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ON NEW PHASES IN FERMI-SYSTEMS WITH DIPOLE INTERACTIONS

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The spontaneous formation of self-organized structures out of states without any order is one of the most fascinating physical phenomena. We have undertaken the task to investigate in detail how Nature can build all the richness of these structures out of such a totally simple microscopic Hamiltonian. That is why we have selected the most simple physical system such as it is represented by a gas or liquid of structureless and chargeless fermions interacting only through their magnetic dipole moments to avoid complications arising from the Coulomb interactions and from an influence of a crystal lattice. This problem is relevant, e.g., for the condensed He^3 or for neutron stars. Intuitively we expected that such simple physical system could exhibit a simple phase transition from the disordered paramagnetic state to the ordered ferromagnetic state and nothing else. However, to our great surprise, the rigorous applications of the first principles and nonperturbative methods have led us directly to the conclusion that the considered system can exist at least in four different phases. These phases follow one after another by decreasing the temperature of the system. Two out of these four phases exhibit totally unique phenomena typical for superconductivity with an unusual anisotropy spontaneously self-generated in the system, as well as a new kind of structural singularities associated with topological properties of boundaries of the system. Therefore these two phases behave like A and B phases He^3 /1,2/.

To carry out this program we consider a system of N interacting structureless and chargeless fermions confined to a box of the volume V at the temperature T. Each fermion has the mass m and the magnetic moment $\vec{\mu}$.
The Hamiltonian operator H of the system is

$$H = \int d^3\vec{x}\ \mathcal{H}(\psi^+,\psi) , \qquad (1)$$

where the Hamiltonian density $\mathcal{H}$ is expressed by the formula

$$\mathcal{H}(\psi^+,\psi) = \frac{1}{2m}\psi^+(\vec{x},t)(-i\hbar\vec{\nabla})^2\psi(\vec{x},t) - 2\pi\mu^2 :\left\{[\psi^+(\vec{x},t)\vec{\sigma}\psi(\vec{x},t)][\psi^+(\vec{x},t)\vec{\sigma}\psi(\vec{x},t)]_{tr}\right\}: \qquad (2)$$

Here $\psi^+(\vec{x},t)$ and $\psi(\vec{x},t)$ are the creation and annihilation Heisenberg operators for the two component fermion field, the symbol :{ }: stands for the normal product ordering, $\vec{\sigma}$ are the Pauli matrices, μ is the magneton of the fermion and $[\psi^+(\vec{x},t)\vec{\sigma}\psi(\vec{x},t)]_{tr}$ is the shorthanded notation for the vector $\psi^+(\vec{x},t)\vec{\sigma}\psi(\vec{x},t)$ which is explicitly given by

$$[\psi^+(\vec{x},t)\vec{\sigma}\psi(\vec{x},t)]_{tr} \equiv \vec{\nabla}\times\vec{\nabla}\times\frac{1}{4\pi}\int d^3\vec{x}'\,\frac{\psi^+(\vec{x}',t)\vec{\sigma}\psi(\vec{x}',t)}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{1}{4\pi}\int d^3\vec{x}'\Big\{3\,\frac{[\psi^+(\vec{x}',t)\vec{\sigma}\psi(\vec{x}',t).(\vec{x}-\vec{x}')]}{|\vec{x}-\vec{x}'|^5}(\vec{x}-\vec{x}') - \frac{\psi^+(\vec{x}',t)\vec{\sigma}\psi(\vec{x}',t)}{|\vec{x}-\vec{x}'|} +$$

$$+ \frac{8\pi}{3}\psi^+(\vec{x}',t)\vec{\sigma}\psi(\vec{x}',t)\,\delta(\vec{x}-\vec{x}')\Big\}.$$

Perhaps at this point we may intuitively explain why the considered system may acquire properties similar to those associated with superconductivity. The Hamiltonian density

$$\mathcal{H}'(\psi^+,\psi) = \frac{1}{2m}\,\psi^+(\vec{x},t)(-i\hbar\vec{\nabla})^2\psi(\vec{x},t) - 2\pi\mu^2 : \Big\{\big[\psi^+(\vec{x},t)\vec{\sigma}\,\psi(\vec{x},t)\big]^2_{tr}\Big\}: \qquad (2a)$$

is completely equivalent to the Hamiltonian density (2), because it gives the same Hamiltonian (1), for in the integral (1) the contribution arising from the scalar product of the transversal and longitudinal parts of the vector $\psi^+(\vec{x},t)\vec{\sigma}\psi(\vec{x},t)$ vanishes. The Hamiltonian density (2a) shows explicitly the attractive four fermion interaction which reminds us distantly the attractive four fermion interaction in the BCS theory of superconductivity /3/.

