объединенный
институт
ядерных
исследований
дубна

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I.V.Polubarinov

ON TRANSFORMATION OF SUMS IN ISING MODEL

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1. Introduction. The Ising model has a long history (see refs.[1-24] and further references therein). One rigorous method for a lattice on a 2-dimensional torus (the transfer matrix method) has been developed by Onsager, Kaufman, Yang and others[1-4,10,11] and recently it has been considerably simplified by Valuev[18,19]. Other ways due to Hurst and Green avoid the matrix formulation and directly introduce a representation of partition functions via Fermi creation and annihilation operators or Grassmann variables[8,9,13-16]. A method of Fradkin and Steingradt [15] (they employ Grassmann variables) is self-consistent and especially simple. A calculation of the partition function in the 2-dimensional Ising model according to this method seems to be most concise. The method excludes a complicated combinatorial analysis, carried out earlier [6-9].

Below we translate the Fradkin and Steingradt method completely into terms of creation and annihilation operators that provides with some additional freedom (although both the languages are equivalent as a whole, see Appendix). In these terms, e.g., there is no need in "external" Clifford variables τ_i, used in ref.[15]. Instead we use the "interior" operator $\eta = (-1)^N$, where N is an operator of number of "particles". After exposing in sects. 2 and 3 the 1-dimensional Ising model without and with a magnetic field (in both the cases possible simplifications are indicated) we turn in sect.4 to the 2-dimensional model. It is demonstrated that the Fradkin-Steingradt transformation (F.-S. transformation) is rigorous for a 2-dimensional rectangular bounded lattice (as in fig.1, not on a torus or a cylinder) and in fact for bounded lattices of any dimensionality. Fradkin and Steingradt themselves started with a 2-dimensional lattice on a torus, for which their transformation is not rigorous (however, justifiable in a thermodynamic limit).

Finally, it is interesting to represent via the creation and annihilation operators partition functions in unsolved problems: the 2-dimensional Ising model with a magnetic field and 3-dimensional Ising model without and with a magnetic field. That is performed in sects. 5-7.

The partition function for the Ising model on a 2-dimensional rectangular bounded lattice (fig.1) is known to be representable as follows [9,13,14]

$$Z=G\langle 0|:\prod_{n=1}^{N}\prod_{m=1}^{M}\{1+\sqrt{t_1t_2}\,a^+(m-1,n)b^+(m,n-1)+\sqrt{t_1t_2}\,a(m,n)b(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_2}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)+\sqrt{t_2}\,b(m,n)]+$$
$$+t_1t_2\,a^+(m-1,n)b^+(m,n-1)a(m,n)b(m,n)\}:|0\rangle= \qquad (1.1.a)$$
$$=G\langle 0|:e^{\sum_{m=1}^{M}\sum_{n=1}^{N}A(m,n)}:|0\rangle, \qquad (1.1.b)$$
$$A(m,n)=\sqrt{t_1t_2}\,a^+(m-1,n)b^+(m,n-1)+\sqrt{t_1t_2}\,a(m,n)b(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_1}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)+\sqrt{t_2}\,b(m,n)],$$

where a, a^+, b, and b^+ are the Fermi annihilation and creation operators with the commutation relations

$$\{a(m,n),a^+(m',n')\}=\delta_{mm'}\delta_{nn'},\quad \{b(m,n),b^+(m',n')\}=\delta_{mm'}\delta_{nn'}, \qquad (1.2)$$

$\{\ \}$ means the anticommutator, other anticommutators equal zero. For the coefficient G see eq. (4.10) below. Here and in what follows $:\ :$ means anti-N-ordering (for the definition see Appendix). An original expression for the partition function (see eq. (4.1) below) can be represented in the form [9,13]

$$Z=G\sum_{k,l=0}^{\infty}c_{MN}(k,l)\,t_1^k t_2^l, \qquad (1.3)$$

where $c_{MN}(k,l)$ has a combinatorial sense of the number of different possible positions of one or several allowed closed polygons (having two- or four-tail vertices, see fig.3), consisting together of k horizontal bounds and l vertical bonds (k and l are even integer). Each position contributes to $c_{MN}(k,l)$ addend +1. In particular, each solitary polygon (one polygon on the whole lattice) contributes +1 (we call it the weight of polygon in what follows). Expression (1.1.a) is also representable in form (1.3) and has the same combinatorial meaning and therefore equals to eq. (4.1). Earlier this statement was proved by combinatorial methods [9,13]. It is almost evident from the construction of eq. (1.1.a) (see figs. 3 and 4), except for the problem on parity of polygons [9,13]: Is there excluded such a situation that some polygons have weight +1, while others −1?

Fradkin and Steingradt [15] have given a direct algebraic transformation of the original expression (4.1) into a form of type (1.1) (in terms of the Grassmann variables), and the combinatorial analysis of the above kind becomes absolutely needless. However, for a torus, which Fradkin and Steingradt started with, their transformation is only approximate unlike the bounded lattice (see Sect.4).

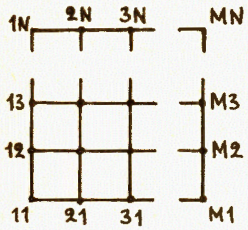

Fig.1.

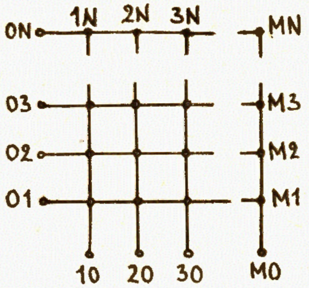

Fig.2.

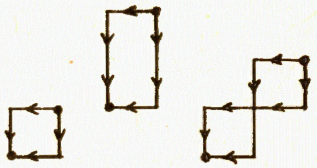

Fig.3.

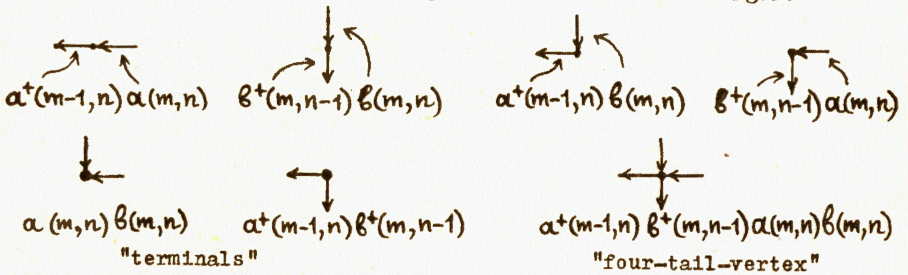

Fig.4. Seven possible vertices at the point mn according to eq.(1.1).

In the case of the Ising model on a rectangular 3-dimensional bounded lattice an original partition function (see eq. (6.1) below) also is a generating function for numbers of possible positions of allowed closed polygons on the space lattice. It seems easy to guess its analog in terms of Fermi creation and annihilation operators, e.g.

$$Z=G_3\langle 0|:\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}[1+t_1a^+(l-1,m,n)a(l,m,n)]$$
$$[1+t_2b^+(l,m-1,n)b(l,m,n)][1+t_3c^+(l,m,n-1)c(l,m,n)]$$
$$\{1+\sqrt{t_1t_2}\,[a^+(l-1,m,n)+a(l,m,n)][b^+(l,m-1,n)+b(l,m,n)]\}$$
$$\{1+\sqrt{t_2t_3}\,[b^+(l,m-1,n)+b(l,m,n)][c^+(l,m,n-1)+c(l,m,n)]\}$$
$$\{1+\sqrt{t_3t_1}\,[c^+(l,m,n-1)+c(l,m,n)][a^+(l-1,m,n)+a(l,m,n)]\}:|0\rangle. \qquad (1.4)$$

The expansion of the six factors of the general term of the product in eq. (1.4) bears 1 plus a sum of 31 possible vertices at a point lmn of the lattice. This expression generates weights +1 for some closed polygons and −1 for others (what is demonstrated by simplest examples) instead of correct +1 for all polygons according to eq. (6.1). There is no possibility of saving the situation by changing signs of the above 31 terms. For some answer how to correct eq.(1.4) see Sect.6.

Note that expressions (1.1.a) and (1.4) are rigorous for the 2- and 3-dimensional bounded lattices and for lattices on tori, if a, a^+, b, b^+, c and c^+ are the Bose annihilation and creation operators and $G_2 = [2\,ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)]^{MN}$, $G_3 = [2\,ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)ch(\beta\varepsilon_3)]^{LMN}$ for the tori. For the bounded lattices eqs. (1.1.a) and (1.4) are also rigorous with Pauli operators a, b, and c (for Pauli operators see Appendix). These statements are almost evident and shall become clear from considerations of sects. 4 and 6. Unfortunately, no further advance is possible.

2. <u>One-dimensional Ising model without field.</u> Let us consider a one-dimensional chain to illustrate the Fradkin-Steingradt method[15], translated into terms of the creation and annihilation operators. As usual, we assume the chain to be closed into a ring, and consider the two cases

A) $\sigma_0 = \sigma_N$ (periodicity condition)

B) $\sigma_0 = -\sigma_N$ (antiperiodicity condition).

The partition function is transformed as follows

$$Z = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} e^{\beta\varepsilon \sum_{n=1}^{N} \sigma_{n-1}\sigma_n} = \tag{2.1}$$

$$= [ch(\beta\varepsilon)]^N \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \prod_{n=1}^{N} (1 + t\sigma_{n-1}\sigma_n) = \tag{2.2}$$

$$= [ch(\beta\varepsilon)]^N \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \langle 0| \prod_{n=1}^{N} (1 + t\sigma_{n-1}\sigma_n) |0\rangle = \tag{2.3}$$

$$= [ch(\beta\varepsilon)]^N \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \langle 0| (1+\sqrt{t}\,a(0)\sigma_0)(1+\sqrt{t}\,a^+(0)\sigma_1)(1+\sqrt{t}\,a(1)\sigma_1)(1+\sqrt{t}\,a^+(1)\sigma_2)$$
$$(1+\sqrt{t}\,a(2)\sigma_2)(1+\sqrt{t}\,a^+(2)\sigma_3)\ldots(1+\sqrt{t}\,a(N-1)\sigma_{N-1})(1+\sqrt{t}\,a^+(N-1)\sigma_N)|0\rangle = \tag{2.4}$$

$$= [2\,ch(\beta\varepsilon)]^N \{ \langle 0|(1+t\,a^+(0)a(1))(1+t\,a^+(1)a(2))\cdots(1+t\,a^+(N-2)a(N-1))|0\rangle \pm$$
$$\pm t \langle 0|a(0)(1+t\,a^+(0)a(1))(1+t\,a^+(1)a(2))\cdots(1+t\,a^+(N-2)a(N-1))a^+(N-1)|0\rangle \} = \tag{2.5}$$

$$= [2\,ch(\beta\varepsilon)]^N [1 \pm t^N], \tag{2.6}$$

where $t = th(\beta\varepsilon)$. Here we start with the standard expression (2.1) and (2.2), put in eq. (2.3) vacua $\langle 0|$ and $|0\rangle$ and split each factor between them as follows

$$1 + t\sigma_{n-1}\sigma_n \rightarrow (1+\sqrt{t}\,a(n-1)\sigma_{n-1})(1+\sqrt{t}\,a^+(n-1)\sigma_n), \tag{2.7}$$

where $a(n-1)$ and $a^+(n-1)$ are annihilation and creation operators, thus obtaining eq. (2.4). To prove the identity of eq. (2.4) with eq. (2.3), let us consider the expansion of some pair of the factors in eq. (2.4) say, entering into the r.h.s. of eq. (2.7). Linear in $a(n-1)$ and $a^+(n-1)$ terms vanish, because these operators can be transferred to vacua $|0\rangle$ and $\langle 0|$, respectively. The product $a(n-1)a^+(n-1)$ can be freely transferred to any of the vacua and converted into 1. Thus, we return to $(1+t\sigma_{n-1}\sigma_n)$. One can transform all pairs of factors in eq. (2.4) analogously, thus returning to eq. (2.3). In eq. (2.4) one can easily carry out the summations over each σ_n [x], using the relation

$$\sum_{\sigma=-1,+1} \sigma^k = \begin{cases} 2 \text{ for even integer } k \\ 0 \text{ for odd integer } k \end{cases}, \tag{2.8}$$

and we are led to eq. (2.5). In fact, each of two terms of (2.5) can be trivially calculated. Only term 1 and terms with pairs $a(n)a^+(n)$ and without unpaired a or a^+ contribute into vacuum expectations. After the expansion of the factors in the first vacuum expectation we find that only 1 contributes, other terms inevitably contain unpaired operators. In the second vacuum expectation the only term contributes, which is proportional to t^N and containes all pairs simultaneously

$$\pm t^N \langle 0|a(0)a^+(0)a(1)a^+(1)\cdots a(N-1)a^+(N-1)|0\rangle = \pm t^N \tag{2.9}$$

Thus, we find expressions (2.6)[x].

