$ be a density matrix corresponding to some initial state of the "atom+field" system. Then, the mean values of the level populations and photon numbers are given by

$$\langle\hat{C}(t)\rangle = \mathrm{Tr}\,\hat{C}(t)\,\hat{\rho}(0), \tag{28}$$

where $\hat{C}$ is $\hat{R}_{jj}$ or $\hat{N}_\alpha$.

First of all, let us consider a simple but interesting case when at the initial moment $t = 0$ the atom is on a level i and the field is in a quantum state with definite occupation numbers $|n_1, n_2\rangle$. Then

$$\hat{\rho}(0) = |\{m_0\}\rangle\langle\{m_0\}|, \quad |\{m_0\}\rangle \equiv |i; n_1, n_2\rangle. \tag{29}$$

One can easily see that the initial state $|\{m_0\}\rangle$ is one of the basis states of the total system. Thus, the density matrix $\hat{\rho}(0)$ has in the basis representation only one nonzero element

$$\rho_{\{m'\},\{m''\}} = \langle\{m'\}|\,\hat{\rho}(0)\,|\{m''\}\rangle = \delta_{\{m'\},\{m_0\}}\,\delta_{\{m''\},\{m_0\}}. \tag{30}$$

On the other hand the operators $\hat{\lambda}_\alpha$ are diagonal in this representation. So, for an arbitrary operator $\hat{C}$ and arbitrary function $f(\cdot)$ we have

$$\langle\hat{C} f(\hat{\lambda}_\alpha)\rangle = \langle\{m_0\}|\,\hat{C} f(\hat{\lambda}_\alpha)\,|\{m_0\}\rangle =$$
$$= \langle\{m_0\}|\hat{C}|\{m_0\}\rangle\, f(\langle\{m_0\}|\hat{\lambda}_\alpha|\{m_0\}\rangle) = \langle\hat{C}\rangle f(\langle\hat{\lambda}_\alpha\rangle). \tag{31}$$

Below we shall use the following notation $\langle\hat{C}\rangle \equiv C$. Now by using the relation (31) we can obtain from eqs. (24) and (27) that

$$R_{11}(t) = -2\mu_+ \sin^2\frac{\lambda_+ t}{2} - 2\mu_- \sin^2\frac{\lambda_- t}{2} - 2\lambda_1^2\mu\sin^2\lambda t + R_{11}(0),$$

$$R_{22}(t) = 2\mu_+ \sin^2\frac{\lambda_+ t}{2} + 2\mu_- \sin^2\frac{\lambda_- t}{2} - 2\lambda_2^2\mu\sin^2\lambda t + R_{22}(0),$$

$$R_{33}(t) = 2\lambda_0^2\mu\sin^2\lambda t + R_{33}(0), \tag{32}$$

$$N_1(t) = -2\mu_+ \sin^2\frac{\lambda_+ t}{2} - 2\mu_- \sin^2\frac{\lambda_- t}{2} - 2\lambda_1^2\mu\sin^2\lambda t + n_1,$$

$$N_2(t) = -2\mu_+ \sin^2\frac{\lambda_+ t}{2} - 2\mu_- \sin^2\frac{\lambda_- t}{2} + 2\lambda_2^2\mu\sin^2\lambda t + n_2.$$

Here the frequencies λ_+, λ_- and 2λ of the two-photon Rabi oscillations in the system are defined by

$$\lambda = \sqrt{\lambda_0^2 + \Delta^2/4}, \quad \lambda_+ = \lambda + \Delta/2, \quad \lambda_- = \lambda - \Delta/2, \tag{33}$$

where

$$\lambda_0 = \sqrt{\lambda_1^2 + \lambda_2^2}, \quad \lambda_1 = g_1\sqrt{n_1 - R_{11}(0) + 1}, \quad \lambda_2 = g_2\sqrt{n_2 + R_{22}(0)}. \tag{34}$$

