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NOISE INDUCED TRANSITION IN A SIMPLE NONLINEAR MODEL DRIVEN BY DICHOTOMOUS MARKOVIAN NOISE

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1. INTRODUCTION

We consider the nonlinear Langevin equation

$$\dot{x}(t) = a\,x(t) - x^3(t) + x(t)\,\xi(t), \tag{1}$$

which represents the simplest spatially homogeneous Landau model subject to an external multiplicative noise $\xi(t)$ coupling to the linear term. This model is often used to study nonequilibrium systems [1-9] and can be experimentally realized in parametrically excited electric circuits [5-7]. Quantities of interest are the stationary probability density $P_s(x)$ and the long time behaviour of correlation functions. Qualitative changes in the behaviour of this quantities, e.g., a change from monomodal to multimodal shape of $P_s(x)$ or from exponential to algebraic decay of the correlations, are called noise induced transitions [10].

The driving process $\xi(x)$ is usually supposed to be a simple stochastic model process. If $\xi(x)$ is a Gaussian white noise with autocorrelation function $\langle \xi(t)\,\xi(t') \rangle = D\,\delta(t-t')$ the stationary probability density can be obtained exactly [1-4] as

$$P_s(x) = (1/2D)^{a/2D}\,\Gamma^{-1}(a/2D)\,|x|^{a/D-1}\,e^{-x^2/2D}, \tag{2}$$

Γ denotes the Gamma function. The maxima of P_s given by

$$x_{max} = \begin{cases} 0 \\ \pm(a-D)^{1/2} \end{cases} \quad a \lessgtr D \tag{3}$$

can be compared with the stationary states of the deterministic case. The value of the control parameter a at which the bifurcation occurs is simply shifted by the strength of the noise D.

If $\xi(t)$ is the simplest continuous stochastic process with finite correlation time, the Ornstein-Uhlenbeck process with autocorrelation $\langle \xi(t)\,\xi(t') \rangle = (\tilde{D}/\tau)\cdot e^{-|t-t'|/\tau}$, one obtains only approximate results. Besides the shift of the bifurcation one finds in this case regions in the parameter plane in which P_s has maxima without deterministic equivalent (noise induced states) [8,9].

In this paper we consider the influence of the simplest discrete stochastic process, the dichotomous markovian process

I_t, jumping randomly between two states. Assuming this process the stationary probability density P_s can be calculated exactly. We obtain an explicit formula for P_s in the general case $\dot{x} = f(x) + g(x) I_t$, where f and g are polynomial functions. For the specific model (1) we discuss the qualitative shape of P_s in the whole parameter region and study the behaviour near the bifurcation line. Finally, the relevance of our results to the problem of the onset of an electrohydrodynamical instability of a nematic liquid crystal under influence of a stochastic voltage is shortly discussed.

2. THE DICHOTOMOUS MARKOVIAN PROCESS

The dichotomous markovian process (DMP) jumps with equal probability between two values $\pm\Delta$. The number of jumps K which occur in the time t is governed by the Poisson distribution

$$P(K,t) = e^{-\alpha t} (\alpha t)^K / K!\,, \tag{4}$$

α denotes the mean number of jumps per unit time. Writing the process as function of K like $I_t = I_K = \Delta(-1)^K$ simple averages can be easily evaluated. For instance, one readily verifies the exponential autocorrelation

$$\langle I_t I_{t'} \rangle = \Delta^2 e^{-2\alpha |t-t'|}. \tag{5}$$

More complicated averages can be calculated exploiting several theorems reflecting the markovian character of the process [11,12].

The results for the Gaussian white noise can be recovered in the white noise limit $\Delta, \alpha \to \infty$, $\Delta^2/2\alpha = D = \text{const}$.