The grand canonical partition function $\mathcal{Z}$ of the system is given by the formula

$$\mathcal{Z} = \mathrm{Tr}\; e^{\beta(\eta N - H)}, \qquad (3)$$

where $\beta = (k_B T)^{-1}$, k_B is the Boltzmann constant, η is the chemical potential and N is the fermion number operator. For the explicit evaluation of the partition function $\mathcal{Z}$ is very convenient to use functional integration methods /4,5/ instead of the second quantization formalism. In quantum statistics formulated in terms of functional integrals the partition function (3) is expressed by the functional integral

$$\mathcal{Z} = \int\mathcal{D}\psi^*\mathcal{D}\psi\,\exp\Big\{\frac{1}{\hbar}S(\psi^*,\psi)\Big\} \equiv \int\mathcal{D}\psi^*\mathcal{D}\psi\,\exp\Big\{\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\int d^3\vec{x}\,\mathcal{L}(\psi^*,\psi)\Big\} \qquad (4)$$

over the anticommuting variable $\psi^*(\vec{x},\tau)$ and $\psi(\vec{x},\tau)$ satisfying the antiperiodic conditions

$$\psi^*(\vec{x},\tau) = -\psi^*(\vec{x},\tau+\beta\hbar)\,; \qquad \psi(\vec{x},\tau) = -\psi(\vec{x},\tau+\beta\hbar).$$

Here the functional $S(\psi^*,\psi)$ is the action associated with the Lagrangian density

$$\mathcal{L}(\psi^*,\psi) = -\hbar\,\psi^*(\vec{x},\tau)\frac{\partial\psi(\vec{x},\tau)}{\partial\tau} - \frac{1}{2m}\,\psi^*(\vec{x},\tau)(-i\hbar\vec{\nabla})^2\psi(\vec{x},\tau)$$

$$+ 2\pi\mu^2\,[\psi^*(\vec{x},\tau)\vec{\sigma}\psi(\vec{x},\tau)]\cdot[\psi^*(\vec{x},\tau)\vec{\sigma}\psi(\vec{x},\tau)]_{tr} + \eta\,\psi^*(\vec{x},\tau)\psi(\vec{x},\tau) \qquad (5)$$

in the four dimensional Euclidean space-time $(\vec{x},\tau)$ which is related to the Minkowski space by the relation $t = -i\tau$ and τ is a real parameter. Thus the anticommuting variables ψ^* and ψ are enumerated by the four vector $x=(\vec{x},\tau)$. The symbol $\mathcal{D}\psi^*\mathcal{D}\psi$ means the infinite product

$$\mathcal{D}\psi^*\mathcal{D}\psi = \prod_x d\psi_+^*(x)\,d\psi_+(x)\,d\psi_-^*(x)\,d\psi_-(x)\;,$$

where the subscripts + and - denote spinorial components of the spinor field, e.g. $\psi^*(x) = (\psi_+^*(x), \psi_-^*(x))$.

Our next strategy in describing the spontaneous formation of macroscopic structures in the considered system is similar to that we have developed for the explanation of self-organized structures in electron plasma /6/. Namely, that any self-generated macroscopic structure is associated with the spontaneous formation of a macroscopic and periodic electromagnetic field in the following way. The interacting fermions may spontaneously generate the macroscopic periodic electromagnetic field. The state with spontaneously generated macroscopic field will be a ground state of the system providing that the energy of the generated field is smaller than that portion of energy by which the energy of the fermions is decreased in this field. Therefore it is necessary to find the electromagnetic field configurations yielding the lowest energy of the system under given external conditions.

One notes that our Lagrangian (5) does not explicitly contain any electromagnetic field which is the field of the Bose type. The electromagnetic field is introduced to our considerations by the following formal transformation. We multiply and divide the partition function (4) by a positive number $\mathcal{Z}_f$. The number $\mathcal{Z}_f$ is chosen in the form of the functional integral

$$\mathcal{Z}_f = \int\mathcal{D}\vec{B}'\,\exp\Big\{\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\int d^3\vec{x}\,\Big[-\frac{1}{8\pi}\vec{B}'^2(x)\Big]\Big\}\;,$$

where we integrate over a space of real vector functions $\vec{B}'(x)$ satisfying the following conditions

$$\vec{\nabla}.\vec{B}' = 0$$

$$\vec{B}'(\vec{x},\tau) = \vec{B}'(\vec{x},\tau+\beta\hbar)\,. \qquad (6)$$

The partition function (4) gets the form

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_f} \int \mathcal{D}\vec{B}' \mathcal{D}\psi^* \mathcal{D}\psi \, \exp\left\{ \frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int d^3\vec{x} \left[\mathcal{L}(\psi^*, \psi) - \frac{1}{8\pi} \vec{B}'^2 \right] \right\}.$$

