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A METHOD OF THE ORDERED GRASSMANN FACTORS IN THE ISING MODEL

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An exact solution of the two-dimensional Ising model in a zero magnetic field has been given first by Onsager in his famous paper [1]. Later on a certain simplification of Onsager's solution has been achieved (see, in particular, refs. [2-12]). However, the available methods continue to be relatively complicated and usually involve one or another nonformal consideration. In the present note a new approach is proposed, which gives a very simple solution of the model. The method may be of interest also for some other Ising-like problems.

We shall use the Grassmann anticommuting variables. Grassmann variables have already been applied to the Ising model in refs. [5-12] (for further comments see at the end of the article).

The model describes a rectangular square lattice of spin variables $X_{mn} = \pm 1$ disposed at sites (m,n), $1 \leq m,n \leq L$, and interacting with the nearest neighbours. L is the length of the lattice side, $N = L^2$ is the total number of spins. Let us consider the general case of inhomogeneous interaction. The Hamiltonian is written as

$$H = -\sum_{mn} \left(J^{(1)}_{m+1n} X_{mn} X_{m+1n} + J^{(2)}_{mn+1} X_{mn} X_{mn+1} \right).$$

The partition function, up to a numerical factor, is given by

$$Q = \underset{(X)}{Sp} \left[\prod_{mn} \left(1 + t^{(1)}_{m+1n} X_{mn} X_{m+1n}\right)\left(1 + t^{(2)}_{mn+1} X_{mn} X_{mn+1}\right) \right], \tag{1}$$

where

$$\underset{(X)}{Sp} [\ldots] = \frac{1}{2^N} \sum_{\{X_{mn} = \pm 1\}} [\ldots], \quad \underset{(X)}{Sp} [1] = 1, \tag{1a}$$

with $t^{(1)}_{mn} = th\,(J^{(1)}_{mn}/\theta)$, $t^{(2)}_{mn} = th\,(J^{(2)}_{mn}/\theta)$, $\theta = kT$ being the temperature (for standard detailes see, for instance, refs. [3,4]).

Let us bring into correspondence to the lattice sites 2N pairs of the conjugated anticommuting Grassmann variables [6]:

$$\{ a_{mn}, a^*_{mn};\ b_{mn}, b^*_{mn} \}. \tag{2}$$

All the variables completely anticommute (to zero); their squares are zeros. Introduce also for every pair an integral with the gaussian weight:

$$\underset{(a_{mn})}{Sp}\,[\ldots] = \int da^*_{mn}\int da_{mn}\, e^{a_{mn}a^*_{mn}}\,[\ldots], \qquad (3a)$$

and analogously for b_{mn}, b^*_{mn}. Introduce the total integral over all pairs:

$$\underset{(a,b)}{Sp}\,[\ldots] = \prod_{mn}\underset{(a_{mn})}{Sp}\,\underset{(b_{mn})}{Sp}\,[\ldots]. \qquad (3b)$$

We use here the standard integrals (see, e.g., ref.[6,8]). Basic relations for one varialle a are: $\int da\cdot 1 = 0$, $\int da\cdot a = 1$ (note that $a^2 = 0$); all the variables and integral symbols anticommute with each other; every pair of Grassmann symboles commutes with any other symbol. The averaging (3b) is completely determined by the averaging over separate pairs (3a) according to the rules:

$$\underset{(a)}{Sp}\,[aa^*] \equiv -\underset{(a)}{Sp}\,[a^*a] = 1, \quad \underset{(a)}{Sp}\,[1] = 1,$$
$$\underset{(a)}{Sp}\,[a^*] = \underset{(a)}{Sp}\,[a] = 0, \qquad (4)$$

where a, a^* is one of the conjugated pairs (2).

Our aim is to pass from discrete variables $x_{mn} = \pm 1$ with averaging (1a) to Grassmann variables (2) with averaging (3).