The amplitudes of the oscillations are found from eqs. (26) to be

$$\mu = [\lambda_1^2 R_{11}(0) + \lambda_2^2 R_{22}(0) - \lambda_0^2 R_{33}(0)]/(2\lambda_0^2\lambda^2),$$
$$\mu_\pm = [R_{11}(0) - R_{22}(0)]\lambda_1^2\lambda_2^2/(\lambda_0^2\lambda\lambda_\pm), \quad \beta = \beta_+ = \beta_- = 0. \tag{35}$$

For the sake of eliminating the above-mentioned fast oscillations and obtaining the time-average values of the mean level populations and photon numbers, we make the following procedure:

$$\bar{\mathcal{O}} = \frac{1}{2T}\int_{t-T}^{t+T} \mathcal{O}(t')\,dt', \qquad T \gg \lambda^{-1} \tag{36}$$

for $\mathcal{O}(t) = R_{jj}(t), N_\alpha(t)$.

Then, in compliance with eqs. (32) we have

$$\begin{aligned}
&\bar R_{11} = -(\mu_+ + \mu_- + \lambda_1^2\mu) + R_{11}(0), \qquad \bar R_{22} = \mu_+ + \mu_- - \lambda_2^2\mu + R_{22}(0),\\
&\bar R_{33} = \lambda_0^2\mu + R_{33}(0),\\
&\bar N_1 = -(\mu_+ + \mu_- + \lambda_1^2\mu) + n_1, \qquad \bar N_2 = -(\mu_+ + \mu_- - \lambda_2^2\mu) + n_2.
\end{aligned} \tag{37}$$

Let us now concretize the initial condition (29) and find the values of the frequencies and amplitudes corresponding to the cases when the atom is initially in the state 1,2,3, respectively.

Case 1. Let at $t = 0$ the atom be in the unexcited state $|1\rangle$, i.e., $|\{m_0\}\rangle = |1; n_1, n_2\rangle$. In this case we have $R_{11}(0) = 1$, $R_{22}(0) = R_{33}(0) = 0$. From eqs. (33-35) it follows that

$$\begin{aligned}
&\lambda = W(\{n_\alpha\}), \quad \lambda_{0,\pm} = W_{0,\pm}(\{n_\alpha\}), \quad \lambda_1 = g_1\sqrt{n_1}, \quad \lambda_2 = g_2\sqrt{n_2},\\
&\mu = \frac{g_1^2 n_1}{2W_0^2(\{n_\alpha\})\,W^2(\{n_\alpha\})}, \qquad \mu_\pm = \frac{g_1^2 g_2^2 n_1 n_2}{W_0^2(\{n_\alpha\})\,W(\{n_\alpha\})\,W_\pm(\{n_\alpha\})},
\end{aligned} \tag{38}$$

where we have introduced the notation

$$\begin{aligned}
&W_0(\{n_\alpha\}) \equiv W_0(n_1, n_2) = \sqrt{g_1^2 n_1 + g_2^2 n_2},\\
&W(\{n_\alpha\}) \equiv W(n_1, n_2) = \sqrt{g_1^2 n_1 + g_2^2 n_2 + \Delta^2/4},\\
&W_\pm(\{n_\alpha\}) \equiv W_\pm(n_1, n_2) = \sqrt{g_1^2 n_1 + g_2^2 n_2 + \Delta^2/4} \pm \Delta/2.
\end{aligned} \tag{39}$$