In this derivation there was absolutely irrelevant whether $a(n)$, $a^+(n)$ are Fermi, Bose or Pauli operators (for the latter see Appendix).

Since in eqs. (2.4) and (2.5) each annihilation operator stands to the left of its creation operator, these expressions are anti-N-ordered ones. Due to this fact we can put the anti-N-ordering symbol $\vdots \;\; \vdots$ (for the anti-N-ordering see Appendix)

$$Z = [2\,ch(\beta\varepsilon)]^N \langle 0| \{ \vdots (1+t\,a^+(0)a(1))(1+t\,a^+(1)a(2))\cdots(1+t\,a^+(N-2)a(N-1)) \vdots \pm$$
$$\pm t \vdots a(0)(1+t\,a^+(0)a(1))(1+t\,a^+(1)a(2))\cdots(1+t\,a^+(N-2)a(N-1))a^+(N-1) \vdots |0\rangle = \tag{2.5'}$$

$$= [2\,ch(\beta\varepsilon)]^N \langle 0| \vdots (1 \pm t\,a(0)a^+(N-1))(1+t\,a^+(0)a(1))\cdots(1+t\,a^+(N-2)a(N-1)) \vdots |0\rangle = \tag{2.10}$$

$$= \langle 0| \vdots e^{t\sum_{n=1}^{N-1} a^+(n-1)a(n) \mp t\,a^+(N-1)a(0)} \vdots |0\rangle = \tag{2.11}$$

$$= \langle 0| \vdots e^{t\sum_{n=1}^{N} a^+(n-1)a(n)} \vdots |0\rangle, \tag{2.12}$$

$$a(N) = \mp a(0). \tag{2.13}$$

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[x] Of course this can be done easily in eq. (1.2) thus giving eqs. (1.6) immediately.

In eqs. (2.5) and (2.10) also Fermi, Bose, and Pauli operators are suitable. However, the exponential form (2.11) and (2.12) is possible only with the Fermi operators. Each factor of eq. (2.10) can be written as $e^{t\,a^+(n-1)a(n)}$, and under $\vdots\ \vdots$ we can join them into one exponential function of eq. (2.11) (see Appendix).

The exponential representation gives one more way to calculate Z (being, however, too complicate for this case). The bilinear form in the exponent of eq. (2.12) can be diagonalized using the Fourier transformations

$$a(n)=\frac{1}{\sqrt{N}}\sum_{p=1}^{N} e^{i\frac{2\pi n}{N}(p+\frac{1}{2})}\alpha(p)\,,\quad a^+(n)=\frac{1}{\sqrt{N}}\sum_{p=1}^{N} e^{-i\frac{2\pi n}{N}(p+\frac{1}{2})}\alpha^+(p) \tag{2.14}$$

$$a(n)=\frac{1}{\sqrt{N}}\sum_{p=1}^{N} e^{i\frac{2\pi n}{N}p}\alpha(p),\quad a^+(p)=\frac{1}{\sqrt{N}}\sum_{p=1}^{N} e^{-i\frac{2\pi n}{N}p}\alpha^+(p) \tag{2.15}$$

in the cases A) and B), respectively. As a result, we obtain for Z

$$Z=[2ch(\beta\varepsilon)]^N\langle 0|\vdots\, e^{\sum_{p=1}^{N} z_p\alpha^+(p)\alpha(p)}\,\vdots|0\rangle= \tag{2.16}$$

$$=[2ch(\beta\varepsilon)]^N\prod_{p=1}^{N}\langle 0|\vdots\, e^{z_p\alpha^+(p)\alpha(p)}\,\vdots|0\rangle= \tag{2.17}$$

$$=[2ch(\beta\varepsilon)]^N\prod_{p=1}^{N}\langle 0|1-z_p\alpha(p)\alpha^+(p)|0\rangle=[2ch(\beta\varepsilon)]^N\prod_{p=1}^{N}(1-z_p)= \tag{2.18}$$

$$=[2ch(\beta\varepsilon)]^N[1\pm t^N] \tag{2.19}$$

with the upper sign in case A), when $z_p=e^{i\frac{2\pi}{N}(p+\frac{1}{2})}$, and with the lower sign in case B), when $z_p=e^{i2\pi p/N}$.

3. <u>One-dimensional Ising model in a magnetic field H.</u> Let us consider once more the chain closed into a ring, and restrict ourselves only by periodicity condition A). Unlike Fradkin and Steingradt[15] we transform the partition function absolutely in the same manner as in the case of H=0:

$$Z=\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1} e^{\beta\varepsilon\sum_{n=1}^{N}\sigma_{n-1}\sigma_n+\beta H\sum_{n=1}^{N}\sigma_n}= \tag{3.1}$$

$$=[ch(\beta\varepsilon)ch(\beta H)]^N\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1}\prod_{n=1}^{N}(1+t\sigma_{n-1}\sigma_n)(1+\bar{t}\sigma_n)= \tag{3.2}$$

$$=[ch(\beta\varepsilon)ch(\beta H)]^N\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1}\langle 0|\prod_{n=1}^{N}(1+t\sigma_{n-1}\sigma_n)(1+\bar{t}\sigma_n)|0\rangle= \tag{3.3}$$

$$=[ch(\beta\varepsilon)ch(\beta H)]^N\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1}\langle 0|(1+\sqrt{t}\,a(0)\sigma_0)(1+\sqrt{t}\,a^+(0)\sigma_1)(1+\bar{t}\sigma_1)\cdot$$

$$(1+\sqrt{t}\,a(1)\sigma_1)(1+\sqrt{t}\,a^+(1)\sigma_2)(1+\bar{t}\sigma_2)\ldots(1+\sqrt{t}\,a(N-1)\sigma_{N-1})(1+\sqrt{t}\,a^+(N-1)\sigma_N)(1+\bar{t}\sigma_N)|0\rangle= \tag{3.4}$$

$$=[2ch(\beta\varepsilon)ch(\beta H)]^N\Big\{\langle 0|\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]|0\rangle+$$

$$+\bar{t}\sqrt{t}\langle 0|a(0)\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]|0\rangle+$$

$$+\bar{t}\sqrt{t}\langle 0|\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]a^+(N-1)|0\rangle+$$

$$+t\langle 0|a(0)\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]a^+(N-1)|0\rangle\Big\}= \tag{3.5}$$

$$=[2ch(\beta\varepsilon)ch(\beta H)]^N\ \langle 0|\vdots\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]\vdots+$$

$$+\bar{t}\sqrt{t}\vdots\, a(0)\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]\vdots+$$

$$+\bar{t}\sqrt{t}\vdots\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]a^+(N-1)\vdots+$$

$$+t\vdots\, a(0)\prod_{n=1}^{N-1}[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)]a^+(N-1)\vdots|0\rangle. \tag{3.6}$$

Here $\bar{t}=th(\beta H)$. Up to eq. (3.4) we follow exactly the above way for H=0. Additional c-number factors $(1+\bar{t}\sigma_n)$ manifest themselves only in the course of summation over σ's to give the result

$$1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))+t\,a^+(n-1)a(n)=[1+\bar{t}\sqrt{t}(a^+(n-1)+a(n))][1+t\,a^+(n-1)a(n)],$$

where the first factor is due to the field H and the latter is the old one. The linear factors make situation more complicated. However one can transform them into bilinear ones in the following ways

$$\langle 0|(1+u_1)(1+u_2)\cdots(1+u_N)|0\rangle=$$

$$=\langle 0|\prod_{n=1}^{N}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n](1+c^+_N)|0\rangle= \tag{3.7.a}$$

$$=\langle 0|c_0\prod_{n=1}^{N}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n](1+c^+_N)|0\rangle= \tag{3.7.b}$$

$$=\langle 0|(1+c_0)\prod_{n=1}^{N}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n]|0\rangle= \tag{3.7.c}$$

$$=\langle 0|(1+c_0)\prod_{n=1}^{N}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n]c^+_N|0\rangle= \qquad (3.7.d)$$

$$=\frac{1}{2}\langle 0|(1+c_0)\prod_{n=1}^{N}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n](1+c^+_N)|0\rangle, \qquad (3.7.e)$$

where u_n are linear in the creation and annihilation operators, in our case $u_n=\bar{t}\sqrt{t}\,(a^+(n-1)+a(n))$. We imply that in (3.7) all factors are arranged in the ascending order of n from the left to the right. Procedure of such a kind for Grassmann binomials has been invented by Fradkin and Steingradt[/15/], but their original derivation seems complicated and contains some artificial tricks. We demonstrate relations (3.7) in the following simple way. By the pairing of c_N with c^+_N in (3.7.a) we get

$$(3.7.a)=\langle 0|\prod_{n=1}^{N-1}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n](1+c^+_{N-1})(1+u_N)|0\rangle .$$

Then after the pairing of c_{N-1} with c^+_{N-1} we obtain

$$(3.7.a)=\langle 0|\prod_{n=1}^{N-2}[1+c^+_{n-1}u_n+(c^+_{n-1}+u_n)c_n](1+c^+_{N-2})(1+u_{N-1})(1+u_N)|0\rangle$$

and so on till after the pairing of c_1 and c^+_N we find finally

$$(3.7.a)=\langle 0|(1+c^+_0)\prod_{n=1}^{N}(1+u_n)|0\rangle=\langle 0|\prod_{n=1}^{N}(1+u_n)|0\rangle .$$

The same steps in case (3.7.b) lead to

$$(3.7.b)=\langle 0|c_0(1+c^+_0)\prod_{n=1}^{N}(1+u_n)|0\rangle=\langle 0|\prod_{n=1}^{N}(1+u_n)|0\rangle .$$

Analogously in cases (3.7.c) and (3.7.d). But now we must move from the left to the right by the pairings of c_0 with c^+_0, then c_1 with c^+_1 and so on. Expression (3.7.e) may be obtained by combining the preceding expressions or its validity can be checked in any of the above ways. In eqs. (3.7) binomials $(1+u_n)$ could alternate with any factors, which commute with operators c. Besides all this is valid under the symbol $\vdots\ \ \vdots$ of anti-N-ordering.

