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A SIMPLE SOLUTION OF THE TWO-DIMENSIONAL ISING MODEL ON A TORUS VIA GRASSMANN INTEGRALS

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The two-dimensional Ising model has first been solved by Onsager in his famous paper[1], where an explicit expression for the free energy of a system (the logarithm of the partition function) has been derived in the limit of an infinite lattice "$N \to \infty$". Onsager's approach was simplified by Kaufman in ref.[2]. In this paper, in particular, an exact solution was first obtained for the model on a finite lattice with periodical boundary conditions (the case being equivalent to the lattice situated on a torus). Later on the theory of the two-dimensional Ising model has been developed by many authors. A detailed discussion of the canonical approaches and references to the original papers one can find, e.g., in the basic monographs[4,5]. In the context of the present paper we would like to point out also refs.[6-10].

As regards the two-dimensional model in a nonzero field and the three-dimensional Ising model, no exact results have been obtained yet for these cases. Moreover, even the availavle canonical solutions of the standard case (dimension two, zero field) seem to be complicated enough and usually involve one or another artificial passage, so making formidable the attempts to analyse unsolved models. For this reason it seems to be of interest to search for simple and sufficiently general approaches in the theory of the Ising-type systems.

Recently in ref.[10] a very simple solution was proposed for the two-dimensional model in a zero field for the case of infinite lattice $(N \to \infty)$. In the present paper we show how the method can be modified in order to obtain an exact solution for finite N.

The problem is solved by constructing the Grassmann integral representation for the partition function Q. We shall derive first a "mixed" representation for Q involving both the original spin variables and new Grassmann variables, in which the spin variables can be easily eliminated (see eq. (17)). To this end we make use of the Grassmann factorization of the Boltzmann weights and the mirror ordering of the arising Grassmann factors[10]. As compared to the case $N \to \infty$, the solution for finite N involves, in particular, a special ordering of the boundary Grassmann factors and makes use of a general identity for Grassmann functions proved in Appendix A. Spin variables being eliminated, the partition function is represented as a sum of four Grassmann integrals of the Gaussian type (20). The integrals are evaluated by standard methods, yielding the final explicit expression for the partition function (21).

Just as in the case $N \to \infty$, the solution of the model on finite lattice appears to be very simple.

Other solutions of the two-dimensional Ising model via Grassmann integrals have been proposed earlier in refs./6-8/ 1) (for comments see the end of the article; for a discussion and additional references see also ref./10/).

The model describes the rectangular planar lattice of spin variables $x_{mn} = \pm 1$ disposed at sites (m,n) and interacting with the nearest neighbours. The sites are labelled by pairs of integers: m,n=1,2,3,...,L; L being the length of the lattice side, $N=L^2$ the total number of spins. The Hamiltonian is given by:

$$H = -\sum_{m=1}^{L}\sum_{n=1}^{L}\left(J_1 x_{mn}x_{m+1n} + J_2 x_{mn}x_{mn+1}\right), \quad (1)$$

where J_1 and J_2 are the energies of spin interaction. The partition function, up to a numerical factor, is given by:

$$Q = \underset{(x)}{Sp}\left\{\prod_{m=1}^{L}\prod_{n=1}^{L}\left(1+t_1 x_{mn}x_{m+1n}\right)\left(1+t_2 x_{mn}x_{mn+1}\right)\right\}, \quad (2a)$$

where $t_1 = th\,(J_1/\theta)$, $t_2 = th\,(J_2/\theta)$, $\theta \equiv kT$ being the temperature, and symbol "Sp" means the normalized summation over the states of spin variables:

$$\underset{(x)}{Sp}\left\{\dots\right\} = \prod_{m=1}^{L}\prod_{n=1}^{L}\frac{1}{2}\sum_{x_{mn}=\pm 1}\left[\dots\right]. \quad (2b)$$

As usual, we shall assume the periodical boundary conditions for spin variables to hold true:

$$\underset{(m=1,2,\dots,L)}{x_{mL+1} \equiv x_{m1}}, \quad \underset{(n=1,2,\dots,L)}{x_{L+1n} \equiv x_{1n}}. \quad (2c)$$

These conditions attach to the lattice the topology of a torus. For a more detailed introduction to the model see, e.g., textbooks/11,12/, and other papers at the list of references.

The problem is to perform the averaging over spin variables $\{x_{mn}\}$ and to find an explicit expression for Q as a function of parameters t_1, t_2 and the total number of spins $N \equiv L^2$.