3. THE STATIONARY PROBABILITY DENSITY

The stationary probability density for the more general process

$$\dot{x} = f(x) + g(x) I_t = F(x, I_t), \tag{6}$$

f and g being arbitrary functions, can be obtained using different methods in a semiexplicit form [13-18] as

$$P_s(x) \propto (1/F_+(x) - 1/F_-(x))\, e^{-\alpha \int^x dy\,(1/F_+(y) + 1/F_-(y))}, \tag{7}$$

with the shorthand $F_\pm(x) = F(x, \pm\Delta)$. In the important case that f and g are polynomial functions we can evaluate (7) explicitly. For $F_\pm(x)$ being a polynomial of order n with simple

zeros $a_\nu^\pm$ we obtain

$$P_s(x) = N g(x) \prod_{\nu=1}^{n} (x - a_\nu^+)^{\lambda_\nu^+} (x - a_\nu^-)^{\lambda_\nu^-} \tag{8}$$

with $\lambda_\nu^\pm = -\alpha / F'_\pm(a_\nu^\pm) - 1$, and N being the normalization. This result shows that the zeros of $\lambda_\nu^\pm$ separate the parameter space in regions characterized by different combinations of signs of the exponents $\lambda_\nu^\pm$. For positive (negative) values of these exponents $P_s(x)$ vanishes (diverges) as the corresponding bases tend to zero. Differentiating (7) the extreme values of P_s are determined by [15,16]

$$f - (\Delta^2/2\alpha) g g' + f f'/\alpha - f^2 g'/(2\alpha g) = 0. \tag{9}$$

The first term corresponds to the deterministic states and survives in the fast motion limit (FML: $\alpha \to \infty$). The second term survives in the white noise limit (WNL) and contributes only for multiplicative stochastic processes for which $g' \neq 0$. The following both terms represent corrections due to the finite correlation time $1/\alpha$.

Analyzing the possible numbers of extreme values given by (9) and the behaviour near the boundaries of the support one can decide whether new noise induced states occur or not.

4. RESULTS FOR THE SPECIFIC MODEL

To determine the support of $P_s(x)$ for the specific model (1) we consider the realizations corresponding to $I_t = \pm\Delta$ separately and perform a linear stability analysis of the stationary states given by

$$\dot{x} = F_\pm(x) = x(a \pm \Delta - x^2) = 0. \tag{10}$$

Results are shown in Fig. 1. For $t \to \infty$ the systems will flow into the hatched region (respectively to $x = 0$ for $a < -\Delta$). Once catched there the system cannot escape. Thus for $-\Delta < a$ the support consists of the hatched region in Fig. 1.

From (9) we obtain the stationary probability distribution

$$P_s(x) = N |x|^{-2(\lambda+\mu)-1} (a + \Delta - x^2)^{\lambda - 1} (x^2 - (a - \Delta))^{\mu - 1} \tag{11}$$

with $2\lambda = \alpha/(a + \Delta)$ and $2\mu = \alpha/(a - \Delta)$. $P_s(x)$ is obviously a symmetric function. The normalization N is determined by

$$\int_{\text{support}} dx\, P_s(x) = 1. \tag{12}$$

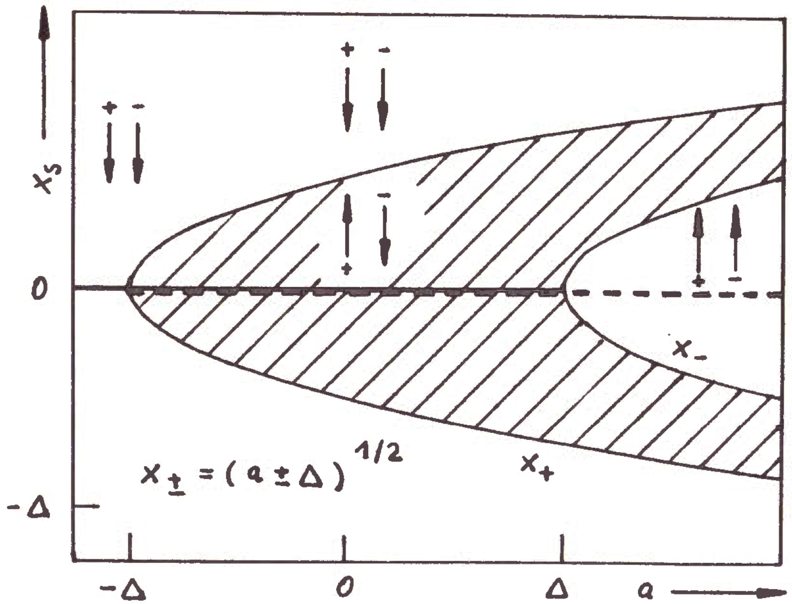

Fig. 1. Linear stability analysis of $\dot{x} = F_\pm(x)$. Solid (broken) lines correspond to stable (unstable) stationary states. The arrows indicate the flow for the different realizations of I_t.