Case 2: Let at $t = 0$ the atom be in the upper state 2, thus $|\{m_0\}\rangle = |2; n_1, n_2\rangle$. Then, we have $R_{22}(0) = 1$, $R_{11}(0) = R_{33}(0) = 0$. Equations (33-35) in this case give

$$\lambda = W(\{n_\alpha + 1\}), \quad \lambda_{0,\pm} = W_{0,\pm}(\{n_\alpha + 1\}), \quad \lambda_1 = g_1\sqrt{n_1 + 1}, \quad \lambda_2 = g_2\sqrt{n_2 + 1},$$

$$\mu = \frac{g_2^2(n_2 + 1)}{2W_0^2(\{n_\alpha + 1\})\,W^2(\{n_\alpha + 1\})}, \qquad \mu_\pm = -\frac{g_1^2 g_2^2 (n_1 + 1)(n_2 + 1)}{W_0^2(\{n_\alpha + 1\})\,W(\{n_\alpha + 1\})\,W_\pm(\{n_\alpha + 1\})}. \tag{40}$$

Case 3: Let at $t = 0$ the atom be in the immediate state 3, i.e., $|\{m_0\}\rangle = |3; n_1, n_2\rangle$. In this case we have $R_{33}(0) = 1$, $R_{11}(0) = R_{22}(0) = 0$. From eqs. (33-35) one can obtain that

$$\begin{aligned}
&\lambda = W(n_1 + 1, n_2), \quad \lambda_{0,\pm} = W_{0,\pm}(n_1 + 1, n_2),\\
&\lambda_1 = g_1\sqrt{n_1 + 1}, \quad \lambda_2 = g_2\sqrt{n_2}, \quad \mu = -\frac{1}{2W^2(n_1 + 1, n_2)}, \quad \mu_\pm = 0.
\end{aligned} \tag{41}$$

Note that the expressions (32) together with (38), (40) and (41) are in compliance with the results of Kancheva et al.[10] and Radmore and Knight[11].

To determine the transition probabilities of the atom, let us introduce the Schrödinger representation with a wave-function of the total system $|\Psi(t)\rangle$, where $|\Psi(0)\rangle = |i; n_1, n_2\rangle$. Then, the probability of finding the atom on its jth level at time t as a result of the transition $i \to j$ initiated by the $n_1 \oplus n_2$ - photon field can be defined by the formula

$$P(t; i \to j) = \sum_{n_1', n_2'} |\langle \Psi(t) | j; n_1', n_2' \rangle|^2, \tag{42}$$

where

$$|\Psi(0)\rangle = |i; n_1, n_2\rangle. \tag{43}$$

It is seen that under the initial condition (43' the population $R_{jj}(t)$ of level j is equal to the probability $P(t; i \to j)$. Hence, by using eqs. (32) together with (38), (40) and (41) one can determine the probabilities of various transitions in the system. In particular, for the two-photon processes of absorption $(1 \to 2)$ and emission $(2 \to 1)$ one obtains[23]

$$\begin{aligned}
P(t; 1 \to 2) = {} & \frac{2g_1^2 g_2^2 n_1 n_2}{W_0^2(\{n_\alpha\})\,W(\{n_\alpha\})} \left\{ \frac{1}{W_+(\{n_\alpha\})} \sin^2 \frac{W_+(\{n_\alpha\})\,t}{2} + \right.\\
& \left. + \frac{1}{W_-(\{n_\alpha\})} \sin^2 \frac{W_-(\{n_\alpha\})\,t}{2} - \frac{1}{2W(\{n_\alpha\})} \sin^2 [W(\{n_\alpha\})\,t] \right\},\\
P(t; 2 \to 1) = {} & \frac{2g_1^2 g_2^2 (n_1 + 1)(n_2 + 1)}{W_0^2(\{n_\alpha + 1\})\,W(\{n_\alpha + 1\})} \left\{ \frac{1}{W_+(\{n_\alpha + 1\})} \sin^2 \frac{W_+(\{n_\alpha + 1\})\,t}{2} + \right.
\end{aligned} \tag{44}$$

$$+\frac{1}{W_-(\{n_\alpha+1\})}\sin^2\frac{W_-(\{n_\alpha+1\})\,t}{2}-\frac{1}{2W(\{n_\alpha+1\})}\sin^2[W(\{n_\alpha+1\})\,t]\Bigg\}.$$