Let us bring into correspondence to the lattice sites $2N$ pairs of the conjugated anticommuting Grassmann variables:

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1) In view of the forthcoming consideration let us note that refs./6,7/ contain a detailed discussion on the standard properties of Grassmann variables and integrals.

$$\left\{a_{mn},\, a^*_{mn}\, ;\, b_{mn},\, b^*_{mn}\right\}, \quad (3)$$

Introduce for every pair an integral with the Gaussian weight:

$$\underset{(a_{mn})}{Sp}\left[\dots\right] = \int da^*_{mn}\int da_{mn}\, e^{a_{mn}a^*_{mn}}\left[\dots\right], \quad (4a)$$

and analogously for b_{mn}, b^*_{mn}. Introduce also the total integral over all pairs:

$$\underset{(a,b)}{Sp}\left\{\dots\right\} = \prod_{m=1}^{L}\prod_{n=1}^{L}\underset{(a_{mn})}{Sp}\;\underset{(b_{mn})}{Sp}\left[\dots\right]. \quad (4b)$$

We use here the standard variables and integrals of the Grassmann analysis (for details see, e.g., refs./6,7/). All the variables completely anticommute (to zero), their squares are zeros. Basic relations for integral over one variable are: $\int da\cdot 1 = 0$, $\int da\cdot a = 1$. The integral symbols anticommute with each other and with the variables. Note that the doubled symbols like aa^*, ab, $\int da^*\int da, \dots$ totally commute with any other symbol.

From the basic relations for one variable, $\int da\cdot 1 = 0$, $\int da\cdot a = 0$, one can easily obtain the rules for evaluating averages (4):

$$\underset{(a)}{Sp}\left[aa^*\right] = -\underset{(a)}{Sp}\left[a^*a\right] = 1, \quad \underset{(a)}{Sp}\left[1\right] = 1,$$
$$\underset{(a)}{Sp}\left[a\right] = \underset{(a)}{Sp}\left[a^*\right] = 0, \quad (5)$$

where (a, a^*) is one of the conjugated pairs in (3). When checking (5) it should be taken into account that $\exp\,(aa^*) = 1 + aa^*$, since $(aa^*)^2 \equiv 0$.

Let us introduce a short-hand notation for the Boltzmann weights occurring in eq. (2a):

$$\Psi^{(1)}_{mn} = 1 + t_1 x_{mn}x_{m+1n}, \quad \Psi^{(2)}_{mn} = 1 + t_2 x_{mn}x_{mn+1}. \quad (6a)$$

We shall use the selection rules given by (5) in order to represent these weights in a factorized form. Introduce Grassmann factors:

$$A_{mn} = 1 + a_{mn}x_{mn}, \quad A^*_{mn} = 1 + t_1 a^*_{m-1n}x_{mn},$$
$$B_{mn} = 1 + b_{mn}x_{mn}, \quad B^*_{mn} = 1 + t_2 b^*_{mn-1}x_{mn}, \quad (6b)$$

here indices mn are in a correspondence to those of x_{mn}. Now, it is easily seen that due to the rules (5) the Boltzmann weights (6a) can be represented in the following factorized form:

$$\Psi^{(1)}_{mn} = \underset{(a_{mn})}{Sp}\left[A_{mn}A^*_{m+1n}\right], \quad \Psi^{(2)}_{mn} = \underset{(b_{mn})}{Sp}\left[B_{mn}B^*_{mn+1}\right]. \quad (6c)$$

Our aim now is to pass from the original representation for Q(2a) with the averaging over discrete spin variables (2b), to the representation expressed in terms of the Grassmann variables with averaging (4). Starting with the factorization like in (6c), we shall obtain first a "mixed" representation, in which spin variables can be eliminated. To this end it is necessary to achieve a special ordering of the Grassmann factors to have four factors with the same variable X_{mn} situated near each other when summing over $\{X_{mn}\}$.

A general motivation of the approach is exhibited especially well when treating the case $N\to\infty$ (ref./10/). So, we shall not stop here in more detail on a general trend of the method, making some comments when necessary.

Let us start with the preparation of the boundary Boltzmann weights involving indices $m = L+1$ and $n = L+1$ [2]. In this case it is convenient to use a slightly differing factorization

$$\Psi_{Ln}^{(1)} = \underset{(a_{Ln})}{Sp}\,[A_{1n}^{*}A_{Ln}],\quad \Psi_{mL}^{(2)} = \underset{(b_{mL})}{Sp}\,[B_{m1}^{*}B_{mL}], \tag{7a}$$

with A_{Ln}, B_{mL} given by standard relations (6b), and A_{1n}^{*}, B_{m1}^{*} defined as follows:

$$A_{1n}^{*} = 1+t_1 a_{on}^{*}X_{1n},\quad B_{m1}^{*} = 1+t_2 b_{mo}^{*}X_{m1}, \tag{7b}$$

with

$$a_{on}^{*} = -a_{Ln}^{*},\quad b_{mo}^{*} = -b_{mL}^{*}. \tag{7c}$$

Here we have taken into account the boundary conditions (2c) and have replaced in (7b) X_{L+1n}, X_{mL+1} by X_{1n}, X_{m1}. The sign "minus" in (7c) is due to inverse order of a, a^{*} and b, b^{*} in (7a) as compared to (6b). We have introduced the new notation a_{on}^{*}, b_{mo}^{*} to put factors (7b) into a standard form of eq. (6b), all the indices in eq. (7b) being arranged in the same manner as in eq. (6b).

