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EXACT RESULTS FOR A MODEL OF A THREE-LEVEL ATOM

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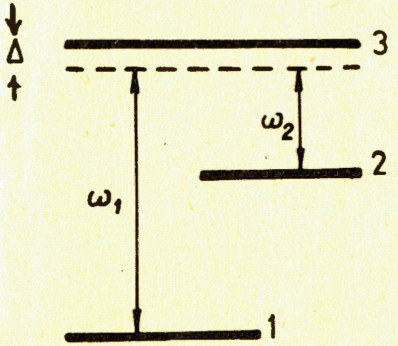
The problem of dynamics of three-level atoms interacting with the electromagnetic field has been the subject of gained active research for last ten years (see[1-11] and references therein). It is central to discussions of two-photon coherence[1,2] resonance Raman scattering and double-resonance processes[3], three-level superradiance[4], two-mode laser[5], three-level echoes[6], etc.

A number of recent papers[5,7-11] has been dedicated to a careful consideration of the problem of dynamics of a single three-level atom interacting with two resonant modes of the radiation field. The semiclassical formalism for the treatment of this problem has been discussed in refs.[7-9]. In another series of articles[5,10,11] the fully quantized theory was studied. Exact Schrödinger wave functions were obtained for some special initial states[5]. In ref.[10] the explicit expression of the evolution operator in the interaction picture was derived. The rigorous examinations of the dynamical behaviour of level populations and photon numbers have been realized in the Heisenberg picture[11]. In the present letter we shall derive the time dependence of atomic transition operators and photon amplitudes as an exact solution of the equations of motion, taking into account a detuning of near-resonant modes. We shall discuss also some interesting consequences of this solution.



Energy-level and transition structure of the model considered.

Consider a three-level atom (see the Figure) in which nonzero dipole moments exist only between levels 1 and 3, and 2 and 3. Let the atom be at rest in a lossless cavity and interact with two modes of the quantized radiation field. The model Hamiltonian of the system under consideration is[10,11]

$$H = H_A + H_F + H_{AF}. \qquad (1)$$

Here H_A, H_F describe the free atom and free field, respectively, and H_{AF} describes the atom-field interaction in the dipole and rotating wave approximations

$$H_A = \sum_{j=1}^{3} \hbar \Omega_j R_{jj}\,, \quad H_F = \sum_{a=1}^{2} \hbar \omega_a b_a^+ b_a, \quad H_{AF} = \sum_{a=1}^{2} \hbar g_a (b_a R_{3a} + b_a^+ R_{a3}) \qquad (2)$$

The operator $R_{jj} \equiv |j\rangle\langle j|$ describes the population of level j. The operator $R_{ij} \equiv |i\rangle\langle j|$ $(i \neq j)$ describes the atomic transition from level j to level i. The vectors of atomic eigenstates $|j\rangle$, $j = 1,2,3$, form the basis of the state space of the three-level atom

$$H_A|j\rangle = \hbar\Omega_j|j\rangle, \quad \langle i|j\rangle = \delta_{ij}, \quad \sum_{j=1}^{3} |j\rangle\langle j| = 1. \tag{3}$$

The operators R_{ij} $(i, j = 1,2,3)$ obey the following relations/4,11/:

$$R_{ij}R_{k\ell} = R_{i\ell}\,\delta_{jk}, \quad \sum_{j=1}^{3} R_{jj} = 1. \tag{4}$$

The operators b^{+}_{a}, b_a $(a = 1,2)$ are the photon amplitudes (or the so-called creation and annihilation operators) corresponding to the mode with the energy $\hbar\omega_a$ of the quantized radiation field. We shall consider the case of exact two-photon resonance/8/ $\Omega_3 - \Omega_1 - \omega_1 = \Omega_3 - \Omega_2 - \omega_2 = \Delta$.

Parameters g_a are the constants of atom-mode coupling.

It can be shown that the operators

$$M_a = b^{+}_{a}b_a - R_{aa}, \quad C = \sum_{a=1}^{2} g_a(b_a R_{3a} + b^{+}_{a}R_{a3}) - \Delta(R_{11} + R_{22}) \tag{5}$$

commute with the Hamiltonian (1) and with each other. They are therefore constant operators. Let us introduce the following auxiliary operators:

$$V_a = g_a b_a R_{aa} + g_{\bar{a}} b_{\bar{a}} R_{a\bar{a}}, \tag{6}$$

where $\bar{a} = 2$ for $a = 1$ and $\bar{a} = 1$ when $a = 2$. Using (4) one can obtain now the equations of motion for the operators R_{a3} and V_a

$$\frac{d}{dt}R_{a3} + i(\omega_a - C)R_{a3} = -iV_a, \quad \frac{d}{dt}V_a + i(\omega_a - C - \Delta)V_a = -i(\lambda_0^2 + g_a^2)R_{a3}, \tag{7}$$

where λ_0 is determined by

$$\lambda_0^2 \equiv \lambda_1^2 + \lambda_2^2, \quad \lambda_a \equiv g_a(M_a + 1)^{1/2}. \tag{8}$$