For the one-photon transitions $3 \rightleftarrows \alpha\ (\alpha=1,2)$ we find

$$P(t;1\to 3)=\frac{g_1^2 n_1}{W^2(\{n_\alpha\})}\sin^2[W(\{n_\alpha\})t],\quad P(t;2\to 3)=\frac{g_2^2(n_2+1)}{W^2(\{n_\alpha+1\})}\sin^2[W(\{n_\alpha+1\})t],$$

$$P(t;3\to 1)=\frac{g_1^2(n_1+1)}{W^2(n_1+1,n_2)}\sin^2[W(n_1+1,n_2)\,t], \tag{45}$$

$$P(t;3\to 2)=\frac{g_2^2 n_2}{W^2(n_1+1,n_2)}\sin^2[W(n_1+1,n_2)\,t].$$

The expressions (44) and (45) are in compliance with the results of Kancheva et al.[10] and Radmore and Knight[11].

5. TIME EVOLUTION OF PHOTON NUMBERS AND LEVEL POPULATIONS IN THE CASE OF ARBITRARY INITIAL FIELD

Now we consider the case when the field is initially in some state described by the density matrix $\hat\rho_F$ whereas the atom is on level i. The total density matrix of the "atom+field" system is

$$\hat\rho(0)=|i\rangle\langle i|\otimes\hat\rho_F. \tag{46}$$

In the case of initially coherent field the matrix $\hat\rho_F$ takes the from [24]

$$\hat\rho_F=|z_1,z_2\rangle\langle z_1,z_2|, \tag{47}$$

where the coherent state $|z_1,z_2\rangle$ is defined by

$$|z_1,z_2\rangle=\sum_{n_1,n_2}\exp\left(-\frac{|z_1|^2+|z_2|^2}{2}\right)\frac{z_1^{n_1}z_2^{n_2}}{\sqrt{n_1!\,n_2!}}|n_1,n_2\rangle. \tag{48}$$

In the case of initially chaotic field $\hat\rho_F$ is

$$\hat\rho_F=\exp(-\beta\hat H_F)/\mathop{\mathrm{Tr}}_{(F)}\exp(-\beta\hat H_F). \tag{49}$$

Here, the field Hamiltonian $\hat H_F$ is given by eq. (3) and β^{-1} is the temperature of the initial field.

It is seen from eqs. (24-27) that the operators $\hat R_{jj}(t)$ and $\hat N_\alpha(t)$ are diagonal in the state subspaces $\{|1;n_1,n_2\rangle\}$, $\{|2;n_1,n_2\rangle\}$ and $\{|3;n_1,n_2\rangle\}$, i.e., for $\hat{\mathcal{C}}=\hat R_{jj}(t),\ \hat N_\alpha(t)$ one has

$$\langle i;n_1',n_2'|\hat{\mathcal{C}}|i;n_1'',n_2''\rangle=\delta_{n_1',n_1''}\delta_{n_2',n_2''}\langle i;n_1',n_2'|\hat{\mathcal{C}}|i;n_1',n_2'\rangle. \tag{50}$$

Hence, the mean value of $\hat{\mathcal{C}}$ in the case of the initial state (46) is found to be

$$\langle\hat{\mathcal{C}}\rangle=\mathrm{Tr}\hat{\mathcal{C}}\hat\rho(0)=\sum_{n_1,n_2}\langle\hat{\mathcal{C}}\rangle_{in_1n_2}P(n_1,n_2), \tag{51}$$

where $\langle\hat{\mathcal{C}}\rangle_{in_1n_2}$ is the mean value of $\hat{\mathcal{C}}$ in the case of the initial state (29)

$$\langle\hat{\mathcal{C}}\rangle_{in_1n_2}=\langle i;n_1,n_2|\hat{\mathcal{C}}|i;n_1,n_2\rangle, \tag{52}$$