Exploiting properties of the hypergeometric function we obtain

$$N = \begin{cases} (2\Delta)^{1-\lambda-\mu}\,(\Delta - a)\; 2^{-1}(\Delta + a)^{\mu+1} B^{-1}(\lambda, -\lambda-\mu) \\ (2\Delta)^{1-\lambda-\mu}\,(a - \Delta)\; a\,(\Delta + a)^{\mu} B^{-1}(\lambda, \mu) \end{cases} \quad (13)$$

for $|a| < \Delta$ and $\Delta < a$, respectively. B denotes the Beta function.

Now we analyze the qualitative shape of P_s in the parameter plane a/Δ, $\alpha/2\Delta$. The plane is separated in regions characterized by different combinations of signs of the exponents in (11). For positive (negative) values of the exponents $P_s(x)$ vanishes (diverges) as the corresponding base tends to zero.

According to (9) the extreme values of P_s are given by a biquadratic equation with the solution

$$x^2_{1,2} = (\alpha + 3a \pm ((\alpha - 2a)^2 + 5\Delta^2))^{1/2})/5. \quad (14)$$

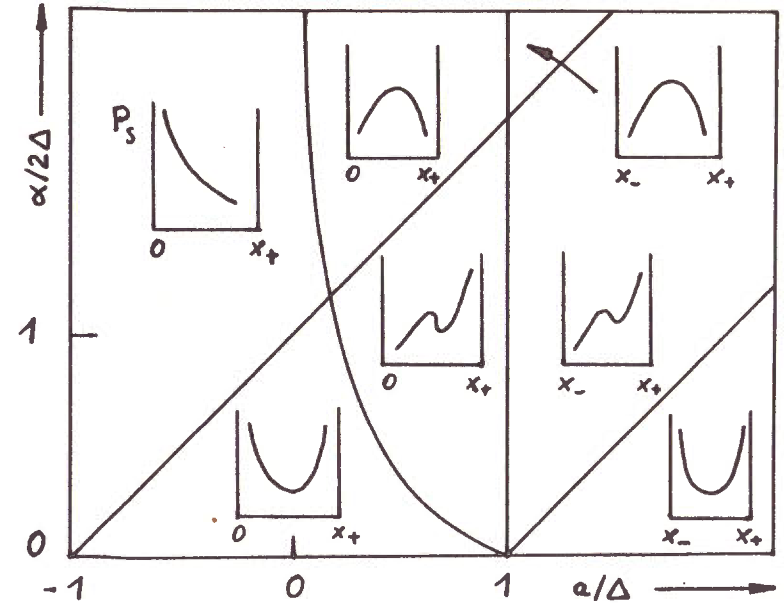

Fig. 2. Qualitative shape of $P_s(x > 0)$ in the parameter plane (phase diagram).

These extremal values can be inside or outside the support of P_s. Correspondingly P_s has the qualitative shape shown in Fig.2.

5. DISCUSSION

The system is governed by three parameters which play a different role. The control parameter a causes when changing its sign in the deterministic case a bifurcation of the stable stationary state. The dichotomous noise I_t couples like a to the linear term and describes therefore stochastic changes of this parameter. I_t is characterized by the jump width Δ and the mean number of jumps in unit time α. These two parameters compete in the following way.