Since in eq. (3.4) and (3.5) each annihilation operator stands to the left of its creation operator, these expressions are anti-N-ordered ones and one can put the symbol $\vdots\ \ \vdots$ to obtain eq. (3.6). Let us transform the linear factors like in eq. (3.7.e). Under $\vdots\ \ \vdots$ extreme factors can be collected as follows

$$(1+c(0))(1+c^+(N-1))+\bar{t}\sqrt{t}\,a(0)(1+c(0))(1+c^+(N-1))+\bar{t}\sqrt{t}(1+c(0))(1+c^+(N-1))a^+(N-1)+$$
$$+t\,a(0)(1+c(0))(1+c^+(N-1))a^+(N-1)=$$

$$=\{1+\bar{t}\sqrt{t}\,[a^+(N-1)+a(0)]+c(0)+c^+(N-1)+2(\bar{t}^2-1)t\,a(0)a^+(N-1)[c^+(N-1)+c(0)]\}\times$$
$$\times\{1+\bar{t}\sqrt{t}\,c^+(N-1)[a^+(N-1)-a(0)]-[c^+(N-1)+\bar{t}\sqrt{t}(a^+(N-1)-a(0))]c(0)\}[1+t\,a(0)a^+(N-1)]. \qquad (3.8)$$

The validity of the latter decomposition into three factors can be easily checked. The partition function takes the form

$$Z=\frac{1}{2}[2\,ch(\beta\varepsilon)ch(\beta H)]^N\langle 0|\vdots\{1+\bar{t}\sqrt{t}\,[a^+(N-1)+a(0)]+$$
$$+c^+(N-1)+c(0)+2(\bar{t}^2-1)t\,a(0)a^+(N-1)[c^+(N-1)+c(0)]\}$$
$$\prod_{n=1}^{N}\{1+\bar{t}\sqrt{t}\,c^+(n-1)[a^+(n-1)+a(n)]+[c^+(n-1)+\bar{t}\sqrt{t}(a^+(n-1)+a(n))]c(n)\}$$
$$[1+t\,a^+(n-1)a(n)]\vdots|0\rangle= \qquad (3.9)$$

$$=\frac{1}{2}[2ch(\beta\varepsilon)ch(\beta H)]^N\langle 0|\vdots\prod_{n=1}^{N}\{1+\bar{t}\sqrt{t}\,c^+(n-1)[a^+(n-1)+a(n)]+$$
$$+[c^+(n-1)+\bar{t}\sqrt{t}(a^+(n-1)+a(n))]c(n)\}[1+t\,a^+(n-1)a(n)]\vdots|0\rangle= \qquad (3.10)$$

$$=\frac{1}{2}[2ch(\beta\varepsilon)ch(\beta H)]^N\langle 0|\vdots\, e^{\sum_{n=1}^{N}A(n)}\vdots|0\rangle, \qquad (3.11)$$

where

$$A(n)=\bar{t}\sqrt{t}\,c^+(n-1)[a^+(n-1)+a(n)]+[c^+(n-1)+\bar{t}\sqrt{t}(a^+(n-1)+a(n))]c(n)+$$
$$+t\,a^+(n-1)a(n), \qquad (3.12)$$

$$a(0)=-a(N),\quad c(0)=-c(N). \qquad (3.13)$$

Expression (3.9) differs from the corresponding one in ref.[/15/] in form of the operator terms of the first factor. However, these terms, being odd in number of operators, do not contribute to the vacuum expectation, since other terms are even. One can demonstrate this, representing $|0\rangle=\eta|0\rangle$ with $\eta=(-1)^N$, $N=\sum(a^+a+c^+c)$, transferring η to $\langle 0|$ and taking halfsum of the original and new expressions. Therefore, we are led to eqs. (3.10) and (3.11). The bilinear form in the exponent of (3.11) can be "diagonalized" through Fourier transformations of type (2.14), which imply the antiperiodicity relations $a(n+N)=-a(n)$, $c(n+N)=-c(n)$, and therefore ensure conditions (3.13). Further one obtains a well-known expression for Z in the 1-dimensional Ising model in the magnetic field H (see ref.[/15/]).

4. Two-dimensional Ising model without field (H=0). Let us consider the 2-dimensional flat rectangular bounded lattice (fig.1). The partition function can be transformed as follows

$$Z = \sum_{\{\sigma_{mn}=\pm 1\}} e^{\beta \sum\limits_{m=1}^{M} \sum\limits_{n=1}^{N} (\varepsilon_1 \sigma_{m-1\,n}\sigma_{mn} + \varepsilon_2 \sigma_{m\,n-1}\sigma_{mn})} = \tag{4.1}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \prod_{n=1}^{N}\prod_{m=1}^{M} (1+t_1\sigma_{m-1\,n}\sigma_{mn})(1+t_2\sigma_{m\,n-1}\sigma_{mn}) = \tag{4.2}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M} (1+t_1\sigma_{m-1\,n}\sigma_{mn})(1+t_2\eta\,\sigma_{m\,n-1}\sigma_{mn})|0\rangle = \tag{4.3}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\Big\{\prod_{m=1}^{M} [1+\sqrt{t_1}\,a(m-1,n)\sigma_{m-1\,n}][1+\sqrt{t_1}\,a^+(m-1,n)\sigma_{mn}]\cdot$$
$$[1+\sqrt{t_2}\,\eta\, b(m,n-1)\sigma_{m\,n-1}][1+\sqrt{t_2}\, b^+(m,n-1)\sigma_{mn}]\Big\}|0\rangle = \tag{4.4}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0|[1+\sqrt{t_2}\,\eta\, b(1,0)\sigma_{10}][1+\sqrt{t_2}\,\eta\, b(2,0)\sigma_{20}]\cdots[1+\sqrt{t_2}\,\eta\, b(M,0)\sigma_{M0}]\cdot$$
$$\prod_{n=1}^{N-1}\Big\{[1+\sqrt{t_1}\,a(0,n)\sigma_{0n}]\prod_{m=1}^{M-1}\Big\{[1+\sqrt{t_1}\,a^+(m-1,n)\sigma_{mn}][1+\sqrt{t_2}\,\eta\, b(m,n)\sigma_{mn}][1+\sqrt{t_2}\, b^+(m,n-1)\sigma_{mn}][1+\sqrt{t_1}\,a(m,n)\sigma_{mn}]\Big\}$$
$$[1+\sqrt{t_1}\,a^+(M-1,n)\sigma_{Mn}][1+\sqrt{t_2}\,\eta\, b(M,n)\sigma_{Mn}][1+\sqrt{t_2}\, b^+(M,n-1)\sigma_{Mn}]$$
$$[1+\sqrt{t_1}\,a(0,N)\sigma_{0N}]\prod_{m=1}^{M-1}\Big\{[1+\sqrt{t_1}\,a^+(m-1,N)\sigma_{mN}][1+\sqrt{t_2}\, b^+(m,N-1)\sigma_{mN}][1+\sqrt{t_1}\,a(m,N)\sigma_{mN}]\Big\}$$
$$[1+\sqrt{t_1}\,a^+(M-1,N)\sigma_{MN}][1+\sqrt{t_2}\, b^+(M,N-1)\sigma_{MN}]|0\rangle = \tag{4.5}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\Big\{\prod_{m=1}^{M} [1+\sqrt{t_1}\,a^+(m-1,n)\sigma_{mn}][1+\sqrt{t_2}\,\eta\, b(m,n)\sigma_{mn}]\cdot$$
$$[1+\sqrt{t_2}\, b^+(m,n-1)\sigma_{mn}][1+\sqrt{t_1}\,a(m,n)\sigma_{mn}]\Big\}|0\rangle = \tag{4.6}$$

$$= G\langle 0|\prod_{n=1}^{N}\Big\{\prod_{m=1}^{M}\{1+\sqrt{t_1t_2}\,a^+(m-1,n)b^+(m,n-1)+\sqrt{t_1t_2}\,\eta\, b(m,n)a(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_2}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)+\sqrt{t_2}\,\eta\, b(m,n)]+$$
$$+t_1t_2\,a^+(m-1,n)\eta\, b(m,n)\, b^+(m,n-1)\, a(m,n)\}\Big\}|0\rangle = \tag{4.7}$$

$$= G\langle 0|:\prod_{n=1}^{N}\prod_{m=1}^{M}\{1+\sqrt{t_1t_2}\,a^+(m-1,n)b^+(m,n-1)+\sqrt{t_1t_2}\,b(m,n)a(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_2}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)-\sqrt{t_2}\,b(m,n)]-$$
$$-t_1t_2\,a^+(m-1,n)b^+(m,n-1)a(m,n)b(m,n)\}:|0\rangle = \tag{4.8}$$

$$= G\langle 0|:e^{\sum\limits_{n=1}^{N}\sum\limits_{m=1}^{M}A(m,n)}:|0\rangle, \tag{4.9}$$

where $\qquad t_1 = th(\beta\varepsilon_1),\ t_2 = th(\beta\varepsilon_2)$,

$$G = 2^{MN}F = 2^{MN}[ch(\beta\varepsilon_1)]^{N(M-1)}[ch(\beta\varepsilon_2)]^{M(N-1)}, \tag{4.10}$$

$$A(m,n) = \sqrt{t_1t_2}\,a^+(m-1,n)\,b^+(m,n-1) - \sqrt{t_1t_2}\,a(m,n)\,b(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_2}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)-\sqrt{t_2}\,b(m,n)], \tag{4.11}$$

summations are over σ_{mn} with m=1, ..., M; n=1, ..., N, corresponding to points of the lattice of fig.1. We put equal to zero other σ's encountered in eqs. (4.1)-(4.5) and drawn in fig.2

$$\sigma_{0n} = \sigma_{m0} = 0 \quad (m=1,\dots,M;\ n=1,\dots,N) \tag{4.12}$$

for the bounded lattice, as proposed. The factors $(1+t_1\sigma_{m-1\,n}\sigma_{mn})$ and $(1+t_2\sigma_{m\,n-1}\sigma_{mn})$ correspond to the "bits" $\underset{m-1\,n}{\bullet}\!\!-\!\!\!-\!\!\underset{mn}{\bullet}$ and $\begin{matrix}\bullet\ mn\\ |\\ \bullet\ m\,n-1\end{matrix}$ on the lattice, respectively. Exponents of $ch(\beta\varepsilon_1)$ and $ch(\beta\varepsilon_2)$ equal the numbers of these bits.

Often (in particular, in ref. /15/) the lattice is considered on a torus

$$\sigma_{m0} = \sigma_{mN},\quad \sigma_{0n} = \sigma_{Mn}\ (m=1,\dots,M;\ n=1,\dots,N) \tag{4.13}$$

(see fig.2). In this case eqs. (4.4) and (4.5) are also valid with $G = 2^{MN}F = [2\,ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)]^{MN}$. However, we could not achieve eq. (4.6). That is why we consider the bounded lattice (fig.1). For obtaining the possibility of transfering in what follows some factors into other places, we insert into eq. (4.3) the operator

$$\eta = (-1)^N,\ N = N_a + N_b,\ N_a = \sum_{m=1}^{M}\sum_{n=1}^{N} a^+(m,n)a(m,n),\ N_b = \sum_{m=1}^{M}\sum_{n=1}^{N} b^+(m,n)b(m,n), \tag{4.14}$$

$$\{\eta a\} = \{\eta a^+\} = \{\eta b\} = \{\eta b^+\} = 0. \tag{4.15}$$

Between $\langle 0|$ and $|0\rangle$ we can split, using creation and annihilation operators, the factors $(1+t_1\sigma_{m-1\,n}\sigma_{mn})$ and $(1+t_2\eta\,\sigma_{m\,n-1}\sigma_{mn})$ into factors linear in σ_{mn}:

$$1+t_1\sigma_{m-1n}\sigma_{mn}\to[1+\sqrt{t_1}\,\alpha(m-1,n)\sigma_{m-1n}][1+\sqrt{t_1}\,\alpha^+(m-1,n)\sigma_{mn}] \qquad (4.16.a)$$

$$1+t_2\eta\sigma_{mn-1}\sigma_{mn}\to[1+\sqrt{t_2}\,\eta\, b(m,n-1)\sigma_{mn-1}][1+\sqrt{t_2}\, b^+(m,n-1)\sigma_{mn}]^{x)} \qquad (4.16.b)$$

Since the factors between vacua in eq. (4.4) do not commute in general we arrange them in the following way. The product is subdivided into N segments, "lines" (n is the number of a line, n=1,2,...,N), with M terms in each line (m is the number of a term, m=1,2,...,M). The "term" is a product of 4 factors written in eq. (4.4). The terms are supposed to be arranged in the ascending order of n and m from the left to the right.

To check the identity of eq. (4.4) with eq. (4.3) let us consider the expansion of a pair of factors, say, those entering into the r.h.s. of eq. (4.16.a). The linear in $\alpha(m-1,n)$ and $\alpha^+(m-1,n)$ terms vanish, because these operators can be transferred to $|0\rangle$ and $\langle 0|$, respectively. This is due to the fact that these operators are not met anywhere else (except in η). The product $\alpha(m-1,n)\alpha^+(m-1,n)$ can freely be transferred to any of the vacua and converted into 1. Thus, we return to $(1+t_1\sigma_{m-1n}\sigma_{mn})$. A similar transformation of all other pairs of brackets turns us back to eq. (4.3).