and $P(n_1,n_2)$ is the weight factor defined by the field density matrix ρ_F

$$P(n_1,n_2)=\langle n_1,n_2|\hat\rho_F|n_1,n_2\rangle. \tag{53}$$

Thus, by using the relation (51) and eqs. (32) together with (38), (40) and (41) one can obtain the mean values of the level populations and photon numbers in the general case. In particular, we find

$$R_{11}(t)=N_1(t)-N_1(0)+1=$$

$$=1-\sum_{n_1,n_2=0}^{\infty}\frac{2g_1^2g_2^2n_1n_2}{W_0^2(\{n_\alpha\})\,W(\{n_\alpha\})}\Bigg\{\frac{1}{W_+(\{n_\alpha\})}\sin^2\frac{W_+(\{n_\alpha\})\,t}{2}+$$

$$+\frac{1}{W_-(\{n_\alpha\})}\sin^2\frac{W_-(\{n_\alpha\})\,t}{2}\Bigg\}P(n_1,n_2)\ -$$

$$-\sum_{n_1,n_2=0}^{\infty}\frac{g_1^4n_1^2}{W_0^2(\{n_\alpha\})\,W^2(\{n_\alpha\})}\sin^2[W(\{n_\alpha\})t]\,P(n_1,n_2),$$

$$R_{22}(t)=N_2(0)-N_2(t)=$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{2g_1^2 g_2^2 n_1 n_2}{W_0^2(\{n_\alpha\})\, W(\{n_\alpha\})} \left\{ \frac{1}{W_+(\{n_\alpha\})} \sin^2 \frac{W_+(\{n_\alpha\})\, t}{2} + \right.$$

$$\left. + \frac{1}{W_-(\{n_\alpha\})} \sin^2 \frac{W_-(\{n_\alpha\})\, t}{2} - \frac{1}{2W(\{n_\alpha\})} \sin^2 [W(\{n_\alpha\})\, t] \right\} P(n_1, n_2) , \tag{54}$$

$$R_{33}(t) = \sum_{n_1,n_2=0}^{\infty} \frac{g_1^2 n_1}{W^2(\{n_\alpha\})} \sin^2 [W(\{n_\alpha\})\, t]\, P(n_1, n_2)$$

for the case $\hat\rho(0) = |1\rangle\langle 1| \otimes \hat\rho_F$, when the atom is initially on the lower level 1. For the other case when the atom is initially on the upper level 2, and therefore $\rho(0) = |2\rangle\langle 2| \otimes \hat\rho_F$, we obtain

$$R_{11}(t) = N_1(t) - N_1(0) =$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{2g_1^2 g_2^2 (n_1+1)(n_2+1)}{W_0^2(\{n_\alpha+1\})\, W(\{n_\alpha+1\})} \left\{ \frac{1}{W_+(\{n_\alpha+1\})} \sin^2 \frac{W_+(\{n_\alpha+1\})\, t}{2} + \right.$$

$$\left. + \frac{1}{W_-(\{n_\alpha+1\})} \sin^2 \frac{W_-(\{n_\alpha+1\})\, t}{2} - \frac{1}{2W(\{n_\alpha+1\})} \sin^2 [W(\{n_\alpha+1\})\, t] \right\} P(n_1, n_2) .$$

$$R_{22}(t) = 1 + N_2(0) - N_2(t) =$$

$$= 1 - \sum_{n_1,n_2=0}^{\infty} \frac{2g_1^2 g_2^2 (n_1+1)(n_2+1)}{W_0^2(\{n_\alpha+1\})\, W(\{n_\alpha+1\})} \left\{ \frac{1}{W_+(\{n_\alpha+1\})} \sin^2 \frac{W_+(\{n_\alpha+1\})\, t}{2} + \right.$$

$$\left. + \frac{1}{W_-(\{n_\alpha+1\})} \sin^2 \frac{W_-(\{n_\alpha+1\})\, t}{2} \right\} P(n_1, n_2) \tag{55}$$