For large α the system can hardly follow the noise. P_s changes with increasing a from monomodal to bimodal shape. The bifurcation occurs crossing the line

$$a_c = (\alpha^2 + \Delta^2)^{1/2} - \alpha \approx \Delta^2/2\alpha - \Delta^4/8\alpha^3. \quad (15)$$

Thus the shift of the bifurcation is smaller than for the corresponding white noise, where $a_c = \Delta^2/2\alpha = D$. For $\alpha \to \infty$ (FML) the system lives on the deterministic states, $x_2 \to \pm a^{1/2}$. For small α only few jumps occur in unit time. The system has time enough to reach the stationary states given by $F_\pm(x) = 0$, the number of which is larger than in the deterministic case (cf. Fig.1). The stationary states without deterministic equivalent can be considered as noise induced. With increasing the corresponding maxima of P_s are shifted and change its relative weight. For large α P_s has maxima near the deterministic states as discussed above. For $a/\Delta > 1$ the support consists of two unconnected branches due to the discrete nature of the DMP. Only this region of the phase diagram has been recently discussed in an experimental context[7]. It is suggestive to compare the maxima of P_s with the order parameter of equilibrium phase transitions. This is supported by the behaviour near the bifurcation line

$$x_{max} = A(\Delta,\alpha)(a - a_c(\Delta,\alpha))^{1/2}, \tag{16}$$

analogous to a continuous phase transition. Both the critical value a_c and the amplitude A depend on the noise parameters Δ and α but not the critical exponent $\beta = 1/2$ which in determined only by the special choice of the functions f and g. The same behaviour has been found for the Gaussian white noise[4] and approximatively for the Ornstein-Uhlenbeck process (OUP)[8,9]. Thus, in a certain sense, β can be considered as an universal quantity. Deviations from the mean field value 1/2 can be expected in a theory which takes into account the spatial inhomogeneity of the order parameter. It would be of considerable interest to establish further analogies to equilibrium phase transitions.

For both DMP and OUP the autocorrelation functions are of the same form and one finds a transition from monomodal to bimodal shape of P_s corresponding to the deterministic transition. The noise induced states however, are different due to the different nature of the driving processes. According to the appearance of noise induced states we find more complicated bifurcation patterns than in the deterministic case.

In this paper we considered only the stationary probability distribution. The long time behaviour of higher moments of the order parameter in several nonlinear models has been investigated by different authors[3,4,19-22]. It seems to be an open question whether there exist a sequence of transitions changing the long time behaviour and how such transitions are connected with the transitions concerning the stationary probability distribution.

6. CONCLUDING REMARKS

Besides of their principal interest the results are relevant in connection with the problem of the influence of a stochastic voltage on the inset of an electrohydrodynamic instability in nematic liquid crystals which is subject of some recent experimental and theoretical work[23-27]. In[23] a nonlinear equation is proposed which reads in reduced variables

$$\dot{x} = ax - x^3 + xI_t - d\cdot x^3 I_t. \tag{17}$$

The corresponding stationary probability density is

$$P_s(x) = N(1 - dx^2)|x|^{-2(\lambda+\mu)-1}\left(\frac{a+\Delta}{1+d\Delta} - x^2\right)^{\lambda-1}\left(x^2 - \frac{a-\Delta}{1-d\Delta}\right)^{\mu-1}, \tag{18}$$

with λ, μ given by (11). Since the exponents are identical to that of the previous problem the parameter plane is analogously separated in regions of qualitative different shape. The instability occurs, when nonzero values of x become probable, i.e., crossing the line $2(\lambda+\mu)+1 = 0$. A more detailed discussion will be given in a forthcoming paper.

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Фазовый переход в простой нелинейной модели, индуцируемый марковским дихотомическим шумом

Получено явное выражение плотности стационарного распределения P_s для некоторого класса стохастических процессов, управляемых дихотомическим марковским шумом I_t. Для модели $\dot{x} = ax - x^3 + xI_t$ качественный анализ формы P_s проведен во всей области параметров. Показано, что максимум распределения P_s имеет поведение, сходное с поведением параметра порядка для непрерывных фазовых переходов. Кратко обсуждаются возможные приложения к вопросу об электродинамической неустойчивости нематических жидких кристаллов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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U.Behn, K.Schiele E17-84-300

Noise Induced Transition in a Simple Nonlinear Model Driven by Dichotomous Markovian Noise

The stationary probability density $P_s(x)$ for a class of stochastic processes driven by the dichotomous markovian noise I_t is calculated explicitly. For the specific model $\dot{x} = ax - x^3 + xI_t$ the qualitative shape of P_s is discussed in the whole parameter region. The maxima of P_s show a behaviour similar to order parameters in continuous phase transitions. An application to electrohydrodynamical instabilities in nematic liquid crystals is shortly discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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