Each σ_{mn} enters into four factors of the product of eq. (4.4). We wish to collect such factors together. Till now the general term of the product contained only two such factors. One more is near-by: it is the first factor $[1+\sqrt{t_1}\,\alpha(m,n)\sigma_{mn}]$ of the next (m+1)th term. The fourth factor $[1+\sqrt{t_2}\,\eta\, b(m,n)\sigma_{mn}]$ is precisely one line below (on (n+1)th line) of its analog in eq. (4.4). We include the factor $[1+\sqrt{t_1}\,\alpha(m,n)\sigma_{mn}]$ of (m+1)th term into mth term, the first term of which being included into (m-1)th term, etc. All the factors $[1+\sqrt{t_2}\,\eta\, b(m,n)\sigma_{mn}]$ can be simultaneously extracted and transferred exactly on one line up (to the left) without changing their mutual order. Along their ways they commute with all encountered factors due to the operator η anticommuting with any creation and annihilation operator (see eqs. (4.15)). Thus, we come to eq. (4.5).

Expression (4.5) is valid for a lattice on a torus too (see eqs.) (4.13))$^{xx)}$ However, summations over σ's should bear in this case a rather complicated result instead of eqs. (4.7) and (4.8).

For the bounded lattice, eq. (4.5) equals eq. (4.6) where the general term contains only σ_{mn}. Note, that the factors $[1+\sqrt{t_1}\,\alpha(0,n)\sigma_{0n}]$

x) One can also split without the radicals, e.g.,

$$1+t_1\sigma_{m-1n}\sigma_{mn}\to[1+\alpha(m-1,n)\sigma_{m-1n}][1+t_1\alpha^+(m-1,n)\sigma_{mn}].$$

xx) Some known expressions /9,13,14/ correspond to the latter. And also for a lattice with free ends.

at the beginning of each line and the factors $[1+\sqrt{t_2}\,\eta\, b(m,0)\sigma_{m0}]$ pushed from 1th line into a space between $\langle 0|$ and 1th line, have been missed out in eq. (4.6) as compared to eq. (4.5). But all these factors equal 1 due to eq.(4.12). Besides, the additional factors $[1+\sqrt{t_1}\,\alpha(M,n)\sigma_{Mn}]$ at the end of each line (n = 1, ..., N) and $[1+\sqrt{t_2}\,\eta\, b(m,N)\sigma_{mN}]$ $(m=1,\dots,M)$ at the last line have been inserted into eq. (4.6) as compared to eq.(4.5). These factors equal to 1 too, due to the absence between vacua the mate operators $\alpha^+(M,n)$ and $b^+(m,N)$ (except possibly in η). In eq.(4.6) it is easy to compute sum over each σ_{mn}. The expansion of the four factors of the general term in eq.(4.6) gives $C_4^0+C_4^2+C_4^4=8$ addends with even powers of σ_{mn} and $C_4^1+C_4^3=8$ addends with odd ones. Using eq.(2.8), we obtain eq.(4.7). Here the operator η commutes with each term of the product in eq.(4.7). Therefore, each η in eq.(4.7) can be transferred to any of vacua and converted into 1 thus giving eq. (4.8) without : :. We can put the symbol of anti-N-ordering : :, since in the expression obtained each annihilation operator stands to the left of its creation operator (this is valid from eq.(4.4) to eq. (4.7) too, if one ignores η). Each term of the product in eq.(4.8) can be represented as $e^{A(m,n)}$ with no matter : : stands or not, but only under : : all $e^{A(m,n)}$ can be brought together into one exponential function of eq. (4.9). (See Appendix).

For completeness let us accomplish calculation of the partition function along one of standard ways (cf.,e.g., ref./13/), Neglecting of influence of some extreme terms, we replace now the bilinear form $\sum\sum A(m,n)$ by a new one $\sum\sum A'(m,n)$, where $\alpha^+(0,n)$ and $b^+(m,0)$ are identified with $\alpha^+(M,n)$ and $b^+(m,N)$, respectively (a form on a torus). This is justified in the thermodynamical limit (see, e.g., refs./9,13/). The bilinear form $\sum\sum A'(m,n)$ can be "diagonalized" by the Fourier transformation $^{x)}$:

$$\alpha(m,n)=\frac{1}{\sqrt{MN}}\sum_{p=1}^{M}\sum_{q=1}^{N}e^{2\pi i\left(\frac{mp}{M}+\frac{nq}{N}\right)}\alpha(p,q),$$

$$b(m,n)=\frac{1}{\sqrt{MN}}\sum_{p=1}^{M}\sum_{q=1}^{N}e^{2\pi i\left(\frac{mp}{M}+\frac{nq}{N}\right)}\beta(p,q),$$

$$\alpha^+(m,n)=\frac{1}{\sqrt{MN}}\sum_{p=1}^{M}\sum_{q=1}^{N}e^{-2\pi i\left(\frac{mp}{M}+\frac{nq}{N}\right)}\alpha^+(p,q),$$

$$b^+(m,n)=\frac{1}{\sqrt{MN}}\sum_{p=1}^{M}\sum_{q=1}^{N}e^{-2\pi i\left(\frac{mp}{M}+\frac{nq}{N}\right)}\beta^+(p,q), \qquad (4.17)$$

x) A transformation of type (2.14) is also suitable.

which incorporates the periodicity conditions

$$a(m+M,n)=a(m,n+N)=a(m,n) \tag{4.18}$$

and similar ones for other operators. When transforming, the completeness relations

$$\frac{1}{M}\sum_{m=1}^{M} e^{-\frac{2\pi i}{M}(p-p')m}=\delta_{p'p} \tag{4.19a}$$

$$\frac{1}{M}\sum_{m=1}^{M} e^{\pm\frac{2\pi i}{M}(p+p')m}=\delta_{p'\,M-p} \tag{4.19b}$$

are used, with $1\leq p\leq M$ and $1\leq p'\leq M$ (p and p' are integer). They follow from the more general relation

$$\frac{1}{M}\sum_{m=1}^{M} e^{\frac{2\pi i}{M}(p-p')m}=\sum_{k=-\infty}^{\infty}\delta_{p'\,p+kM}\ , \tag{4.20}$$

which is valid for all positive and negative integer p and p' , k runs over all these values. After the Fourier transformation one obtains

$$\sum_{m=1}^{M}\sum_{n=1}^{N}A'(m,n)=\sum_{p=1}^{M}\sum_{q=1}^{N}B(p,q), \tag{4.21}$$

$$B(p,q)=\xi_1\xi_2^{*}\alpha^{+}(p,q)\beta^{+}(-p,-q)\pm\sqrt{t_1t_2}\,\alpha(p,q)\beta(-p,-q)+$$
$$+[\xi_1\alpha^{+}(p,q)+\xi_2\beta^{+}(p,q)][\sqrt{t_1}\,\alpha(p,q)\pm\sqrt{t_2}\,\beta(p,q)],$$

$$\xi_1\equiv\xi_1(p)=\sqrt{t_1}\,e^{2\pi i\frac{p}{M}}\ ,\qquad \xi_2\equiv\xi_2(q)=\sqrt{t_2}\,e^{2\pi i\frac{q}{N}}\ , \tag{4.22}$$

where upper and lower signs correspond to eqs. (1.1.b) and (4.9), respectively. Although eqs. (1.1) and (4.9) equal each other (see below), their approximate expressions with exponents (4.21), (4.22) do not (see eq. (4.27) below). The equality will be restored only in the thermodynamical limit (see eq. (4.28.b) below). Further we have

$$Z\approx G\langle 0|\!:\!\prod_{p=1}^{M}\prod_{q=1}^{N}e^{B(p,q)}\!:\!|0\rangle=G\prod_{p=1}^{M}\prod_{q=1}^{N}\langle 0|\!:\!e^{B(p,q)+B(-p,-q)}\!:\!|0\rangle^{\frac{1}{2}} \tag{4.23}$$

due to splitting of the transformed product into independent factors: $:e^{B(p,q)+B(-p,-q)}:$ (with no common operators), which commute with each other, so that their vacuum expectation values can be calculated separately. Each of these factors can be ordered explicitly as follows

$$:e^{B(p,q)+B(-p,-q)}:=$$
$$=e^{\pm\sqrt{t_1t_2}\,\alpha(p,q)\beta(-p,-q)\pm\sqrt{t_1t_2}\,\alpha(-p,-q)\beta(p,q)}$$
$$:e^{[\xi_1\alpha^{+}(p,q)+\xi_2\beta^{+}(p,q)][\sqrt{t_1}\,\alpha(p,q)\pm\sqrt{t_2}\,\beta(p,q)]}:$$
$$:e^{[\xi_1^{*}\alpha^{+}(-p,-q)+\xi_2^{*}\beta^{+}(-p,-q)][\sqrt{t_1}\,\alpha(-p,-q)\pm\sqrt{t_2}\,\beta(-p,-q)]}:$$
$$e^{\xi_1\xi_2^{*}\alpha^{+}(p,q)\beta^{+}(-p,-q)+\xi_1^{*}\xi_2\alpha^{+}(-p,-q)\beta^{+}(p,q)}=$$
$$=\{1\pm\sqrt{t_1t_2}\,\alpha(p,q)\beta(-p,-q)\pm\sqrt{t_1t_2}\,\alpha(-p,-q)\beta(p,q)+t_1t_2\alpha(p,q)\beta(-p,-q)\alpha(-p,-q)\beta(p,q)\}$$
$$\{1-[\sqrt{t_1}\,\alpha(p,q)\pm\sqrt{t_2}\,\beta(p,q)][\xi_1\alpha^{+}(p,q)+\xi_2\beta^{+}(p,q)]\}$$
$$\{1-[\sqrt{t_1}\,\alpha(-p,-q)\pm\sqrt{t_2}\,\beta(-p,-q)][\xi_1^{*}\alpha^{+}(-p,-q)+\xi_2^{*}\beta^{+}(-p,-q)]\}$$
$$\{1+\xi_1\xi_2^{*}\alpha^{+}(p,q)\beta^{+}(-p,-q)+\xi_1^{*}\xi_2\alpha^{+}(-p,-q)\beta^{+}(p,q)+|\xi_1|^2|\xi_2|^2\alpha^{+}(p,q)\beta^{+}(-p,-q)\alpha^{+}(-p,-q)\beta^{+}(p,q)\}. \tag{4.24}$$

Remove the braces of the first and fourth factors. Among 16 terms thus obtained only 6 with equal numbers of creation and annihilation operators can contribute to the vacuum expectation. In the second and third factors we take into account only the relations $\alpha^2=\alpha^{+2}=\beta^2=\beta^{+2}=0$. Then in the notation

$$\alpha(+)=\alpha(p,q),\ \ \alpha(-)=\alpha(-p,-q),\ \ \beta(+)=\beta(p,q),\ \ \beta(-)=\beta(-p,-q),$$
$$z_1=\sqrt{t_1}\,\xi_1=t_1e^{2\pi i\frac{p}{M}}\ ,\qquad z_2=\sqrt{t_2}\,\xi_2=t_2e^{2\pi i\frac{q}{N}} \tag{4.25}$$

one obtains for the above 6 terms

$$\langle 0|\!:\!e^{B(p,q)+B(-p,-q)}\!:\!|0\rangle=$$
$$=\langle 0|\{1-[\sqrt{t_1}\,\alpha(+)\pm\sqrt{t_2}\,\beta(+)][\xi_1\alpha^{+}(+)+\xi_2\beta^{+}(+)]\}\{1-[\sqrt{t_1}\,\alpha(-)\pm\sqrt{t_2}\,\beta(-)][\xi_1^{*}\alpha^{+}(-)+\xi_2^{*}\beta^{+}(-)]\}\mp$$
$$\mp z_1z_2^{*}\alpha(+)[1\mp z_2\beta(+)\beta^{+}(+)]\alpha^{+}(+)\beta(-)[1-z_1^{*}\alpha(-)\alpha^{+}(-)]\beta^{+}(-)\pm$$
$$\pm z_1^{*}z_2\alpha(+)[1\mp\xi_1\sqrt{t_2}\,\beta(+)\alpha^{+}(+)]\beta^{+}(+)\beta(-)[1\mp\xi_2^{*}\sqrt{t_1}\,\alpha(-)\beta^{+}(-)]\alpha^{+}(-)\pm$$
$$\pm z_1z_2^{*}\beta(+)[1-\xi_2\sqrt{t_1}\,\alpha(+)\beta^{+}(+)]\alpha^{+}(+)\alpha(-)[1\mp\xi_1^{*}\sqrt{t_2}\,\beta(-)\alpha^{+}(-)]\beta^{+}(-)\mp$$
$$\mp z_1^{*}z_2\beta(+)[1-z_1\alpha(+)\alpha^{+}(+)]\beta^{+}(+)\alpha(-)[1\mp z_2^{*}\beta(-)\beta^{+}(-)]\alpha^{+}(-)+$$
$$+|z_1|^2|z_2|^2\alpha(+)\beta(+)\alpha^{+}(+)\beta^{+}(+)\beta(-)\alpha(-)\beta^{+}(-)\alpha^{+}(-)|0\rangle=$$

$$=(1-z_1\mp z_2)(1-z_1^*\mp z_2^*)\mp z_1 z_2^*(1\mp z_2)(1-z_1^*)+|z_1|^2|z_2|^2+$$
$$+|z_1|^2|z_2|^2\mp z_1^* z_2(1-z_1)(1\mp z_2^*)+|z_1|^2|z_2|^2=$$
$$=1-z_1-z_1^*\mp(z_2+z_2^*)+|z_1|^2+|z_2|^2\pm|z_1|^2(z_2+z_2^*)+|z_2|^2(z_1+z_1^*)+|z_1|^2|z_2|^2. \tag{4.26}$$