It is convenient to assume the free boundary conditions in the initial form (1). Let us also introduce, in addition to the boundary terms $(1+t^{(1)}_{L+1n}x_{Ln}x_{L+1n})$, $(1+t^{(2)}_{mL+1}x_{mL}x_{mL+1})$, already present in (1), analogous terms on the opposite sides of the lattice, $(1+t^{(1)}_{1n}x_{0n}x_{1n})$, $(1+t^{(2)}_{m1}x_{m0}x_{m1})$, with the corresponding extension of a set $\{x_{mn}\}$ and variables (2).

Introduce now Grassmann factors:

$$A_{mn} = 1 + a_{mn}x_{mn}, \qquad A^*_{mn} = 1 + t^{(1)}_{mn}a^*_{m-1n}x_{mn},$$
$$B_{mn} = 1 + b_{mn}x_{mn}, \qquad B^*_{mn} = 1 + t^{(2)}_{mn}b^*_{mn-1}x_{mn}. \qquad (5)$$

Indices (m,n) here are in a correspondence to those of x_{mn}. Making use of the selection rules (4), we may represent the terms occurring in (1) in a factorized form:

$$1 + t^{(1)}_{m+1n}x_{mn}x_{m+1n} = \underset{(a_{mn})}{Sp}\,A_{mn}A^*_{m+1n}, \qquad (6a)$$

$$1 + t^{(2)}_{mn+1}x_{mn}x_{mn+1} = \underset{(b_{mn})}{Sp}\,B_{mn}B^*_{mn+1}. \qquad (6b)$$

By making use of (6), we shall sum over x_{mn} in (1). To get this end it is necessary to achieve a special ordering of the Grassmann factors to obtain four factors with given x_{mn} situated near each other.

Using step by step factorization (6b), we may write down the following representation:

$$\prod_{m=1}^{L}\left(1+t^{(2)}_{mn+1}x_{mn}x_{mn+1}\right) = \underset{(b)}{Sp}\left\{\overset{\xleftarrow{m}}{\prod_{m=1}^{L}} B_{mn}\cdot\overset{\xrightarrow{m}}{\prod_{m=1}^{L}} B^*_{mn+1}\right\}, \qquad (7)$$

where the products of B_{mn}, B^*_{mn+1} are ordered with $m = 1,\ldots,L$ increasing in opposite directions; here (and in analogous cases below) Sp is taken over the pairs occurring in the r.h.s. only.

Further, by the ordering multiplication of quantities (7) with n going from left to right we obtain:

$$\prod_{n=0}^{L}\prod_{m=1}^{L}\left(1+t^{(2)}_{mn+1}x_{mn}x_{mn+1}\right) = \qquad (8a)$$

$$= \underset{(b)}{Sp}\left\{\overset{\xleftarrow{m}}{\prod_{m=1}^{L}} B_{m0}\prod_{n=1}^{L}\left[\overset{\xrightarrow{m}}{\prod_{m=1}^{L}} B^*_{mn}\cdot\overset{\xleftarrow{m}}{\prod_{m=1}^{L}} B_{mn}\right]\overset{\xrightarrow{m}}{\prod_{m=1}^{L}} B^*_{mL+1}\right\}. \qquad (8b)$$

In (8b) we have rearranged the product by joining neighbouring terms with the same index n. Since the boundary terms B_{m0}, B^*_{mL+1} become unity after averaging over x_{m0}, x_{mL+1}, we shall omit them below.