The operators λ_a can be considered as the quantum expressions of the one-photon Rabi frequencies in the system/13/. From (7) we obtain for R_{a3} the following equation

$$\{[\frac{d}{dt} + i(\omega_a - C - \Delta)][\frac{d}{dt} + i(\omega_a - C)] + \lambda_0^2 + g_a^2\}R_{a3} = 0. \tag{9}$$

Its solution can be presented in the following form:

$$R_{a3}(t) = e^{i(C+\Delta/2-\omega_a)t}\cdot\{\cos(t\sqrt{\lambda^2 + g_a^2})\,R_{a3}(0) - i(\lambda^2 + g_a^2)^{-1/2}\sin(t\sqrt{\lambda^2 + g_a^2})\,(V_a(0) + \frac{\Delta}{2}R_{a3}(0))\}, \tag{10}$$

where $\lambda^2 = \lambda_0^2 + \frac{\Delta^2}{4}$. From the equation of motion for the photon amplitudes $\frac{d}{dt}b_a = -i\omega_a b_a - ig_a R_{a3}$ combined with expression (10) one can obtain the explicit time dependence for $b_a(t)$. After a number of cumbersome transformations this dependence can be represented in the form

$$b_a(t) = e^{i(C+\Delta-\omega_a)t}\{b_a(0) + B_{-a} + B_a - e^{-i(\frac{\Delta}{2} - \sqrt{\lambda^2+g_a^2})t}B_{-a} - e^{-i(\frac{\Delta}{2} + \sqrt{\lambda^2+g_a^2})t}B_a\}, \tag{11}$$

where for simplicity we have introduced the notation

$$B_{\mp a} = g_a\{(C + \frac{\Delta}{2} \pm \sqrt{\lambda^2 + g_a^2})(2\sqrt{\lambda^2 + g_a^2})\}^{-1}\cdot\{(\sqrt{\lambda^2 + g_a^2} \mp \frac{\Delta}{2})R_{a3}(0) \mp V_a(0)\}. \tag{12}$$

It should be noted that

$$[\lambda^2, R_{a3}] = -g_a^2 R_{a3}, \quad [\lambda^2, V_a] = -g_a^2 V_a. \tag{13}$$

So, for an arbitrary operator function $f(\lambda^2)$ we have

$$f(\lambda^2)R_{a3} = R_{a3}f(\lambda^2 - g_a^2), \quad f(\lambda^2)V_a = V_a f(\lambda^2 - g_a^2). \tag{14}$$

Then, expression (10) can be rewritten in the form

$$R_{a3}(t) = e^{i(C+\frac{\Delta}{2}-\omega_a)t}\cdot\{R_{a3}(0)\cos\lambda t - i(V_a(0) + \frac{\Delta}{2}R_{a3}(0))\frac{\sin\lambda t}{\lambda}\}. \tag{15}$$

Analogously, with the help of eq. (14) we can obtain from expressions (11) and (12)

$$b_a(t) = e^{i(C+\Delta-\omega_a)t}\cdot\{b_a(0) + B_{-a} + B_a - B_{-a}e^{i\lambda_- t} - B_a e^{-i\lambda_+ t}\}, \tag{16}$$

where

$$B_{\mp a} = g_a\{R_{a3}(0)\lambda_{\mp} \mp V_a(0)\}[2\lambda(C \pm \lambda_{\pm})]^{-1} \tag{17}$$

and the operators λ_+, λ_- are determined by $\lambda_{\pm} = \lambda \pm \Delta/2$.

Expressions (15) and (16) represent our main results. They describe the time dependence of the atomic transition and photon operators. Using these results we can calculate various correlation functions and averages of photon numbers, level populations and atomic dipole moments. Below, we shall consider some consequences.

As first, we note that the model of a two-level atom interacting with a single mode radiation field is a particular case of the model considered here. In this case we have either $g_2 = 0$ or $g_1 = 0$ and expressions (15), (16) can be then reduced to the results of paper[12].

For the 3-th level population operator $R_{33}(t)$ from (15) and the relation $R_{33} = R_{3\alpha}R_{\alpha 3}$ we get

$$R_{33}(t) - -\{u(\cos 2\lambda t - 1) + v \sin 2\lambda t\} + R_{33}(0). \tag{18}$$

Here

$$u = \{\lambda_1^2 R_{11}(0) + \lambda_2^2 R_{22}(0) - \lambda_0^2 R_{33}(0) + g_1 g_2 B(0) + \frac{\Delta}{2}(g_1 C_1(0) + g_2 C_2(0))\}(2\lambda^2)^{-1},$$

$$v = -\{g_1 A_1(0) + g_2 A_2(0)\}(2\lambda)^{-1} \tag{19}$$

and

$$C_\alpha \equiv b_\alpha^+ R_{\alpha 3} + b_\alpha R_{3\alpha}, \quad A_\alpha \equiv i(b_\alpha^+ R_{\alpha 3} - b_\alpha R_{3\alpha}), \quad B \equiv b_1^+ b_2 R_{12} + b_1 b_2^+ R_{21}. \tag{20}$$