Hence, the partition function equals

$$Z\approx G\prod_{p=1}^{M}\prod_{q=1}^{N}\left[(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos\frac{2\pi p}{M}\mp 2t_2(1-t_1^2)\cos\frac{2\pi q}{N}\right]^{\frac{1}{2}} \tag{4.27}$$

For the thermodynamical potential we obtain the famous Onsager result

$$\Phi=-T\lim_{M,N\to\infty}\frac{1}{MN}\ln Z=-T\lim\frac{1}{MN}\{MN\ln 2+N(M-1)\ln ch(\beta\varepsilon_1)+M(N-1)\ln ch(\beta\varepsilon_2)$$
$$+\frac{1}{2}\sum_{p=1}^{M}\sum_{q=1}^{N}\ln\left[(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos\frac{2\pi p}{M}\mp 2t_2(1-t_1^2)\cos\frac{2\pi q}{N}\right]\Big\}= \tag{4.28.a}$$
$$=-T\{\ln 2+\ln[ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)]+$$
$$+\frac{1}{8\pi^2}\int_0^{2\pi}d\varphi_1\int_0^{2\pi}d\varphi_2\ln[(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos\varphi_1-2t_2(1-t_1^2)\cos\varphi_2]\Big\}, \tag{4.28.b}$$

where passage to the integral is performed by setting $\varphi_1=\frac{2\pi p}{M}$, $\varphi_2=\frac{2\pi q}{N}$, $d\varphi_1=\frac{2\pi}{M}$, $d\varphi_2=\frac{2\pi}{N}$. In eq. (4.28.b) the dependence on sign disappears: The signs before $\cos\varphi_1$ and $\cos\varphi_2$ can be altered by changing variables. For the same reason eq. (4.28.b) depends only on $|t_1|$ and $|t_2|$, i.e., on $|\varepsilon_1|$ and $|\varepsilon_2|$. The argument of the logarithm takes minimum value $K(\beta)=(1+|t_1|+|t_2|-|t_1t_2|)^2$ at $\varphi_1=\varphi_2=0$, and a critical point is defined from $K(\beta_c)=0$.

Remarks. (1) It was shown that the F.-S. transformation is rigorous for the bounded lattice.

2) Expression (4.8) differs from eq. (1.1.a) in the change of all $b(m,n)$ by $-b(m,n)$ (while $b^+(m,n)$, $a(m,n)$ and $a^+(m,n)$ remain unchanged). In spite of this, eqs. (4.8) and (1.1.a) equal each other since each allowed polygon on the lattice contains even numbers of the vertical and horisontal bonds and therefore is represented by a product of even numbers of a,a+,b and b+ separately.

3) If we split

$$(1+t_2\eta\sigma_{mn-1}\sigma_{mn})\to[1+\sqrt{t_2}\,b(m,n-1)\sigma_{mn-1}][1+\sqrt{t_2}\,b^+(m,n-1)\eta\sigma_{mn}], \tag{4.29}$$

put the factors $[1+\sqrt{t_2}\,b(m,n-1)\sigma_{mn-1}]$ on their places (on one line back), then split factors $(1+t_1\sigma_{m-1n}\sigma_{mn})$, as above, we again collect four factors with σ_{mn} as a general term. Summation over

each σ_{mn} leads to

$$Z=G\langle 0|\prod_{n=1}^{N}\Big\{\prod_{m=1}^{M}\{1+\sqrt{t_1t_2}\,a^+(m-1,n)b^+(m,n-1)\eta+\sqrt{t_1t_2}\,b(m,n)a(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_2}\,b^+(m,n-1)\eta][\sqrt{t_1}\,a(m,n)+\sqrt{t_2}\,b(m,n)]+$$
$$+t_1t_2\,a^+(m-1,n)b(m,n)b^+(m,n-1)\eta\, a(m,n)\}\Big\}|0\rangle=$$
$$=G\langle 0|\prod_{n=1}^{N}\Big\{\prod_{m=1}^{M}\{1+\sqrt{t_1t_2}\,a^+(m-1,n)b^+(m,n-1)+\sqrt{t_1t_2}\,b(m,n)a(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)-\sqrt{t_2}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)+\sqrt{t_2}\,b(m,n)]+$$
$$+t_1t_2\,a^+(m-1,n)b^+(m,n-1)b(m,n)a(m,n)\}\Big\}|0\rangle. \tag{4.30}$$

This differs from (4.7) in signs of the vertices

$$a^+(m-1,n)b(m,n) \quad \text{and} \quad b^+(m,n-1)a(m,n). \tag{4.31}$$

Nevertheless the equality of eqs. (4.7) and (4.30) is ensured by the derivation. We can account for the situation by the fact that each allowed polygon contains equal numbers of the vertices (4.31) (see figs. 3 and 4).

4) One could take the operator $\eta_B=(-1)^{N_B}$ instead of η, thus obtaining expressions of the form (4.7) with η_B substituted for η. However, now one could not remove η_B.

5) We could follow Fradkin and Steingradt[15] more closely. Then instead of the above vacua $\langle 0|$ and $|0\rangle$ we should take the "vacua" $\psi^T\langle 0|$ and $|0\rangle\psi$, where $\psi=\binom{0}{1}$ and $\psi^T=(0,1)$ are auxiliary spinors. Then using Pauli matrices τ_i (Clifford variables) we might split between the new "vacua" as follows

$$(1+t_1\sigma_{m-1n}\sigma_{mn})(1+t_2\sigma_{mn-1}\sigma_{mn})\to(1+t_1\sigma_{m-1n}\sigma_{mn})(1+t_2\tau_3\sigma_{mn-1}\sigma_{mn})$$
$$\to[1+\sqrt{t_1}\,\tau_1 a(m-1,n)\sigma_{m-1n}][1+\sqrt{t_1}\,\tau_1 a^+(m-1,n)\sigma_{mn}]$$
$$[1-i\sqrt{t_2}\,\tau_2 b(m,n-1)\sigma_{mn-1}][1+\sqrt{t_2}\,\tau_1 b^+(m,n-1)\sigma_{mn}] \tag{4.32}$$

and convert like above the general term of the product to the form

$$[1+\sqrt{t_1}\,\tau_1 a^+(m-1,n)\sigma_{mn}][1-i\sqrt{t_2}\,\tau_2 b(m,n)\sigma_{mn}]$$
$$[1+\sqrt{t_2}\,\tau_1 b^+(m,n-1)\sigma_{mn}][1+\sqrt{t_1}\,\tau_1 a(m,n)\sigma_{mn}], \tag{4.33}$$

thus obtaining eq. (4.8) again. Note that now we have anti-N-ordering at all stages, what permits us to reformulate this approach completely in terms of Grassmann variables (see Appendix), used in ref.[15].

6) In the cases of Bose and Pauli operators there are no problems with collecting together factors with the same σ_{mn}. We have no need to insert factors like η, and expressions (4.4)-(4.7) are valid with $\eta = 1$ for the bounded lattice. Under $\vdots$ $\vdots$ they are valid for the lattice on a torus.

5. <u>2-dimensional Ising model with magnetic field H.</u> We can transform the partition function in this case in the same manner as for H=0:

$$Z = \sum_{\{\sigma_{mn}=\pm 1\}} e^{\beta \sum_{m=1}^{M}\sum_{n=1}^{N}(\varepsilon_1 \sigma_{m-1n} + \varepsilon_2 \sigma_{mn-1} + H)\sigma_{mn}} = \tag{5.1}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \prod_{n=1}^{N}\prod_{m=1}^{M}(1+t_1\sigma_{m-1n}\sigma_{mn})(1+t_2\sigma_{mn-1}\sigma_{mn})(1+\bar{t}\sigma_{mn}) = \tag{5.2}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0| \prod_{n=1}^{N}\prod_{m=1}^{M}(1+t_1\sigma_{m-1n}\sigma_{mn})(1+t_2\eta\sigma_{mn-1}\sigma_{mn})(1+\bar{t}\sigma_{mn})|0\rangle = \tag{5.3}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0| \prod_{n=1}^{N}\Big\{\prod_{m=1}^{M}[1+\sqrt{t_1}\,a(m-1,n)\sigma_{m-1n}][1+\sqrt{t_1}\,a^+(m-1,n)\sigma_{mn}]$$
$$[1+\sqrt{t_2}\,\eta\, b(m,n-1)\sigma_{mn-1}][1+\sqrt{t_2}\, b^+(m,n-1)\sigma_{mn}](1+\bar{t}\sigma_{mn})\Big\}|0\rangle = \tag{5.4}$$

$$= F \sum_{\{\sigma_{mn}=\pm 1\}} \langle 0| \prod_{n=1}^{N}\Big\{\prod_{m=1}^{M}[1+\sqrt{t_1}\,a^+(m-1,n)\sigma_{mn}][1+\sqrt{t_2}\,\eta\, b(m,n)\sigma_{mn}]$$
$$[1+\sqrt{t_2}\, b^+(m,n-1)\sigma_{mn}][1+\sqrt{t_1}\,a(m,n)\sigma_{mn}](1+\bar{t}\sigma_{mn})\Big\}|0\rangle = \tag{5.5}$$

$$= G\langle 0| \prod_{n=1}^{N}\Big\{\prod_{m=1}^{M}\Big\{1+\bar{t}\sqrt{t_1}[a^+(m-1,n)+a(m,n)]+\bar{t}\sqrt{t_2}[b^+(m,n-1)+\eta b(m,n)]\Big\}$$
$$\Big\{1+\sqrt{t_1 t_2}\,a^+(m-1,n)b^+(m,n-1)+\sqrt{t_1 t_2}\,\eta\, b(m,n)a(m,n)+$$
$$+[\sqrt{t_1}\,a^+(m-1,n)+\sqrt{t_2}\,b^+(m,n-1)][\sqrt{t_1}\,a(m,n)+\sqrt{t_2}\,\eta\, b(m,n)]+$$
$$+t_1 t_2\, a^+(m-1,n)\eta\, b(m,n)\, b^+(m,n-1)a(m,n)\Big\}\Big\}|0\rangle, \tag{5.6}$$

where $\bar{t} = \mathrm{th}(\beta H)$, and

$$G = 2^{MN}F = 2^{MN}[\mathrm{ch}(\beta\varepsilon_1)]^{(M-1)N}[\mathrm{ch}(\beta\varepsilon_2)]^{M(N-1)}[\mathrm{ch}(\beta H)]^{MN}. \tag{5.7}$$

Like in the one-dimensional model with a field, here also there appear factors linear in creation and annihilation operators due to the field. They do not permit us to remove the operators η. One

can try to convert the linear factors into bilinear ones according to eq. (3.7). However, the presence of η in the bilinear factors of eq. (5.6) prevents this possibility (while η's in the linear terms do not interfere).