Now, making use of factorization (7) and appealing to the idea of the mirror ordering/10/, we may write down the following representa-

2) The point of the matter in the forth-coming transformations with the boundary weights in eqs. (7)-(11) may be seen from a general scheme of the method (ref./10/). The mirror variant of ordering in eq. (8) is needed to place later on factors A_{Ln} at suitable positions. Some other peculiarities are inspired by wishing to get for the Grassmann factors the (symbolic) order $A^{*}B^{*}AB$, resulting from the preparation of the "inner" weights.

tions for the products of the boundary Boltzmann weights:

$$\prod_{n=1}^{L}\Psi_{Ln}^{(1)} = \Big\{\prod_{n=1}^{L}\overleftarrow{A_{1n}^{*}}^{\,n}\cdot\prod_{n=1}^{L}\overrightarrow{A_{Ln}}^{\,n}\Big\}, \tag{8a}$$

$$\prod_{m=1}^{L}\Psi_{mL}^{(2)} = \Big\{\prod_{m=1}^{L}\overrightarrow{B_{m1}^{*}}^{\,m}\cdot\prod_{m=1}^{L}\overleftarrow{B_{mL}}^{\,m}\Big\}, \tag{8b}$$

where the products in every relation are ordered with n and m increasing or decreasing in opposite directions. (That is why we use for such representations the term "mirror ordering"/10/.) For simplicity we have omitted in eqs.(8) as well as in the forthcoming intermediate transformations of eqs.(9)-(16) the symbol of averaging over Grassmann pairs in the r.h. sides[3].

Representation (8a) can be derived as follows: take the Boltzmann weight $(\Psi_{Ln}^{(1)})_{n=1} = A_{11}^{*}A_{L1}$ and sandwich the Boltzmann weight $(\Psi_{Ln}^{(1)})_{n=2} = A_{12}^{*}A_{L2}$ between factors A_{11}^{*} and A_{L1} of $\Psi_{Ln}^{(1)}$. Repeat the procedure step by step for $n = 3,4,\dots,L$, placing every time the next in turn Boltzmann weight taken in a factorized form between already existing Grassmann products. The result will be eq. (8a). Representation (8b) may be obtained in an analogous way.

Let us continue by sandwiching the product (8a) as a whole between the ordered products of eq. (8b). This yields the following representation for the total product of all the boundary Boltzmann weights:

$$\Psi_{\Gamma} \equiv \prod_{n=1}^{L}\Psi_{Ln}^{(1)}\cdot\prod_{m=1}^{L}\Psi_{mL}^{(2)} =$$

$$= \prod_{m=1}^{L}\overrightarrow{B_{m1}^{*}}^{\,m}\cdot\prod_{n=1}^{L}\overleftarrow{A_{1n}^{*}}^{\,n}\cdot\prod_{n=1}^{L}\overrightarrow{A_{Ln}}^{\,n}\cdot\prod_{m=1}^{L}\overleftarrow{B_{mL}}^{\,m}. \tag{9}$$

For convenience it will be suitable to transpose now the first two ordered products in eq. (9). The reason is that instead of (symbolic) ordering $B^{*}A^{*}AB$ of eq. (9) we shall need below the ordering like $A^{*}B^{*}AB$ arising for the inner Boltzmann weights (see also ref./10/). To this end let us make use of the identity (A.7) that is a particular case of a quite general identity for Grassmann functions (A.3) (Appendix A).

3) Alternatively, one may simply assume that all the transformations are realized under the sign of the total averaging (4b). Such an averaging will act only on the Grassmann pairs occurring in the r.h.sides, the l.h.sides being unchanged at all.

The symbolic structure of identities (A.3) and (A.7) is the following:

$$B^+A^+ = \frac{1}{2}\left[A^+B^+ + A^+B^- + A^-B^+ - A^-B^-\right], \tag{10}$$

where the sign "minus", if one applies eq. (10) for transposition of the first two products in eq. (9), will correspond to the alternation of signs of a^*_{on}, b^*_{mo} in the Grassmann forms A^*_{1n}, B^*_{m1} (see eqs. (7)). However, we may note that instead we may simply change the signs in the interpretation of a^*_{on}, b^*_{mo} through a^*_{Ln}, b^*_{mL} in relations (7c), keeping forms A^*_{1n}, B^*_{m1} unchanged.