$$- \sum_{n_1,n_2=0}^{\infty} \frac{g_2^4 (n_2+1)^2}{W_0^2(\{n_\alpha+1\})\, W^2(\{n_\alpha+1\})} \sin^2 [W(\{n_\alpha+1\})\, t]\, P(n_1, n_2) ,$$

$$R_{33}(t) = \sum_{n_1,n_2=0}^{\infty} \frac{g_2^2(n_2+1)}{W^2(\{n_\alpha+1\})} \sin^2 [W(\{n_\alpha+1\})\, t]\, P(n_1, n_2) .$$

For illustration we calculate the time variation of the photon numbers $\delta N_\alpha(t) = N_\alpha(t) - N_\alpha(0)$ for the case when the atom is initially unexcited on level 1 and the field is in the state (47) or (49).

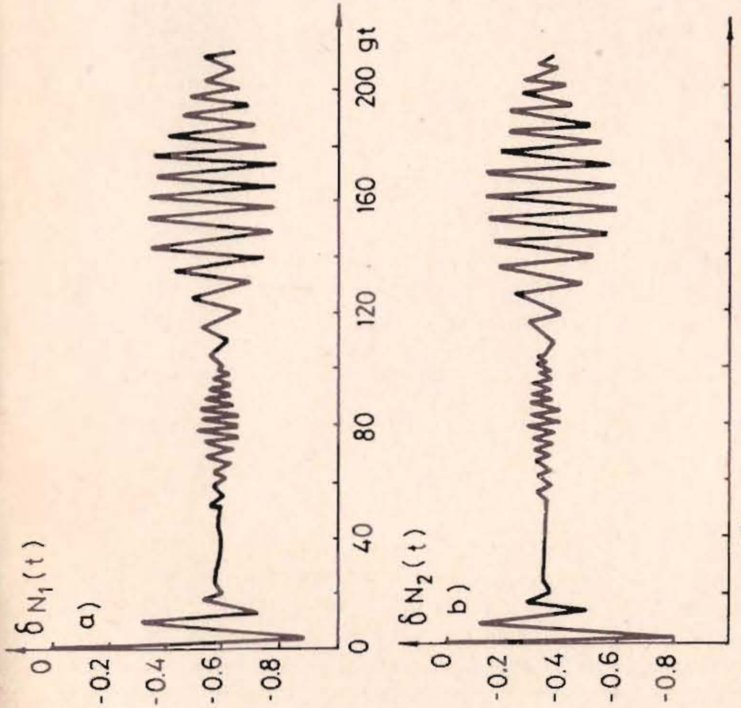

Fig.2. Time evolution of the photon numbers in the case of the initially coherent field (for $g_1 = g_2 = g$, $\Delta = 0$, $\bar{n}_1 = \bar{n}_2 = 5$).