In the last fuctional integral we make the transformation of the integration variables

$$\vec{B}'(x) = \vec{B}(x) - 4\pi\mu \left[\psi^*(x) \vec{\sigma} \psi(x) \right]_{tr} , \tag{7}$$

where $\vec{B}(x)$ are new integration variables satisfying the conditions (6). By the transformation (7) we get the following representation for the partition function

$$\begin{aligned} \mathcal{Z} &= \frac{1}{\mathcal{Z}_f} \int \mathcal{D}\vec{B} \mathcal{D}\psi^* \mathcal{D}\psi \, \exp\left\{ \frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int d^3\vec{x} \, \mathcal{L}_{elm}(\psi^*, \psi, \vec{B}) \right\} = \\ &\equiv \frac{1}{\mathcal{Z}_f} \int \mathcal{D}\vec{B} \mathcal{D}\psi^* \mathcal{D}\psi \, \exp\left\{ \frac{1}{\hbar} S_{elm}(\psi^*, \psi, \vec{B}) \right\} , \end{aligned} \tag{8}$$

where $\mathcal{L}_{elm}(\psi^*, \psi, \vec{B})$ is the Lagrangian density of the electromagnetic interaction given in the form

$$\begin{aligned} \mathcal{L}_{elm}(\psi^*, \psi, \vec{B}) &= -\hbar \psi^*(x) \frac{\partial \psi(x)}{\partial \tau} - \frac{1}{2m} \psi^*(x) (-i\hbar\vec{\nabla})^2 \psi(x) \\ &+ \eta \psi^*(x) \psi(x) + \mu \vec{B}(x) \psi^*(x) \vec{\sigma} \psi(x) - \frac{1}{8\pi} \vec{B}^2(x) . \end{aligned} \tag{9}$$

By the formal transformation (7) one has, in fact, expressed the grandious Faraday idea that the interactions between magnetic moments are mediated by the magnetic field $\vec{B}$. According to our strategy emphasized above, the formation of a macroscopic structure must be associated with the spontaneous generation of the macroscopic periodic magnetic field $\vec{B}_0(x)$ which is determined by the statistical mean value $\langle \vec{B}(x) \rangle$ defined by the formula

$$\vec{B}_0(x) = \langle \vec{B}(x) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\vec{B} \mathcal{D}\psi^* \mathcal{D}\psi \, \vec{B}(x) \exp\left\{ \frac{1}{\hbar} S_{elm}(\psi^*, \psi, \vec{B}) \right\} .$$

The last integral can be formally evaluated with the result

$$\vec{B}_0(x) = 4\pi\mu \, \langle \psi^*(x) \vec{\sigma} \psi(x) \rangle . \tag{10}$$

The explicit expression for the macroscopic and periodic magnetic field $\vec{B}_0(x)$ will be derived in the following way. Suppose one carries

out explicitly the functional integration over the anticommuting variables ψ^* and ψ in the functional integral (8). After this integration the functional integrand in (8) becomes dependent only on the magnetic field $\vec{B}$ and is written down in the form

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_f} \int \mathcal{D}\vec{B} \, \exp\left\{ \frac{1}{\hbar} S_{eff}(\vec{B}) \right\} , \tag{11}$$

where $S_{eff}(\vec{B})$ is an effective action. The last relation tells us that the system of interacting fermions can be described by means of the Bose field $\vec{B}$. All quantum effects due to Fermi statistics are therefore completely implemented in the effective action $S_{eff}(\vec{B})$. If the fermion system is capable to give rise to a self-organized macroscopic magnetic field $\vec{B}_0(x)$, then this field must be a solution to the classical equation of motion

$$\frac{\delta S_{eff}(\vec{B})}{\delta \vec{B}(x)} = 0 . \tag{12}$$

For this reason we calculate the effective action. The afore-mentioned functional integration over ψ^* and ψ in (8) is carried out exactly with the result