On the basis of eqs. (9) and (10) we then find the following representation for the total boundary Boltzmann weight:

$$\Psi_\Gamma = \frac{1}{2}\left[\Psi_\gamma|_{\Gamma_1} + \Psi_\gamma|_{\Gamma_2} + \Psi_\gamma|_{\Gamma_3} - \Psi_\gamma|_{\Gamma_4}\right], \tag{11a}$$

where Ψ_γ is given by

$$\Psi_\gamma = \prod_{n=1}^{L} \overleftarrow{A^*_{1n}}^{\,n} \cdot \prod_{m=1}^{L} \overrightarrow{B^*_{m1}}^{\,m} \cdot \prod_{n=1}^{L} \overrightarrow{A_{Ln}}^{\,n} \cdot \prod_{m=1}^{L} \overleftarrow{B_{mL}}^{\,m}, \tag{11b}$$

and $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ mean the following boundary conditions for Grassmann variables:

$$\begin{aligned}
\Gamma_1 &= \{a^*_{on} = -a^*_{Ln},\ b^*_{mo} = -b^*_{mL}\},\\
\Gamma_2 &= \{a^*_{on} = -a^*_{Ln},\ b^*_{mo} = b^*_{mL}\},\\
\Gamma_3 &= \{a^*_{on} = a^*_{Ln},\ b^*_{mo} = -b^*_{mL}\},\\
\Gamma_4 &= \{a^*_{on} = a^*_{Ln},\ b^*_{mo} = b^*_{mL}\},
\end{aligned} \tag{11c}$$

the factors A^*_{1n}, B^*_{m1} in (11b) being defined as before (eqs. (7b)).

The boundary Boltzmann weights are now completely prepared, and we may proceed to introduce in (11) the remaining inner weights by a method analogous to that of ref. /10/. Since further transformations and averaging over spin variables will not affect the interpretation of boundary conditions in (11c), we shall deal directrly with Ψ_γ (11b).

Starting from (6c) and using again the mirror ordering, we may write the representation:

$$\Psi^{(2)}_n \equiv \prod_{m=1}^{L} \Psi^{(2)}_{mn} = \mathfrak{B}_n \mathfrak{B}^*_{n+1}, \tag{12a}$$

with

$$\mathfrak{B}_n = \prod_{m=1}^{L} \overleftarrow{B_{mn}}^{\,m}, \quad \mathfrak{B}^*_n = \prod_{m=1}^{L} \overrightarrow{B^*_{mn}}^{\,m}. \tag{12b}$$

This representation may be derived by analogy with eqs. (8).

Using short-hand notation (12b), we may now rewrite expression (11b) in the form:

$$\Psi_\gamma = \mathfrak{A}^*_1 \cdot \mathfrak{B}^*_1 \cdot \prod_{n=1}^{L} \overrightarrow{A_{Ln}}^{\,n} \cdot \mathfrak{B}_L, \quad \mathfrak{A}^*_1 \equiv \prod_{n=1}^{L} \overleftarrow{A^*_{Ln}}^{\,n}. \tag{13}$$

Further, let us multiply Ψ_γ by product of weights $\Psi^{(2)}_n$ over $n = 1, 2, 3, \ldots, L-1$, inserting $\Psi^{(2)}_n$ in a factorized form (12a) into the third term of (13) and rearranging the product[4]:

$$\left[\prod_{n=1}^{L-1} \Psi^{(2)}_n\right]\Psi_\gamma \equiv \left[\prod_{m=1}^{L}\prod_{n=1}^{L-1} \Psi^{(2)}_{mn}\right]\Psi_\gamma = \tag{14a}$$

$$= \mathfrak{A}^*_1 \cdot \mathfrak{B}^*_1 \cdot \left[\prod_{n=1}^{L-1} \overrightarrow{A_{Ln}\Psi^{(2)}_n}^{\,n}\right] \cdot A_{LL} \cdot \mathfrak{B}_L = \tag{14b}$$

$$= \mathfrak{A}^*_1 \cdot \mathfrak{B}^*_1 \cdot \left[\prod_{n=1}^{L-1} \overrightarrow{A_{Ln}\mathfrak{B}_n\mathfrak{B}^*_{n+1}}^{\,n}\right] \cdot A_{LL} \cdot \mathfrak{B}_L = \tag{14c}$$

$$= \mathfrak{A}^*_1 \cdot \left[\prod_{n=1}^{L} \overrightarrow{\mathfrak{B}^*_n A_{Ln} \mathfrak{B}_n}^{\,n}\right]. \tag{14d}$$

In (14d) we have rearranged the product joining together the neighbouring terms with the same index n.

We may introduce now the remaining Boltzmann weights $\Psi^{(1)}_{mn}$ in factorized form (6c) into the ordered products $\mathfrak{B}^*_n$ (see (12b)) occurring in (14d). It will be suitable to consider at once the complex $\mathfrak{B}^*_n A_{Ln}$. So, we write:

$$\prod_{m=1}^{L} \Psi^{(1)}_{mn} \cdot \mathfrak{B}^*_n A_{Ln} = \tag{15a}$$

$$= \prod_{m=1}^{L-1} \overrightarrow{B^*_{mn}\left[A_{mn}A^*_{m+1\,n}\right]}^{\,m} \cdot B^*_{Ln} A_{Ln} = \tag{15b}$$

$$= B^*_{1n} A_{1n} \left[\prod_{m=2}^{L} \overrightarrow{A^*_{mn} B^*_{mn} A_{mn}}^{\,m}\right], \tag{15c}$$

as usual, we have jointed together neighbouring terms with the same index mn.