6. <u>Three-dimensional Ising model without field.</u> Let us consider a bounded 3-dimensional rectangular lattice and transform the partition function in terms of annihilation and creation operators in the same manner as in the 1-dimensional and 2-dimensional cases above.

$$Z = \sum_{\{\sigma_{lmn}=\pm 1\}} e^{\beta \sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N}(\varepsilon_1 \sigma_{l-1mn} + \varepsilon_2 \sigma_{lm-1n} + \varepsilon_3 \sigma_{lmn-1})\sigma_{lmn}} = \tag{6.1}$$

$$= F \sum_{\{\sigma_{lmn}=\pm 1\}} \prod_{l=1}^{L}\prod_{m=1}^{M}\prod_{n=1}^{N}(1+t_1\sigma_{l-1mn}\sigma_{lmn})(1+t_2\sigma_{lm-1n}\sigma_{lmn})(1+t_3\sigma_{lmn-1}\sigma_{lmn}) = \tag{6.2}$$

$$= F \sum_{\{\sigma_{lmn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}(1+t_1\zeta\sigma_{l-1mn}\sigma_{lmn})(1+t_2\xi\sigma_{lm-1n}\sigma_{lmn})(1+t_3\chi\sigma_{lmn-1}\sigma_{lmn})|0\rangle = \tag{6.3}$$

$$= F \sum_{\{\sigma_{lmn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}[1+\sqrt{t_1}\,\zeta' a(l-1,m,n)\sigma_{l-1mn}][1+\sqrt{t_1}\,a^+(l-1,m,n)\zeta''\sigma_{lmn}]$$
$$[1+\sqrt{t_2}\,\xi' b(l,m-1,n)\sigma_{lm-1n}][1+\sqrt{t_2}\,b^+(l,m-1,n)\xi''\sigma_{lmn}]$$
$$[1+\sqrt{t_3}\,\chi' c(l,m,n-1)\sigma_{lmn-1}][1+\sqrt{t_3}\,c^+(l,m,n-1)\chi''\sigma_{lmn}]|0\rangle = \tag{6.4}$$

$$= F \sum_{\{\sigma_{lmn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}[1+\sqrt{t_1}\,a^+(l-1,m,n)\zeta''\sigma_{lmn}][1+\sqrt{t_2}\,\xi' b(l,m-1,n)\sigma_{lm-1n}]$$
$$[1+\sqrt{t_2}\,b^+(l,m-1,n)\xi''\sigma_{lmn}][1+\sqrt{t_3}\,\chi' c(l,m,n-1)\sigma_{lmn-1}]$$
$$[1+\sqrt{t_3}\,c^+(l,m,n-1)\chi''\sigma_{lmn}][1+\sqrt{t_1}\,\zeta' a(l,m,n)\sigma_{lmn}]|0\rangle = \tag{6.5}$$

$$= F \sum_{\{\sigma_{lmn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}[1+\sqrt{t_1}\,a^+(l-1,m,n)\zeta''\sigma_{lmn}][1+\sqrt{t_2}\,\xi' b(l,m,n)\sigma_{lmn}]$$
$$[1+\sqrt{t_2}\,b^+(l,m-1,n)\xi''\sigma_{lmn}][1+\sqrt{t_3}\,\chi' c(l,m,n-1)\sigma_{lmn-1}]$$
$$[1+\sqrt{t_3}\,c^+(l,m,n-1)\chi''\sigma_{lmn}][1+\sqrt{t_1}\,\zeta' a(l,m,n)\sigma_{lmn}]|0\rangle = \tag{6.6}$$

$$= F \sum_{\{\sigma_{lmn}=\pm 1\}} \langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}[1+\sqrt{t_1}\,a^+(l-1,m,n)\zeta''\sigma_{lmn}][1+\sqrt{t_2}\,\xi' b(l,m,n)\sigma_{lmn}]$$
$$[1+\sqrt{t_2}\,b^+(l,m-1,n)\xi''\sigma_{lmn}][1+\sqrt{t_3}\,\chi' c(l,m,n)\sigma_{lmn}]$$
$$[1+\sqrt{t_3}\,c^+(l,m,n-1)\chi''\sigma_{lmn}][1+\sqrt{t_1}\,\zeta' a(l,m,n)\sigma_{lmn}]|0\rangle = \tag{6.7}$$

$$
\begin{aligned}
&=G\langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{\ell=1}^{L}\Big\{1+t_1a^+(\ell-1,m,n)\zeta''\zeta'a(\ell,m,n)+t_2\xi'b(\ell,m,n)b^+(\ell,m-1,n)\xi''+\\
&\qquad\qquad +t_3\chi'c(\ell,m,n)c^+(\ell,m,n-1)\chi''+\\
&+\sqrt{t_1t_2}\,a^+(\ell-1,m,n)\zeta''\left[b^+(\ell,m-1,n)\xi''+\xi'b(\ell,m,n)\right]+\\
&+\sqrt{t_1t_2}\left[b^+(\ell,m-1,n)\xi''+\xi'b(\ell,m,n)\right]\zeta'a(\ell,m,n)+\\
&+\sqrt{t_2t_3}\left[b^+(\ell,m-1,n)\xi''+\xi'b(\ell,m,n)\right]\left[c^+(\ell,m,n-1)\chi''+\chi'c(\ell,m,n)\right]+\\
&+\sqrt{t_1t_3}\,a^+(\ell-1,m,n)\zeta''\left[c^+(\ell,m,n-1)\chi''+\chi'c(\ell,m,n)\right]+\\
&+\sqrt{t_1t_3}\left[c^+(\ell,m,n-1)\chi''+\chi'c(\ell,m,n)\right]\zeta'a(\ell,m,n)+\\
&+t_1t_2\,a^+(\ell-1,m,n)\zeta''\xi'b(\ell,m,n)b^+(\ell,m-1,n)\xi''\zeta'a(\ell,m,n)+\\
&+t_2t_3\,\xi'b(\ell,m,n)b^+(\ell,m-1,n)\xi''\chi'c(\ell,m,n)c^+(\ell,m,n-1)\chi''+\\
&+t_3t_1\,a^+(\ell-1,m,n)\zeta''\chi'c(\ell,m,n)c^+(\ell,m,n-1)\chi''\zeta'a(\ell,m,n)+\\
&+t_1\sqrt{t_2t_3}\,a^+(\ell-1,m,n)\zeta''\left[b^+(\ell,m-1,n)\xi''+\xi'b(\ell,m,n)\right]\left[c^+(\ell,m,n-1)\chi''+\chi'c(\ell,m,n)\right]\zeta'a(\ell,m,n)+\\
&+t_2\sqrt{t_1t_3}\Big\{a^+(\ell-1,m,n)\zeta''\xi'b(\ell,m,n)b^+(\ell,m-1,n)\xi''\left[c^+(\ell,m,n-1)\chi''+\chi'c(\ell,m,n)\right]+\\
&\qquad +\xi'b(\ell,m,n)b^+(\ell,m-1,n)\xi''\left[c^+(\ell,m,n-1)\chi''+\chi'c(\ell,m,n)\right]\zeta'a(\ell,m,n)\Big\}+\\
&+t_3\sqrt{t_1t_2}\Big\{a^+(\ell-1,m,n)\zeta''\left[b^+(\ell,m-1,n)\xi''+\xi'b(\ell,m,n)\right]\chi'c(\ell,m,n)c^+(\ell,m,n-1)\chi''+\\
&\qquad +\left[b^+(\ell,m-1,n)\xi''+\xi'b(\ell,m,n)\right]\chi'c(\ell,m,n)c^+(\ell,m,n-1)\chi''\zeta'a(\ell,m,n)\Big\}+\\
&+t_1t_2t_3\,a^+(\ell-1,m,n)\zeta''\xi'b(\ell,m,n)b^+(\ell,m-1,n)\xi''\chi'c(\ell,m,n)c^+(\ell,m,n-1)\chi''\zeta'a(\ell,m,n)\Big\}|0\rangle,
\end{aligned}
\tag{6.8}
$$

where $t_i=\mathrm{th}(\beta\varepsilon_i)\quad(i=1,2,3)$,

$$G=2^{LMN}F=2^{LMN}\left[\mathrm{ch}(\beta\varepsilon_1)\right]^{(L-1)MN}\left[\mathrm{ch}(\beta\varepsilon_2)\right]^{L(M-1)N}\left[\mathrm{ch}(\beta\varepsilon_3)\right]^{LM(N-1)} \tag{6.9}$$

In eq. (6.1) summations are over $\sigma_{\ell mn}$ with $\ell=1,\ldots,L$; $m=1,\ldots,M$; $n=1,\ldots,N$, and superfluous variables σ_{0mn}, $\sigma_{\ell 0n}$, $\sigma_{\ell m0}$, encountered in eqs. (6.1)-(6.4), are zero for the bounded lattice

$$\sigma_{0mn}=\sigma_{\ell 0n}=\sigma_{\ell m0}=0\qquad(\ell=1,\ldots,L;\ m=1,\ldots,M;\ n=1,\ldots,N). \tag{6.10}$$

In eq. (6.3) operators $\zeta=\zeta'\zeta''$, $\xi=\xi'\xi''$, $\chi=\chi'\chi''$ are inserted to facilitate transferring some factors in what follows. It is supposed that $\zeta',\zeta'',\xi',\xi'',\chi',\chi''$ are some products of operators $\eta_a=(-1)^{N_a}$, $\eta_b=(-1)^{N_b}$ and $\eta_c=(-1)^{N_c}$. The operators η_a,η_b, and η_c anticommute with relative operators and commute with others, e.g.,

$$\{\eta_a a\}=\{\eta_a a^+\}=[\eta_a b]=[\eta_a b^+]=[\eta_a c]=[\eta_a c^+]=0. \tag{6.11}$$

Expression (6.4) is obtained by evident splitting of the factors entering into eq. (6.3). New factors do not commute, and we arrange them in the following order. The product is subdivided into N "pages" (n is the number of a page, n=1,...,N), each page is subdivided into M "lines" (m is the number of a line, m=1,...,M) with L "terms" in each line (ℓ is the number of a term in a line, $\ell=1,\ldots,L$). The term is the product of 6 factors, written in eq. (6.4). The terms are arranged in the ascending order of n, m, and ℓ from the left to the right. We wish to collect together all 6 factors with $\sigma_{\ell mn}$. Till now the general term of the product contained only three such factors. One more is near-by: it is the first factor $[1+\sqrt{t_1}\zeta'a(\ell,m,n)\sigma_{\ell mn}]$ of the next (ℓ+1)th term. The fifth factor $[1+\sqrt{t_2}\xi'b(\ell,m,n)\sigma_{\ell mn}]$ is precisely one line below (on (m+1)th line) of its analog in the general term of eq. (6.4). The sixth factor $[1+\sqrt{t_3}\chi'c(\ell,m,n)\sigma_{\ell mn}]$ is precisely one page below (on (n+1)th page) of its copy in the general term of eq.(6.4).