From exp.(16) for the photon-number operator $N_\alpha \equiv b_\alpha^+ b_\alpha$ we obtain

$$N_\alpha(t) = \mu_\alpha(\cos\lambda_+ t - 1) + \beta_\alpha \sin\lambda_+ t + \mu_{-\alpha}(\cos\lambda_- t - 1) + \beta_{-\alpha}\sin\lambda_- t + u_\alpha(\cos 2\lambda t - 1) +$$
$$+ v_\alpha \sin 2\lambda t + N_\alpha(0), \tag{21}$$

where $u_\alpha = (\lambda_\alpha/\lambda_0)^2 u$, $v_\alpha = (\lambda_\alpha/\lambda_0)^2 v$,

$$\mu_{\pm\alpha} = (-1)^\alpha \{(R_{22}(0) - R_{11}(0))\lambda_1^2\lambda_2^2(\lambda\lambda_0^2\lambda_\pm)^{-1} + g_1 g_2 B(0)(\lambda_1^2 - \lambda_2^2)(2\lambda\lambda_0^2\lambda_\pm)^{-1}$$
$$\pm(\lambda_1^2 g_2 C_2(0) - \lambda_2^2 g_1 C_1(0))(2\lambda\lambda_0^2)^{-1}\},$$

$$\beta_{\pm\alpha} = (-1)^\alpha \{(\lambda_2^2 g_1 A_1(0) - \lambda_1^2 g_2 A_2(0))(2\lambda\lambda_0^2)^{-1} \pm g_1 g_2 D(0)(2\lambda\lambda_\pm)^{-1}\}, \tag{22}$$

and $D \equiv i(b_1^+ b_2 R_{12} - b_1 b_2^+ R_{21})$.

Using the conservation law (5): $N_\alpha(t) \equiv R_{\alpha\alpha}(t) + M_\alpha$ and the relation $R_{\alpha\alpha}R_{\alpha\alpha} \equiv R_{\alpha\alpha}$ from (21) we obtain:

$$R_{\alpha\alpha}(t) = \mu_\alpha(\cos\lambda_+ t - 1) + \beta_\alpha \sin\lambda_+ t + \mu_{-\alpha}(\cos\lambda_- t - 1) + \beta_{-\alpha}\sin\lambda_- t + u_\alpha(\cos 2\lambda t - 1) +$$
$$+ v_\alpha \sin 2\lambda t + R_{\alpha\alpha}(0), \tag{23}$$

and also

$$N_\alpha^m(t) = M_\alpha^m + \{(M_\alpha + 1)^m - M_\alpha^m\}R_{\alpha\alpha}(t), \tag{24}$$

here $\alpha = 1,2$ and m is an arbitrary natural number (m = 1,2, 3,...). Expression (24) is useful for the investigation of photon statistics.

It is easily seen that the operators λ_+, λ_-, and 2λ define the frequencies of two-photon Rabi oscillations of level populations and photon numbers in the system[7-11]. At one-photon resonance $\Delta = 0$ expressions (18), (21) and (23) coincide with the results of ref.[11]. In this case there are two branches of the two-photon Rabi frequencies defined by the operators λ and 2λ[11]. It should be noted that the existence of the "soft branch" λ is a characteristic feature of the three-level system. Such a kind of oscillation frequencies is absent in the two-level system[13,14]. Our present results (18), (21) and (23) show that the detuning Δ in the case of two-photon resonance leads to the splitting of the "soft branch" λ to two branches characterized by the frequency operators $\lambda_+ = \lambda + \Delta/2$ and $\lambda_- = \lambda - \Delta/2$. This conclusion of the fully quantized theory is in accord with the results of the semiclassical theory[9].

Thus, in this paper we derived in the Heisenberg picture the explicit expressions of the time dependence for the atomic transition operators, photon amplitudes, and also for the level population and photon-number operators. The frequency operators λ_+, λ_-, 2λ of two-photon Rabi oscillations have been defined. Our results can be generalized to the other types of three-level atoms[8]. Further discussions will be made in a future publication.

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Боголюбов Н.Н. /мл./, Фам Ле Киен, Шумовский А.С. E17-84-637
Точные результаты для модели трехуровневого атома

Строго исследована модель трехуровневого атома, взаимодействующего с двумя модами квантованного поля излучения. Получена временная зависимость операторов переходов, фотонных амплитуд, населенностей уровней и чисел фотонов в представлении Гейзенберга. Показано, что учет наличия расстройки мод приводит к расщеплению при условии двухфотонного резонанса "мягкой ветви" частот нелинейных осцилляций в системе.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1984

Bogolubov N.N., Jr., Fam Le Kien, Shumovsky A.S. E17-84-637
Exact Results for a Model of a Three-Level Atom

For a three-level atom interacting with two modes of the quantized radiation field the time dependence of transition operators, photon amplitudes, and of operators of level populations and photon numbers is examined rigorously. It is shown that the availability of a mode detuning leads to the splitting of the "soft branch" of frequencies of two-photon Rabi oscillations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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