Inserting (15c) into (14d) and taking into account the explicit form of $\mathfrak{A}^*_1$ and $\mathfrak{B}_n$, we find:

$$\left[\prod_{n=1}^{L}\prod_{m=1}^{L-1} \Psi^{(1)}_{mn}\right] \cdot \left[\prod_{n=1}^{L-1}\prod_{m=1}^{L} \Psi^{(2)}_{mn}\right] \cdot \Psi_\gamma = \tag{16a}$$

$$= \mathfrak{A}^*_1 \cdot \prod_{n=1}^{L} \overrightarrow{\left\{B^*_{1n} A_{1n} \left[\prod_{m=2}^{L} \overrightarrow{A^*_{mn} B^*_{mn} A_{mn}}^{\,m}\right] \mathfrak{B}_n\right\}}^{\,n} = \tag{16b}$$

$$= \prod_{n=1}^{L} \overleftarrow{A^*_{1n}}^{\,n} \cdot \prod_{n=1}^{L} \overrightarrow{\left\{B^*_{1n} A_{1n} \left[\prod_{m=2}^{L} \overrightarrow{A^*_{mn} B^*_{mn} A_{mn}}^{\,m} \cdot \prod_{m=2}^{L} \overleftarrow{B_{mn}}^{\,m}\right] B_{1n}\right\}}^{\,n}. \tag{16c}$$

4) A basic tendency in further transformations is to combine factors with the same index mn (i.e., the same variable X_{mn}) that is needed below when eliminating spin variables.

In the square brackets of (16a) all the inner Boltzmann weights are represented, while the product of the boundary Boltzmann weights is related to Ψ_γ through eqs.(11). So, we obtain for the total product of the Boltzmann weights:

$$\Psi \equiv \prod_{m=1}^{L} \prod_{n=1}^{L} (1+t_1 x_{mn} x_{m+1n})(1+t_2 x_{mn} x_{mn+1}) = \quad (17a)$$

$$= \frac{1}{2}\left[\Phi|_{\Gamma_1} + \Phi|_{\Gamma_2} + \Phi|_{\Gamma_3} - \Phi|_{\Gamma_4} \right], \quad (17b)$$

with

$$\Phi \equiv \underset{(a,b)}{Sp} \left\{ \overleftarrow{\prod_{n=1}^{L}} A^*_{1n} \cdot \overrightarrow{\prod_{n=1}^{L}} B^*_{1n} A_{1n} \left[\overrightarrow{\prod_{m=2}^{L}} A^*_{mn} B^*_{mn} A_{mn} \cdot \overleftarrow{\prod_{m=2}^{L}} B_{mn} \right] B_{1n} \right\}, \quad (17c)$$

and with the boundary conditions $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ given by (11c). In (17c) we have written again explicitly the symbol of Grassmann averaging (4).

The product of the Boltzmann weights Ψ is the Gibbs density matrix of the two-dimensional Ising model on a torus in the absence of an external field. The partition function Q may be found by averaging Ψ over spin variables (eq. (2)). Note that eq. (17) is the exact mathematical identity derived without any approximations.

The density matrix is represented in (17) in a special factorized form suitable for summing over $\{x_{mn}\}$. As we shall now see, the task reduces to the independent summing over separate x_{mn} as follows[5)]:

$$\underset{(x_{mn})}{Sp} \left[A^*_{mn} B^*_{mn} A_{mn} B_{mn} \right] = \quad (18a)$$

$$= \frac{1}{2} \sum_{x_{mn}=\pm 1} \left[(1+t_1 a^*_{m-1n} x_{mn})(1+t_2 b^*_{mn-1} x_{mn})(1+a_{mn} x_{mn})(1+b_{mn} x_{mn}) \right] = \quad (18b)$$

$$= (1+t_1 t_2 a^*_{m-1n} b^*_{mn-1})(1+a_{mn} b_{mn}) + (t_1 a^*_{m-1n} + t_2 b^*_{mn-1})(a_{mn} + b_{mn}) = \quad (18c)$$

$$= \exp\left[t_1 t_2 a^*_{m-1n} b^*_{mn-1} + a_{mn} b_{mn} + (t_1 a^*_{m-1n} + t_2 b^*_{mn-1})(a_{mn} + b_{mn}) \right] \equiv G_{mn}. \quad (18d)$$

Really, consider the expression in squared brackets of eq. (17c) for given fixed n [6)]. At the "junction" of the m-products there are four neighbouring factors (18a) with the same index mn (given n, $m = L$). These four factors are the only ones that involve the variable $(x_{mn})_{m=L}$. Summing over $(x_{mn})_{m=L}$, in accordance with eqs. (18) we obtain totally commuting Grassmann term (18d), with the corresponding index, that may be taken "out of brackets". Thereupon the procedure should be repeated for $m = L-1, L-2, \dots, m = 2$ with a given n, and all over again for other n. All the squared brackets will then disappear in (17c), yielding the product of factors (18d) over m=2,3,...,L and n=1,2,...L. The remainder will be:

$$\underset{(a,b)}{Sp} \left\{ \overleftarrow{\prod_{n=1}^{L}} A^*_{1n} \cdot \overrightarrow{\prod_{n=1}^{L}} B^*_{1n} A_{1n} B_{1n} \right\}, \quad (19)$$

and the same method of averaging over $\{x_{1n}\}$ yields the product of factors (18d) over $m = 1$, $n = 1, 2, 3, \dots, L$.