To obtain eq. (6.5), we include the factor $[1+\sqrt{t_1}\zeta'a(\ell,m,n)\sigma_{\ell mn}]$ into the general term, joining the first its factor to preceding term, etc. The factors $[1+\sqrt{t_1}\zeta'a(0,m,n)\sigma_{0mn}]$ at the beginning of each line are missed out and the factors $[1+\sqrt{t_1}\zeta'a(L,m,n)\sigma_{Lmn}]$ are incerted into eq. (6.5) as compared to eq. (6.4). However the equality remains, since all these factors equal 1, the first ones due to eq. (6.10), and the latter due to the absence of the mate operators $a^+(L,m,n)$ between the vacua (except possibly in η). Then we extract all the factors $[1+\sqrt{t_2}\xi'b(\ell,m,n)\sigma_{\ell mn}]$ and transfer them simultaneously on one line up (to the left) without changing their mutual order. The factors $[1+\sqrt{t_2}\xi'b(\ell,0,n)\sigma_{\ell 0n}]$ pushed into spaces between the pages are missed out, and the factors $[1+\sqrt{t_2}\xi'b(\ell,M,n)\sigma_{\ell Mn}]$ at the last line of each page are incerted in eq. (6.6) as compared to eq. (6.5). However, all these factors equal 1 due to eq. (6.10) and the absence of the mate operators $b^+(\ell,M,n)$ between the vacua. Finally, we extract all the factors $[1+\sqrt{t_3}\chi'c(\ell,m,n)\sigma_{\ell mn}]$ and transfer them simultaneously on one page back (to the left) without changing their mutual order. The factors $[1+\sqrt{t_3}\chi'c(\ell,m,0)\sigma_{\ell m0}]$ pushed into a space between $\langle 0|$ and the first page, are missed out, and the factors $[1+\sqrt{t_3}\chi'c(\ell,m,N)\sigma_{\ell mN}]$ on the last page are incerted in eq. (6.7) as compared to eq. (6.6). Again all these factors equal 1 due to eq. (6.10) and the absence of the mate operators $c^+(\ell,m,N)$ between the vacua. In eq. (6.7), where each term depends only on one $\sigma_{\ell mn}$, one can easily carry out summations over all $\sigma_{\ell mn}$. The expansion of 6 factors of the general term contains $C_6^0+C_6^2+C_6^4+C_6^6=32$ addends with even powers of $\sigma_{\ell mn}$ and $C_6^1+C_6^3+C_6^5=32$ addends with odd ones. Using eq. (2.8), we obtain eq. (6.8).

When transferring the factors we suggested tacitly that they commute with all encountered factors along their way due to an appropriate choice of $\zeta', \zeta'', \xi', \xi'', \chi', \chi''$. So, one can choose

$$\zeta'=\zeta''=1,\ \xi'=\eta_a\eta_b=\eta\eta_c,\ \xi''=1,\ \chi'=\eta_a\eta_b\eta_c=\eta,\ \chi''=\eta_a\eta_b=\eta\eta_c. \qquad (6.12)$$

The operator $\eta=\eta_a\eta_b\eta_c$ can be removed like in the 2-dimensional case, and we get

$$\begin{aligned}
Z=G\langle 0|\prod_{n=1}^{N}\prod_{m=1}^{M}\prod_{l=1}^{L}\Big\{&1+t_1a^+(l-1,m,n)a(l,m,n)+t_2\eta_c b(l,m,n)b^+(l,m-1,n)+\\
&+t_3c(l,m,n)c^+(l,m,n-1)\eta_c+\\
&+\sqrt{t_1t_2}\,a^+(l-1,m,n)[b^+(l,m-1,n)-\eta_c b(l,m,n)]+\\
&+\sqrt{t_1t_2}\,[b^+(l,m-1,n)+\eta_c b(l,m,n)]a(l,m,n)+\\
&+\sqrt{t_2t_3}\,[b^+(l,m-1,n)+\eta_c b(l,m,n)][c^+(l,m,n-1)\eta_c-c(l,m,n)]+\\
&+\sqrt{t_1t_3}\,a^+(l-1,m,n)[c^+(l,m,n-1)\eta_c-c(l,m,n)]+\\
&+\sqrt{t_1t_3}\,[-c^+(l,m,n-1)\eta_c+c(l,m,n)]a(l,m,n)-\\
&-t_1t_2a^+(l-1,m,n)\eta_c b(l,m,n)b^+(l,m-1,n)a(l,m,n)+\\
&+t_2t_3b(l,m,n)b^+(l,m-1,n)c(l,m,n)c^+(l,m,n-1)+\\
&+t_3t_1\,a^+(l-1,m,n)c(l,m,n)c^+(l,m,n-1)\eta_c a(l,m,n)+\\
&+t_1\sqrt{t_2t_3}\,a^+(l-1,m,n)[b^+(l,m-1,n)-\eta_c b(l,m,n)][-c^+(l,m,n-1)\eta_c+c(l,m,n)]a(l,m,n)+\\
&+t_2\sqrt{t_1t_3}\{-a^+(l-1,m,n)\eta_c b(l,m,n)b^+(l,m-1,n)[c^+(l,m,n-1)\eta_c-c(l,m,n)]+\\
&\qquad+\eta_c b(l,m,n)b^+(l,m-1,n)[-c^+(l,m-1,n)\eta_c+c(l,m,n)]a(l,m,n)\}+\\
&+t_3\sqrt{t_1t_2}\{a^+(l-1,m,n)[b^+(l,m-1,n)-\eta_c b(l,m,n)]c(l,m,n)c^+(l,m,n-1)\eta_c+\\
&\qquad+[b^+(l,m-1,n)+\eta_c b(l,m,n)]c(l,m,n)c^+(l,m,n-1)\eta_c a(l,m,n)\}-\\
&-t_1t_2t_3a^+(l-1,m,n)b(l,m,n)b^+(l,m-1,n)c(l,m,n)c^+(l,m,n-1)a(l,m,n)\Big\}|0\rangle. \qquad (6.13)
\end{aligned}$$

It is just the operator η_c which ensures for each polygon on the space lattice now to contribute +1. However, η_c prevents further advance. Other possibilities for choice $\zeta', \zeta'', \dots, \chi''$ are, e.g.,

$$\zeta'=\zeta''=\xi'=1,\ \ \xi''=\chi'=\eta_a\eta_b=\eta\eta_c,\ \ \chi''=\eta_a\eta_b\eta_c=\eta, \qquad (6.14)$$

$$\zeta'=\zeta''=1,\ \ \xi'=\eta_b,\ \ \xi''=1,\ \ \chi'=\eta_b\eta_c,\ \ \chi''=\eta_b, \qquad (6.15)$$

$$\zeta'=\zeta''=\xi'=1,\ \ \xi''=\chi'=\eta_b,\ \ \chi''=\eta_b\eta_c. \qquad (6.16)$$

When splitting there are needed at least two of the operators η_a, η_b, η_c, but one cannot remove all of them at the end. In the last two cases (6.15) and (6.16) two operators remain unlike the choices (6.12) and (6.14). One more way (analogous to the Fradkin and Steingradt one for the "-dimensional case) is to replace the vacua $\langle 0|$ and $|0\rangle$ by $\psi^T\langle 0|$ and $|0\rangle\psi$ with $\psi^T=(1\ 0\ 0\ 0)$ and to use the 4x4 Dirac matrices for $\zeta', \dots, \chi''$

$$\zeta'=\zeta''=\gamma_4,\ \xi'=-i\gamma_1,\ \xi''=\gamma_2,\ \chi'=-i\gamma_3,\ \chi''=\gamma_5,$$

$$\zeta=I,\ \ \xi=-i\gamma_1\gamma_2=\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},\quad \chi=-i\gamma_3\gamma_5=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}, \qquad (6.17)$$

where $\gamma_k=\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix}(k=1,2,3)$, $\gamma_4=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$, $\gamma_5=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix}$.

The choice is not unique. In any case, however, the γ-matrices do not fall out completely. Note that now expression between $\langle 0|$ and $|0\rangle$ occurs to be anti-N-ordered.

7. <u>3-dimensional Ising model with magnetic fields H.</u> Due to the field the factors $(1+\bar{t}\,\sigma_{lmn})$ appear in the partition function. We leave them to be c-number. Other factors are transformed precisely as above. Now after summation over all σ_{lmn} one obtains a product with a general term, which contains 64 addends and falls into a linear in a, b, c, a^+, b^+, c^+ factor with $\bar{t}$ and old bilinear factor (such as in eq. (6.8)).

8. Let us give some alternative representations of partition functions in the Ising model. Thus, in terms of transfer matrices they can in 1-, 2- and 3-dimensional cases (on rectangular lattices) be represented as follows

$$Z=[2\,ch(\beta\varepsilon)]^N\, tr\Big\{\Big[(th(\beta\varepsilon))^{\sigma_N^+\sigma_N^-}\, e^{H\sigma_N^x}\Big]^N\Big\}, \qquad (8.1)$$

$$Z=[2\,ch(\beta\varepsilon_2)]^{MN}\, tr\Big\{\Big[(th(\beta\varepsilon_2))^{\sum_{m=1}^{M}\sigma_{mN}^+\sigma_{mN}^-}\, e^{\beta\varepsilon_1\sum_{m=1}^{M}\sigma_{m-1N}^x\sigma_{mN}^x}\Big]^N\Big\}, \qquad (8.2)$$

$$Z=[2\,ch(\beta\varepsilon_3)]^{LMN}\, tr\Big\{\Big[(th(\beta\varepsilon_3))^{\sum_{l=1}^{L}\sum_{m=1}^{M}\sigma_{lmN}^+\sigma_{lmN}^-}\, e^{\beta\sum_{l=1}^{L}\sum_{m=1}^{M}(\varepsilon_1\sigma_{l-1mN}+\varepsilon_2\sigma_{lm-1N})\sigma_{lmN}}\Big]^N\Big\}, \qquad (8.3)$$

where σ^+, σ^- and σ^x are the 2x2 matrices $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, labelled by some indices, and periodicity conditions are taken into account. For derivation of eqs. (8.1) and (8.2) and further transformations see paper /11/ by Schultz, Mattis and Lieb. We obtain expression (8.3) for the 3-dimensional case analogously.

Note, that other approaches are possible for splitting of factors instead of the way used in Secs.2-6 . For example, one can use

σ- or γ-matrices instead of the creation and annihilation operators, and the trace, instead of taking the vacuum expectation. One more possibility for splitting is using variables $\sigma(m,n)$ and $\tau(m,n)$ (or $\rho(\ell,m,n)$, $\sigma(\ell,m,n)$ and $\tau(\ell,m,n)$ in the 3-dimensional case) of the same nature (i.e., having the values ± 1) as original variables σ_{mn} (or $\sigma_{\ell mn}$). In the 1-dimensional case expression (2.2) passes merely into itself. In 2- and 3-dimensional cases we obtain

$$1+t_1\sigma_{m-1n}\sigma_{mn}=\frac{1}{2}\sum_{\sigma(m-1,n)=\pm 1}[1+\sqrt{t_1}\,\sigma(m-1,n)\sigma_{m-1n}][1+\sqrt{t_1}\,\sigma(m-1,n)\sigma_{mn}],$$

$$1+t_2\sigma_{mn-1}\sigma_{mn}=\frac{1}{2}\sum_{\tau(m,n-1)=\pm 1}[1+\sqrt{t_2}\,\tau(m,n-1)\sigma_{mn-1}][1+\sqrt{t_2}\,\tau(m,n-1)\sigma_{mn}], \tag{8.4}$$

$$Z=\left[\frac{1}{2}ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)\right]^{MN}\sum_{\{\sigma(m,n)=\pm 1\}}\;\sum_{\{\tau(m,n)=\pm 1\}}\;\prod_{m=1}^{M}\prod_{n=1}^{N}$$
$$[1+t_1\sigma(m-1,n)\sigma(m,n)][1+t_2\tau(m,n-1)\tau(m,n)]$$
$$\left\{1+\frac{\sqrt{t_1t_2}}{(1+t_1)(1+t_2)}[\sigma(m-1,n)+\sigma(m,n)][\tau(m,n-1)+\tau(m,n)]\right\}; \tag{8.5}$$

$$1+t_1\sigma_{\ell-1mn}\sigma_{\ell mn}=\frac{1}{2}\sum_{\rho(\ell-1,m,n)=\pm 1}[1+\sqrt{t_1}\,\rho(\ell-1,m,n)\sigma_{\ell-1mn}][1+\sqrt{t_1}\,\rho(\ell-1,m,n)\sigma_{\ell mn}],$$

$$1+t_2\sigma_{\ell m-1n}\sigma_{\ell mn}=\frac{1}{2}\sum_{\sigma(\ell,m-1,n)=\pm 1}[1+\sqrt{t_2}\,\sigma(\ell,m-1,n)\sigma_{\ell m-1n}][1+\sqrt{t_2}\,\sigma(\ell,m-1,n)\sigma_{\ell mn}],$$