Now we are able to introduce by an appropriate way the remaining terms from (1) in grassmannized form (6a). So, we write:

$$Q = \underset{(x)}{Sp}\left[\prod_{mn}(1+t^{(1)}_{m+1n}x_{mn}x_{m+1n})(1+t^{(2)}_{mn+1}x_{mn}x_{mn+1})\right] = \qquad (9a)$$

$$= \underset{(x)}{Sp}\,\underset{(a,b)}{Sp}\left\{\prod_{n=1}^{L}\left[A_{0n}A^*_{1n}\overset{\xrightarrow{m}}{\prod_{m=1}} B^*_{mn}A_{mn}A^*_{m+1n}\cdot\overset{\xleftarrow{m}}{\prod_{m=1}^{L}} B_{mn}\right]\right\} = \qquad (9b)$$

$$= \underset{(x)}{Sp}\,\underset{(a,b)}{Sp}\left\{\prod_{n=1}^{L}\left[\overset{\xrightarrow{m}}{\prod_{m=1}^{L}} A^*_{mn}B^*_{mn}A_{mn}\cdot\overset{\xleftarrow{m}}{\prod_{m=1}^{L}} B_{mn}\right]\right\}. \qquad (9c)$$

In (9c) we have rearranged the m-products joining together the terms with the same index m (in an analogy with (8b)) and have omitted the boundary terms A_{on}, A^*_{L+1n} which become unity after averaging over X_{on}, X_{L+1n}.

Now we are able to eliminate X_{mn}. It is easily seen that the task reduces to the independent summing over every given X_{mn}:

$$\mathop{\mathrm{Sp}}_{(X_{mn})} A^*_{mn} B^*_{mn} A_{mn} B_{mn} = \tag{10a}$$

$$= \frac{1}{2} \sum_{X_{mn}=\pm 1} \Big[\big(1 + t^{(1)}_{mn} a^*_{m-1n} X_{mn}\big)\big(1 + t^{(2)}_{mn} b^*_{mn-1} X_{mn}\big) \cdot \tag{10b}$$

$$\cdot \big(1 + a_{mn} X_{mn}\big)\big(1 + b_{mn} X_{mn}\big)\Big] =$$

$$= \big(1 + t^{(1)}_{mn} t^{(2)}_{mn} a^*_{m-1n} b^*_{mn-1}\big)\big(1 + a_{mn} b_{mn}\big) + \tag{10c}$$

$$+ \big(t^{(1)}_{mn} a^*_{m-1n} + t^{(2)}_{mn} b^*_{mn-1}\big)\big(a_{mn} + b_{mn}\big) =$$

$$= \exp\Big[t^{(1)}_{mn} t^{(2)}_{mn} a^*_{m-1n} b^*_{mn-1} + a_{mn} b_{mn} + \tag{10d}$$

$$+ \big(t^{(1)}_{mn} a^*_{m-1n} + t^{(2)}_{mn} b^*_{mn-1}\big)\big(a_{mn} + b_{mn}\big)\Big].$$

Indeed, for given n at the "junction" of the m-products in (9c) there are four neigbouring factors with the same X_{mn} $(m=L)$. Summing over $X_{m=L,n}$, one obtains the totally commuting term (10c) that may be taken "out of brakets". There upon the procedure should be repeated for $m = L-1, L-2, \dots, 1$ with a given n, and all over again for other n. The equivalence of (10c) and (10d) can be checked by a series expansion of the exponential.

As a result one comes to a purely Grassmann representation for the partition function of the two-dimensional Ising model with inhomogeneous interaction and in a zero field as the product of factors (10d) averaging by the rule (3b):

$$Q = \int \prod_{mn} da^*_{mn}\, da_{mn}\, db^*_{mn}\, db_{mn} \exp\Big\{ \sum_{mn} \Big[a_{mn} a^*_{mn} +$$

$$+ b_{mn} b^*_{mn} + a_{mn} b_{mn} + t^{(1)}_{mn} t^{(2)}_{mn} a^*_{m-1n} b^*_{mn-1} +$$

$$+ \big(t^{(1)}_{mn} a^*_{m-1n} + t^{(2)}_{mn} b^*_{mn-1}\big)\big(a_{mn} + b_{mn}\big)\Big]\Big\}. \tag{11}$$

This is a Gaussian integral over Grassmann variables, which can be expressed generally through the determinant of the quadratic form in the exponential/6,8/.