We thus obtain the following exact purely Grassmann representation for the partition function:

$$Q = \frac{1}{2}\left[G|_{\Gamma_1} + G|_{\Gamma_2} + G|_{\Gamma_3} - G|_{\Gamma_4} \right], \quad (20a)$$

with

$$G = \underset{(x)}{Sp}\{\Phi\} = \underset{(a,b)}{Sp}\left\{ \prod_{m=1}^{L} \prod_{n=1}^{L} G_{mn} \right\} =$$

$$= \int \prod_{m=1}^{L} \prod_{n=1}^{L} da^*_{mn}\, da_{mn}\, db^*_{mn}\, db_{mn} \cdot \exp\left\{ \sum_{m=1}^{L} \sum_{n=1}^{L} \left[a_{mn} a^*_{mn} + b_{mn} b^*_{mn} + a_{mn} b_{mn} + t_1 t_2 a^*_{m-1n} b^*_{mn-1} + (t_1 a^*_{m-1n} + t_2 b^*_{mn-1})(a_{mn} + b_{mn}) \right] \right\}, \quad (20b)$$

and with the boundary conditions $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ given by (11c).

As is seen, the partition function appears to be a sum of four Gaussian integrals over Grassmann variables. In accordance with a general theory (see, e.g., refs./6,7/) every integral is expressed as the determinant of the matrix of the quadratic form in the exponential. Boundary conditions (11c) say that the matrix in all cases is the exactly cyclic or anticyclic block matrix, and hence it can be brought into the block-diagonal form by the Fourier transformation. It may be noted that the problem is reduced to the diagonalization of such matrices in all the other approaches to the model as well[/1-9/ 7)].

So, the evaluation of integrals (20) is a standard technical task (for analogous calculations see, e.g., ref./6,7/). We shall not here reproduce these simple and standard manipulations in detail, restricting ourselves only noting some their peculiarities/6,7/. The Fourier

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5) Equivalence of (18c) and (18d) can be checked by a series expansion of the exponential.

6) In all four terms of (17b) summing over x_{mn} is performed in the same manner, independently of the boundary conditions.

7) For an exact solution in the case of N finite see refs./2,7,9/ and refs./4,5/.

transformation may be realized by going to the momentum space for Grassmann variables. For the sake of symmetry of the transformation it is convenient to consider first the squared integrals, paying later on attention to the correct choice of sign of their square roots. The transformation should be performed in agreement with the boundary conditions (11c). Finally, when doing all these it is suitable, for technical reasons, to enumerate the discrete indices m, n and their momentum counter-parts p, q by numbers $0,1,2,\dots,(L-1)$ (instead of $1,2,\dots,L$), and for the squared integrals by numbers $\pm 0, \pm 1, \pm 2, \dots, \pm(L-1)$.

The operation of "shifting" indices, $m \to m-1$, $n \to n-1$, in the momentum space is replaced by the phase factors like $\exp(\pm i 2\pi p/L)$, $\exp(\pm i 2\pi q/L)$, ensuring the factorization of the (squared) integrals into the products of the same-type integrals of low dimensionality which may be easily evaluated. As a result, we obtain the following explicit expression for the partition function of the two-dimensional Ising model on a torus in a zero external field:

$$Q = \frac{1}{2}\prod_{p=0}^{L-1}\prod_{q=0}^{L-1}\left[\lambda_0 - \lambda_1\cos\frac{2\pi p+\pi}{L} - \lambda_2\cos\frac{2\pi q+\pi}{L}\right]^{\frac{1}{2}} + \tag{21a}$$
$$+\frac{1}{2}\prod_{p=0}^{L-1}\prod_{q=0}^{L-1}\left[\lambda_0 - \lambda_1\cos\frac{2\pi p+\pi}{L} - \lambda_2\cos\frac{2\pi q}{L}\right]^{\frac{1}{2}} +$$
$$+\frac{1}{2}\prod_{p=0}^{L-1}\prod_{q=0}^{L-1}\left[\lambda_0 - \lambda_1\cos\frac{2\pi p}{L} - \lambda_2\cos\frac{2\pi q+\pi}{L}\right]^{\frac{1}{2}} -$$
$$-\,\mathrm{sgn}\left(\frac{\theta-\theta_c}{\theta_c}\right)\cdot\frac{1}{2}\prod_{p=0}^{L-1}\prod_{q=0}^{L-1}\left[\lambda_0 - \lambda_1\cos\frac{2\pi p}{L} - \lambda_2\cos\frac{2\pi q}{L}\right]^{\frac{1}{2}},$$

where

$$\lambda_0 = (1+t_1^2)(1+t_2^2),\quad \lambda_1 = 2t_1(1-t_2^2),\quad \lambda_2 = 2t_2(1-t_1^2). \tag{21b}$$