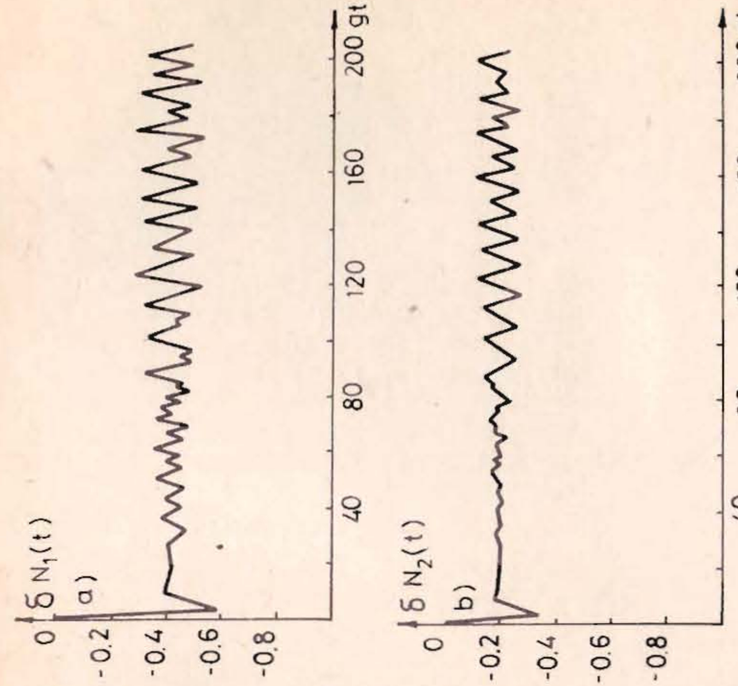

Fig.3. Time evolution of the photon numbers in the case of the initially chaotic field (for $g_1 = g_2 = g$, $\Delta = 0$, $\hbar\omega_1\beta = \hbar\omega_2\beta = 0.2$.

5.1. Initially Coherent Field

In this case, according to (47) and (53) we have

$$P(n_1, n_2) = \exp[-(\bar{n}_1 + \bar{n}_2)]\bar{n}_1^{n_1}\bar{n}_2^{n_2}/(n_1!n_2!). \qquad (56)$$

Here $\bar{n}_1 = |z_1|^2$, $\bar{n}_2 = |z_2|^2$ are the mean photon numbers in modes 1 and 2, respectively, at the initial time $t = 0$.

Substituting (56) into (54) and by using (39) we can now calculate the time evolution of $\delta N_1(t)$, $\delta N_2(t)$. The results of calculation for the case $g_1 = g_2 = g$, $\Delta = 0$, $\bar{n}_1 = \bar{n}_2 = 5$ are shown in fig.2. The revivals and collapses of the two-photon Rabi oscillations are exhibited. Such a behaviour has been prediced for the coherent-state Jaynes-Cummings model (see Eberly et al [25] and references therein) and also for the three-level two-mode lambda configuration by Li and Bei [15] and Fam Le Kien et al. [17].

5.2. Initially Chaotic Field

In this case the weight factor $P(n_1, n_2)$ according to (49) and (53), is given by

$$P(n_1, n_2) = Z^{-1}\exp[-\beta(\hbar\omega_1 n_1 + \hbar\omega_2 n_2)], \qquad (57)$$

where

$$Z^{-1} = [1 - \exp(-\beta\hbar\omega_1)][1 - \exp(-\beta\hbar\omega_2)]. \qquad (58)$$

The time behaviour of $\delta N_1(t)$ and $\delta N_2(t)$ has been calculated for $g_1 = g_2 = g$, $\hbar\omega_1\beta = \hbar\omega_2\beta = 0.2$, $\Delta = 0$, and is plotted in fig.3. The long-time and chaotic character of the revival is noted.

More detailed investigation of the revivals and collapses is the subject of a future publication.

6. CONCLUSION

Thus, in this paper the operator equations for the three-level atom of the ladder configuration interacting with two-mode radiation field detuned from one-photon resonance has been solved explicitly. The quantum electrodynamic expressions of two-photon Rabi frequencies have been found. The time evolution of the photon numbers and level populations has been examined.

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Боголюбов Н.Н. /мл./, Фам Ле Киен, Шумовский А.С. E17-84-822
Об одной точно решаемой модели квантовой электродинамики

Точно решаются уравнения движения, описывающие динамику трехуровневого атома типа каскада, взаимодействующего с двумя модами квантованного поля излучения. Исследована эволюция населенностей уровней и чисел фотонов при различных начальных условиях.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1984

Bogolubov N.N. (Jr.), Fam Le Kien, Shumovsky A.S. E17-84-822
On Exactly Soluble Model in Quantum Electrodynamics

The model of a ladder-configuration three-level atom interacting with a two-mode near-resonant radiation field is treated. It is shown that the operator equations of motion can be solved explicitly. The dynamical behaviour of the photon numbers and level populations is studied for various initial conditions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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