$$1+t_3\sigma_{\ell mn-1}\sigma_{\ell mn}=\frac{1}{2}\sum_{\tau(\ell,m,n-1)=\pm 1}[1+\sqrt{t_3}\,\tau(\ell,m,n-1)\sigma_{\ell mn-1}][1+\sqrt{t_3}\,\tau(\ell,m,n-1)\sigma_{\ell mn}], \tag{8.6}$$

$$Z=\left[\frac{1}{4}ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)ch(\beta\varepsilon_3)\right]^{LMN}\sum_{\{\rho(\ell,m,n)=\pm 1\}}\;\sum_{\{\sigma(\ell,m,n)=\pm 1\}}\;\sum_{\{\tau(\ell,m,n)=\pm 1\}}$$
$$\prod_{\ell=1}^{L}\prod_{m=1}^{M}\prod_{n=1}^{N}[1+t_1\rho(\ell-1,m,n)\rho(\ell,m,n)][1+t_2\sigma(\ell,m-1,n)\sigma(\ell,m,n)][1+t_3\tau(\ell,m,n-1)\tau(\ell,m,n)]$$
$$\left\{1+\frac{\sqrt{t_1t_2}}{(1+t_1)(1+t_2)}[\rho(\ell-1,m,n)+\rho(\ell,m,n)][\sigma(\ell,m-1,n)+\sigma(\ell,m,n)]+\right.$$
$$+\frac{\sqrt{t_2t_3}}{(1+t_2)(1+t_3)}[\sigma(\ell,m-1,n)+\sigma(\ell,m,n)][\tau(\ell,m,n-1)+\tau(\ell,m,n)]+$$
$$\left.+\frac{\sqrt{t_1t_3}}{(1+t_1)(1+t_3)}[\tau(\ell,m,n-1)+\tau(\ell,m,n)][\rho(\ell-1,m,n)+\rho(\ell,m,n)]\right\}. \tag{8.7}$$

Applying the procedure of Schultz, Mattis and Lieb to these expressions we obtain

$$Z=[2\,ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)]^{MN}\operatorname*{tr}_{\substack{\sigma(n)\\ \tau(m)}}\prod_{n=1}^{N}\left\{\prod_{m=1}^{M}[S(n)T(m)+\sqrt{t_1t_2}\,\sigma^x(n)\tau^x(m)]\right\} \tag{8.8}$$

where

$$S(n)=[th(\beta\varepsilon_1)]^{\sigma^+(n)\sigma^-(n)}=\begin{bmatrix}t_1 & 0\\ 0 & 1\end{bmatrix}_n,\quad T(m)=[th(\beta\varepsilon_2)]^{\tau^+(m)\tau^-(m)}=\begin{bmatrix}t_2 & 0\\ 0 & 1\end{bmatrix}_m, \tag{8.9}$$

$$Z=[2\,ch(\beta\varepsilon_1)ch(\beta\varepsilon_2)ch(\beta\varepsilon_3)]^{LMN}\operatorname{tr}\prod_{n=1}^{N}\left\{\prod_{m=1}^{M}\left\{\prod_{\ell=1}^{L}\right.\right.$$
$$[R(m,n)S(\ell,n)T(\ell,m)+\sqrt{t_1t_2}\,\rho^x(m,n)\sigma^x(\ell,n)T(\ell,m)+$$
$$\left.\left.+\sqrt{t_2t_3}\,R(m,n)\sigma^x(\ell,n)\tau^x(\ell,m)+\sqrt{t_1t_3}\,\rho^x(m,n)S(\ell,m)\tau^x(\ell,m)]\right\}\right\}, \tag{8.10}$$

where

$$R(m,n)\equiv R(L,m,n)=[th(\beta\varepsilon_1)]^{\rho^+(L,m,n)\rho^-(L,m,n)}=\begin{bmatrix}t_1 & 0\\ 0 & 1\end{bmatrix}_{Lmn},$$
$$S(\ell,n)\equiv S(\ell,M,n)=[th(\beta\varepsilon_2)]^{\sigma^+(\ell,M,n)\sigma^-(\ell,M,n)}=\begin{bmatrix}t_2 & 0\\ 0 & 1\end{bmatrix}_{\ell Mn},$$
$$T(\ell,m)\equiv T(\ell,m,N)=[th(\beta\varepsilon_3)]^{\tau^+(\ell,m,N)\tau^-(\ell,m,N)}=\begin{bmatrix}t_3 & 0\\ 0 & 1\end{bmatrix}_{\ell mN}. \tag{8.11}$$

The matrices $\sigma(n)\equiv\sigma(M,n)$ and $\tau(m)\equiv\tau(m,N)$ commute mutually, and so is for $\rho(L,m,n)$, $\sigma(\ell,M,n)$ and $\tau(\ell,m,N)$. The factors [ ] of the products in eq. (8.8) and (8.10) are arranged like above, but in the descending order of n and m (or n , m and l) from the left to the right.

Appendix. The anti-N-product may be defined as that in which each annihilation operator stands to the left from its creation operator. Properties of the anti-N-ordered expressions are:

a) Under anti-N-ordering symbol ⁝ ⁝ the creation and annihilation operators can be transferred as purely commuting quantities (like c-numbers) in the Bose case, and as purely anticommuting quantities (like Grassmann variables) in the Fermi case. The situation is similar to those with T- and N-orderings (and with some other orderings).

b) An analog of the Wick theorem is valid for a decomposition of the anti-N-product into N-products with the pairings

$$\underbrace{a(n)a^+(n)}=1\ , \quad \underbrace{a^+(n)a(n)}=\begin{cases} 1 \text{ in Bose case} \\ -1 \text{ in Fermi case}\end{cases} \tag{A.1}$$

Other pairing equals zero.

In the Fermi case under $\vdots\quad\vdots$ the square of any combination linear in the creation and annihilation operators equals zero, e.g.,

$$[a(n)]^2 = [a^+(n)]^2 = [\alpha a^+(m-1,n) + \beta\, b(m,n)]^2 = 0$$

$$[\alpha a^+(m,n) + \beta\, a(m,n)]^2 = 0 \tag{A.2}$$

(while without ordering $[\alpha a^+(m,n)+\beta\, a(m,n)]^2 = \alpha\beta\{a(m,n)a^+(m,n)\} = \alpha\beta$). If there are no ordering one cannot bring together two factors $e^{A(n)}$ and $e^{A(n+1)}$ into one exponential function $e^{A(n)+A(n+1)}$ since the exponent contain noncommuting operators (for A(n) see, e.g., eq. (3.12)). However this is possible under $\vdots\quad\vdots$ (like under T- or N-ordering symbols).

Pauli operators are defined as creation and annihilation operators, which obey the mixed set of commutation-anticommutation rules

$$\{a_n, a_n\} = \{a^+_n, a^+_n\} = 0, \quad \{a_n, a^+_n\} = 1 \qquad \text{(no summation)}$$

$$[a_m, a_n] = [a^+_m, a^+_n] = [a_m, a^+_n] = 0 \qquad \text{for } m \neq n. \tag{A.3}$$

The Pauli operators can be represented through 2x2 matrices

$$a_n = 1\otimes 1\otimes\cdots\otimes 1\otimes\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes 1\otimes\cdots\otimes 1,\quad a^+_n = 1\otimes 1\otimes\cdots\otimes 1\otimes\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes 1\otimes\cdots\otimes 1, \tag{A.4}$$

where $1\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ are n th factors. Vacuum is $|0\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}0\\1\end{pmatrix}$. For the Pauli operators one can also define anti-N-ordering (and N- and T-orderings too) with pure anticommutation of operators with the same numbers and commutation of operators with different numbers.

Let us demonstrate the equivalence of the Grassmann-variable formalism with the operator one. Let a_n and a^+_n be at first the Bose operators. Any operator $\hat{F}$ may be written in N-ordered, symmetrized and anti-N-ordered forms

$$\hat{F} = :\mathbb{F}_1(a,a^+): = \text{sym}\,\mathbb{F}_2(a,a^+) = \vdots\,\mathbb{F}_3(a,a^+)\,\vdots \tag{A.5}$$

We are interested in the latter. Let $|\alpha\rangle = |\alpha_1, \alpha_2, \dots, \alpha_N\rangle$ be coherent states

$$a_n|\alpha\rangle = \alpha_n|\alpha\rangle\ , \quad \langle\alpha|a^+_n = \alpha^+_n\langle\alpha|\ , \quad n = 1,\dots,N\ , \tag{A.6}$$

where $\alpha_1, \alpha_2, \dots, \alpha_N$ are complex eigenvalues. Then the following relations are valid (see, e.g., ref. /25/, Appendix)

$$\langle\alpha|\,\vdots\,\mathbb{F}_3(a,a^+)\,\vdots\,|\alpha\rangle = \Lambda^2\langle\alpha|:\mathbb{F}_3(a,a^+):|\alpha\rangle = \Lambda^2\mathbb{F}_3(\alpha,\alpha^*) = \tag{A.7.a}$$

$$= \left(\frac{i}{\pi}\right)^N \int \prod_{n=1}^{N} d\alpha'_n\, d\alpha'^*_n\, |\langle\alpha|\alpha'\rangle|^2\, \mathbb{F}_3(\alpha',\alpha'^*), \tag{A.7.b}$$

where

$$\Lambda = e^{\frac{1}{2}\sum\frac{\partial}{\partial\alpha_n}\frac{\partial}{\partial\alpha^*_n}}, \quad |\langle\alpha|\alpha'\rangle|^2 = e^{-\sum|\alpha_n-\alpha'_n|^2}. \tag{A.8}$$

Note that (A.7.a) and (A.7.b) are two possible representations of the Gauss transformation. From eq. (A.7) there follows for the vacuum expectation of the operator

$$\langle 0|\,\vdots\,\mathbb{F}_3(a,a^+)\,\vdots\,|0\rangle = \Lambda^2\,\mathbb{F}_3(\alpha,\alpha^*)\big|_{\alpha=\alpha^*=0} = \tag{A.9.a}$$

$$= \left(\frac{i}{\pi}\right)^N \int \prod_{n=1}^{N} d\alpha'_n\, d\alpha'^*_n\, |\langle 0|\alpha'\rangle|^2\, \mathbb{F}_3(\alpha',\alpha'^*), \tag{A.9.b}$$

$$|0\rangle \equiv |\alpha\rangle\big|_{\alpha=0}, \quad |\langle 0|\alpha'\rangle|^2 = e^{-\sum\alpha'^*_n\alpha'_n}. \tag{A.10}$$

Let now a_n and a^+_n be Fermi operators, then all these constructions are valid with α and α^* being Grassmann variables (Feynman, Schwinger, Beresin /13/). In their treatments of the Ising model Beresin /13/ used expression (A.9.b), and Fradkin and Steingradt /15/ exploited mainly expression (A.9.a).

We stress that the above Gauss transformation performs N-ordering to compute expectations (A.7) and (A.9). Operators similar to Λ^2 were introduced by Hori /26/ in quantum electrodynamics for N-ordering of T-products of Bose and Fermi fields.

Note that the above Grassmann variable formalism corresponds only to operators, given in an anti-N-ordered form. However the formalism of creation and annihilation operators is more general and flexible, since it permits operators in any form, e.g., including operators $\eta = (-1)^N$ which destroy anti-N-ordered form.

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Полубаринов И.В. E17-84-413

О преобразовании сумм в модели Изинга

Метод Фрадкина и Штейнградта изложен на языке операторов рождения и уничтожения, что дает некоторую дополнительную свободу действия. По этому методу преобразованы статистические суммы в модели Изинга на 1-, 2- и 3-мерных решетках без магнитного поля и с магнитным полем. Подробно изложен 2-мерный случай без поля.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Polubarinov I.V. E17-84-413

On Transformation of Sums in Ising Model

The method by Fradkin and Steingradt is exposed in terms of creation and annihilation operators what provides with some additional freedom for action. Partition functions in the Ising model on 1-, 2- and 3-dimensional lattices without and with a magnetic field are transformed according to this approach. Two-dimensional case without the field is presented in detail .

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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