The four terms in eq. (21a) differ from each other only by the quasi-discrete arguments of the cosines. These differences are in a correspondence to different boundary conditions in (20a) and vanish as $N\to\infty$. The sign factor before the last term is +1 for $\theta > \theta_c$ and -1 for $\theta < \theta_c$, θ being the temperature of the system, θ_c being the critical temperature given by the well-known relation:

$$t_1 = \frac{1-t_2}{1+t_2}. \tag{21c}$$

This sign factor arises according to the standard reasonings when extracting the square root of $(G|_{\Gamma_4})^2$ /2,7,9/.

Passing in (21) to the limit of an infinite lattice, $N\equiv L^2\to\infty$, one obtains the famous Onsager expression for the free energy/1/:

$$f = \frac{1}{N}\ln Q\Big|_{N\to\infty} = \frac{1}{2}\int_0^{2\pi}\frac{d\varphi_1}{2\pi}\int_0^{2\pi}\frac{d\varphi_2}{2\pi}\ln\left[\lambda_0 - \lambda_1\cos\varphi_1 - \lambda_2\cos\varphi_2\right], \tag{22}$$

whence there follow the known properties of the model, in particular the phase transition at point (21c) with the logarithmic singularity in specific heat.

The method of the present paper can be applied also to compute the correlation functions, to solve the model on a triangular lattice, etc.[8)]

In all the canonical approaches to the two-dimensional Ising model the main and most labour-consuming part of the solution is to reduce the problem to evaluation of the determinant of a cyclic-type matrix or the corresponding Gaussian integral. In the present method the basic part of solution is the derivation of representation (20) for Q. This representation is obtained here in a very simple and elementary manner on the basis of the selection rules given by eqs. (5), without using any auxiliary mathematical tools. The main points of the solution are the Grassmann factorization of the Boltzmann weights and the principle of mirror ordering of the Grassmann factors. The third basic point of the method, needed in the considered case of a finite lattice on a torus, is the universal identity for Grassmann functions (A.3).

Grassmann representations similar to (20) have been derived earlier by several authors in refs./6-8/. However, the methods proposed in these papers were more complicated. Berezin/6/ has used the direct combinatorial approach (an analog of the classical combinatorial method/3/, closely related also to the fermionic approach by Green and Hurst/4/). The solution was given in ref./6/ for the case $N\to\infty$, and the problem of generalization of the solution to the case $N\neq\infty$ was pointed out by Berezin as a difficult task. An exact combinatorial solution for a finite lattice on a torus has been given by Popov in his book/7/. Such a solution indeed involves some nontrivial topological conjectures about closed path on a grassmannized torus. Another

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8) The computation of the correlation function can be reduced to the evaluation of the partition function in a particular case of inhomogeneous interaction. In the most general inhomogeneous case the Boltzmann weights in the original spin representation for Q (see eq. (2a)) are : $1+t^{(1)}_{m+1n}X_{mn}X_{m+1n}$, $1+t^{(2)}_{mn+1}X_{mn}X_{mn+1}$, where in view of eqs. (2c) $t^{(1)}_{L+1n}\equiv t^{(1)}_{1n}$, $t^{(2)}_{mL+1}\equiv t^{(2)}_{m1}$. One can then obtain the corresponding Grassmann representation for Q by replacing in the intermediate transformations and final expression (20) the complexes $t_1 a^*_{m-1n}$, $t_2 b^*_{mn-1}$, by $t^{(1)}_{mn} a^*_{m-1n}$, $t^{(2)}_{mn} b^*_{mn-1}$ (see also ref./10/).

approach was proposed by Fradkin and Shteingradt in ref.[8], whose first steps are very similar to the method of the present paper. As a starting point there was used a variant of the factorization, involving the Grassmann differentiation, closely related in its sense to factorization (6c). However, the remaining part of the solution was realized here in a different way, by introducing auxiliary Clifford variables in order to rearrange arising Grassmann factors. In ref.[8] the solution was given for the case $N \to \infty$.

It may be noted that the generalization of the method working for $N \to \infty$ to the case $N \neq \infty$ usually makes the solution much more complicated. A subtle nature of the exact solution in the case $N \neq \infty$ manifests itself outwardly in a characteristic expanding of Q into four terms (cf. eqs. (20), (21)). In the algebraic approach (see, e.g., refs.[2,9]) such an expansion arises from the analysis of eigenvalues of the transfer-matrix, in the combinatorial approach (see, e.g., ref.[7]) this follows from the analysis of the topology of closed paths. It is interesting to note that in the present solution the four terms in Q arise as a manifestation of a simple and quite general property of the Grassmann functions given by the identity (A.3)

In conclusion let us make some remarks concerning the problem in a nonzero field and the three-dimensional Ising model. One can apply the factorization principles in these cases as well. However, difficulties arise here with an appropriate ordering of the Grassmann factors, and straightforward attempts lead to non-Gaussian representations with "four-particle" terms in the exponential; such representations are not exactly solvable (in the sense that there are no regular calculational methods). We may note that in approaching nonsolved problems one may try to exploit as a starting point the exact factorized representation for the density matrix (17).