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_f} \int \mathcal{D}\vec{B} \, \exp\left\{ -\frac{1}{8\pi\hbar} \int_0^{\beta\hbar} d\tau \int d^3\vec{x} \, \vec{B}^2(x) \right\} \det M(\vec{B}),$$

where $M(\vec{B})$ is the infinite-dimensional matrix, the rows and columns of which are numerated by the four vector x and spinorial index a. In the Dirac notation the matrix elements of $M(\vec{B})$ are given by

$$\langle x, a | M(\vec{B}) | y, b \rangle = \frac{1}{\hbar} \left[\left(\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \vec{\nabla}^2 - \eta \right) \delta_{ab} - \mu \vec{B}(x) \cdot \vec{\sigma}_{ab} \right] \delta(x-y) .$$

By exploiting the well-known identity det M = exp (Tr ln M), the effective action can be written down in the form

$$S_{eff}(\vec{B}) = \int_0^{\beta\hbar} d\tau \int d^3\vec{x} \left\{ -\frac{1}{8\pi} \vec{B}^2 + \hbar \sum_a \langle x, a | \ln M(\vec{B}) | x, a \rangle \right\} .$$

The relation (12) gives us the equation

$$\frac{1}{4\pi} \vec{B}_0(x) + \mu \sum_a \langle x, a | M^{-1}(\vec{B}_0) \vec{\sigma} | x, a \rangle = 0 \tag{13}$$

for the spontaneously organized macroscopic magnetic field $\vec{B}_o(x)$. Although the last equation is a complicated non-linear integral equation, we will construct its exact solution. Suppose we know the function $\vec{B}_o(x)$ in an explicit form. Then the inverse matrix $M^{-1}(B_o)$ is, apart from the minus sign, the Green function D of the non-interacting fermion gas embeded in the given external field $\vec{B}_o(x)$, for the Green function $D(x_1,x_2)$ is defined by the relation

$$\frac{1}{\hbar}\left[\hbar\frac{\partial}{\partial\tau_1} - \frac{\hbar^2}{2m}\vec{\nabla}_1^2 - \eta - \mu\vec{B}_o(x_1)\cdot\vec{\sigma}\right]D(x_1,x_2) = -\delta(x_1-x_2)\,.$$

The Green function $D(x_1,x_2)$ is the matrix in the spinor space with the following matrix elements

$$D(x_1,x_2) = \begin{pmatrix} \langle\psi_+^*(x_2)\psi_+(x_1)\rangle_o & \langle\psi_-^*(x_2)\psi_+(x_1)\rangle_o \\ \langle\psi_+^*(x_2)\psi_-(x_1)\rangle_o & \langle\psi_-^*(x_2)\psi_-(x_1)\rangle_o \end{pmatrix}.$$

Here the subscript o denotes the statistical mean value with respect to the action $S_o(\psi^*,\psi,\vec{B}_o)$ associated with the Lagrangian density

$$\mathcal{L}_o(\psi^*,\psi,\vec{B}_o) = -\hbar\,\psi^*(x)\frac{\partial\psi(x)}{\partial\tau} - \frac{1}{2m}\psi^*(x)(-i\hbar\vec{\nabla})\psi(x) + \eta\,\psi^*(x)\psi(x) + \mu\,\vec{B}_o(x)\cdot\psi^*(x)\vec{\sigma}\psi(x) \tag{14}$$

of the non-interacting fermions embeded in the given external field $\vec{B}_o(x)$. By using the algebraic reduction

$$\sum_a\langle x_1,a|M^{-1}(\vec{B}_o)\vec{\sigma}|x_2,a\rangle = -\sum_a\left[D(x_1,x_2)\vec{\sigma}\right]_{aa} = -\langle\psi^*(x_2)\vec{\sigma}\psi(x_1)\rangle_o$$

the equation (13) can be rewritten in the form

$$\vec{B}_o = 4\pi\mu\,\langle\psi^*(x)\,\vec{\sigma}\,\psi(x)\rangle_o\,. \tag{15}$$

The last equation tells us that the magnetic field $\vec{B}_o(x)$ is a solution to Eq. (13) if $4\pi\mu$ times the magnetization of the fermions in this field is the same as this field. We have found that these requirements can be satisfied by the field $\vec{B}_o(x)$ which is given by a superposition of a static circularly polarised spin wave and a homogenous magnetic field as it is explicitly expressed by the formula

$$\vec{B}_o(\vec{x}) = \frac{b_1}{\sqrt{2}}\left(\vec{\varepsilon}\,e^{i\lambda\vec{q}\cdot\vec{x}} + \vec{\varepsilon}^*e^{-i\lambda\vec{q}\cdot\vec{x}}\right) + \frac{\vec{q}}{q}b_2\,, \tag{16}$$

where b_1 and b_2 are constant parameters, $\vec{q}$ is the wave vector, $\lambda=\pm1$ is the helicity of the spin wave, $\vec{\varepsilon}$ and $\vec{\varepsilon}^*$ are the polarization vectors satisfying the relations

$$\vec{\varepsilon}\cdot\vec{\varepsilon} = \vec{\varepsilon}^*\cdot\vec{\varepsilon}^* = \vec{q}\cdot\vec{\varepsilon} = \vec{q}\cdot\vec{\varepsilon}^* = 0\,, \qquad \vec{\varepsilon}\cdot\vec{\varepsilon}^* = 1\,.$$