In the homogeneous interaction case, $t^{(1)}_{mn} \equiv t_1$, $t^{(2)}_{mn} \equiv t_2$, the calculation of integral (11) is a simple technical task. After passing to the momentum space (Fourier-transformation) the integral can be evaluated explicitly, yielding the famous Onsager result for the free energy in the limit of infinite lattice $N \equiv L^2 \to \infty$ *):

$$f = \lim_{N\to\infty} \Big(\frac{1}{N} \ln Q\Big) = \frac{1}{2} \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{2\pi} \frac{d\varphi_2}{2\pi} \ln\Big[1 + t_1^2 +$$

$$+ t_2^2 + t_1^2 t_2^2 - 2t_1(1-t_2^2)\cos\varphi_1 - 2t_2(1-t_1^2)\cos\varphi_2\Big] \tag{12}$$

(for technical details see, e.g., refs./6,8/).

The generalized representation (11) can be used also to calculate the correlation functions.

Grassmann representation in the homogeneous case has first been directly derived by a formiddable combinatorial method by Berezin in ref./6/. The same result can be obtained by grassmannization of the fermionic representation by Green and Hurst/7/ (see also/5/). An improved version of the combinatorial approach has been developed by Popov/8/. A factorization closely related to that of (6), involving the Grassmann differentiation, has been applied to the two-dimensional Ising model by Fradkin and Shteingradt in ref./9/ (among other approaches the method of the present paper is most close to that of ref./9/). Some indirect analogs of the factorization can be found also in ref./3/ (see, in particular, p.p. 158, 165 therein). In recent years the Grassmann representations in the Ising model have been discussed in a context of the field-theoretical problems in refs./10-12/. Such representations have been derived here by traditional algebraic or combinatorial methods.

The two basic points of the present method are the factorization (6) and "mirror" ordering of the arising Grassmann factors (7)-(9). Except the evaluation of the partition function (free energy), the method can be used to deal with other questions in the two-dimensional Ising model. However, it may be more important that due to its very simplicity it may provide grounds to make easily first steps in trying to analyse unsolved models like the model in a field and the

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*) The evaluation of the integral requires some approximations in the boundary terms of the Grassmann quadratic form, this does not matter as $N \to \infty$ /6/. Also we would like to note that a slightly modified above method provides an exact solution for finite N, as well, both for the free and periodical boundary conditions; the details will be given elsewhere.

three-dimensional Ising model. In these cases, however, the problems arising from the noncommutativity of the Grassmann factors seem to be much more difficult, and strightforward attempts yield nongaussian "four-particle" terms in the exponential, arising also under the combinatorial analysis /3,5,10-12/.

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Плечко В.Н. E17-84-23
Метод упорядоченных грассмановых множителей в модели Изинга

Предложен новый подход к двумерной модели Изинга, не связанный с традиционными матричными или комбинаторными построениями, существенно упрощающий решение задачи. Метод использует грассмановы антикоммутирующие переменные и основан на специальных приемах - грассмановой факторизации и зеркальном способе упорядочения возникающих грассмановых множителей. Получено грассманово представление для статсуммы в общем случае неоднородного взаимодействия, из которого, в частности, следует результат Онсагера. Метод может представлять интерес и для других задач изинговского типа.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Plechko V.N. E17-84-23
A Method of the Ordered Grassmann Factors in the Ising Model

A new approach to the two-dimensional Ising model is proposed, which does not deal with traditional matrix or combinatorial methods and provides essential simplifications in the solution. The method makes use of Grassmann variables and a special technique of the Grassmann factorization and mirror ordering of the arising Grassmann factors. A generalizied Grassmann representation for the partition function in the inhomogeneous interaction case is obtained, which yields, in particular, the Onsager result. The method may be of interest also for other Ising-type problems.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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