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Appendix A

Here we prove some universal identity for Grassmann functions, whose particular case is used in the text.

Consider a set of anticommuting Grassmann variables:

$$\{a_1, a_2, a_3, \ldots \ldots, a_{R-1}, a_R\}. \qquad (A.1)$$

Since the second and higher orders of these variables are zeros, any natural function defined on a set (A.1) (any "analytic" function) will be a finite polynomial with numerical coefficients. Consider two such functions: $f = f(a_1, \ldots, a_R)$ and $g = g(a_1, \ldots, a_R)$. Every function may involve in fact not all but only a subset of the variables. Let us also introduce the functions arising from f, g by changing the signs of the variables: $a_1 \to -a_1, \ldots, a_R \to -a_R$. It is convenient to introduce the short notation:

$$f^{\pm} = f(\pm a_1, \ldots, \pm a_R), \quad g^{\pm} = g(\pm a_1, \ldots, \pm a_R). \qquad (A.2)$$

Our statement is that the following identity holds:

$$f^+ g^+ = \frac{1}{2}\left[g^+ f^+ + g^+ f^- + g^- f^+ - g^- f^-\right]. \qquad (A.3)$$

Really, we may write: $f = \lambda_f + C_f$, $g = \lambda_g + C_g$, where λ_f, λ_g are sums of the odd terms of the polynomials (the terms involving products of an odd number of the variables), and C_f, C_g are sums of even terms. Evidently, we may write:

$$f^{\pm} = \lambda_f \pm C_f, \quad g^{\pm} = \lambda_g \pm C_g. \qquad (A.4)$$

On the other hand, we may note that C_f and C_g will anticommute with each other, while λ_f and λ_g will commute with each other and with C_f, C_g .

So, we may write:

$$\begin{aligned} f^+ g^+ &= (\lambda_f + C_f)(\lambda_g + C_g) = (\lambda_g + C_g)\lambda_f + (\lambda_g - C_g)C_f = \\ &= (\lambda_g + C_g)\frac{(\lambda_f + C_f) + (\lambda_f - C_f)}{2} + \\ &\qquad + (\lambda_g - C_g)\frac{(\lambda_f + C_f) - (\lambda_f - C_f)}{2} = \\ &= \frac{1}{2} g^+ (f^+ + f^-) + \frac{1}{2} g^- (f^+ - f^-), \end{aligned} \qquad (A.5)$$

where the last line proves the identity (A.3).

Keeping in mind applications to the Ising model, let us consider, for instance, the following expressions:

$$A^{\pm} = \prod_{m=1}^{M} (1 \pm \alpha_m), \quad B^{\pm} = \prod_{n=1}^{N} (1 \pm \beta_n), \qquad (A.6)$$

where α_m, β_n are some linear superpositions of the variables (A.1), and the products are assumed to be ordered in one or another fixed mode. Identity (A.3) then yields the relation:

$$\begin{aligned} &\prod_{m=1}^{M} (1 + \alpha_m) \cdot \prod_{n=1}^{N} (1 + \beta_n) = \\ &= \frac{1}{2}\Big[\prod_{n=1}^{N}(1+\beta_n) \cdot \prod_{m=1}^{M}(1+\alpha_m) + \prod_{n=1}^{N}(1+\beta_n) \cdot \prod_{m=1}^{M}(1-\alpha_m) + \\ &+ \prod_{n=1}^{N}(1-\beta_n) \cdot \prod_{m=1}^{M}(1+\alpha_m) - \prod_{n=1}^{N}(1-\beta_n) \cdot \prod_{m=1}^{M}(1-\alpha_m)\Big], \end{aligned} \qquad (A.7)$$

where the products on the r.h.s. are ordered just in the same manner as the products on the l.h.s. The identity has been used in the text when passing from eq. (9) to eq. (11).

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Простое решение двумерной модели Изинга на торе через грассмановы интегралы

Предложен простой способ точного решения двумерной модели Изинга на торе /решетка конечных размеров с периодическими граничными условиями/. Задача решается через построение грассманова представления для статистической суммы. Применяется зеркальная грассманова факторизация для больцмановских весов и специальное тождество для грассмановых функций. Метод обладает полной математической строгостью.

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Plechko V.N. E17-84-469
A Simple Solution of the Two-Dimensional Ising Model on a Torus via Grassman Integrals

A simple solution of the two-dimensional Ising model on a torus /a finite-size lattice with periodical boundary conditions/ is proposed. The problem is solved by constructing Grassmann representation for the partition function. To this end the mirror Grassmann factorization of the Boltzmann weights and a special identity for Grassmann functions are used. The method is completely rigorous in a mathematical sense.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1984