The vectors $\vec{\varepsilon}$, $\vec{\varepsilon}^*$ and $\vec{q}$ can be parametrized as follows

$$\vec{\varepsilon} = \frac{1}{\sqrt{2}}(1,i,0)\,;\quad \vec{\varepsilon}^* = \frac{1}{\sqrt{2}}(1,-i,0)\,;\quad \vec{q} = (0,0,q)\,.$$

The proof that function (16) is the solution to Eq. (13) or (15) will be made by the direct calculation. That is why we evaluate the partition function

$$\begin{aligned} \mathcal{Z}_o &= \int\mathcal{D}\psi^*\mathcal{D}\psi\,\exp\left\{\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\int d^3\vec{x}\,\mathcal{L}_o(\psi^*,\psi,\vec{B})\right\} \equiv \\ &\equiv \int\mathcal{D}\psi^*\mathcal{D}\psi\,\exp\left\{\frac{1}{\hbar}S_o(\psi^*,\psi,\vec{B})\right\} \end{aligned} \tag{17}$$

with the field $\vec{B}_o(x)$ given by the form (16). To carry out the functional integration in (17) explicitly it is very convenient to make the following unitary transformation of the integration variables $\psi^*(x)$ and $\psi(x)$

$$\begin{aligned} \psi_+^*(x) &= \sum_{\nu=-\infty}^{+\infty}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left[u(\vec{k})a_+^*(\vec{k},\nu)+v(\vec{k})a_-^*(\vec{k},\nu)\right]\exp\left[i\omega_\nu\tau - i\left(\vec{k}+\frac{\lambda\vec{q}}{2}\right)\cdot\vec{x}\right] \\ \psi_-^*(x) &= \sum_{\nu=-\infty}^{+\infty}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left[u(\vec{k})a_-^*(\vec{k},\nu)-v(\vec{k})a_+^*(\vec{k},\nu)\right]\exp\left[i\omega_\nu\tau - i\left(\vec{k}-\frac{\lambda\vec{q}}{2}\right)\cdot\vec{x}\right] \\ \psi_+(x) &= \sum_{\nu=-\infty}^{+\infty}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left[u(\vec{k})a_+(\vec{k},\nu)+v(\vec{k})a_-(\vec{k},\nu)\right]\exp\left[-i\omega_\nu\tau + i\left(\vec{k}+\frac{\lambda\vec{q}}{2}\right)\cdot\vec{x}\right] \\ \psi_-(x) &= \sum_{\nu=-\infty}^{+\infty}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left[u(\vec{k})a_-(\vec{k},\nu)-v(\vec{k})a_+(\vec{k},\nu)\right]\exp\left[-i\omega_\nu\tau + i\left(\vec{k}-\frac{\lambda\vec{q}}{2}\right)\cdot\vec{x}\right] \end{aligned} \tag{18}$$

to the new anticommuting variables $a^*(\vec{k},\nu)$ and $a(\vec{k},\nu)$. Here we have used the following abbreviations: $\omega_\nu = (\pi/\beta\hbar)(2\nu+1)$, ν is an integer number,

$$u(\vec{k}) = -\frac{1}{\sqrt{2}}\left[1-\frac{\lambda\vec{q}\cdot\vec{k}-\gamma_2}{\left[(\lambda\vec{q}\cdot\vec{k}-\gamma_2)^2+\gamma_1^2\right]^{1/2}}\right]^{1/2};\quad v(\vec{k}) = -\frac{1}{\sqrt{2}}\left[1-\frac{\lambda\vec{q}\cdot\vec{k}-\gamma_2}{\left[(\lambda\vec{q}\cdot\vec{k}-\gamma_2)^2+\gamma_1^2\right]^{1/2}}\right]^{1/2}$$

$$\gamma_1 = \frac{2m}{\hbar^2}\mu b_1\,;\qquad \gamma_2 = \frac{2m}{\hbar^2}\mu b_2$$

By the transformation (18) we get Z_0 in the form

$$\mathcal{Z}_0 = \int \mathcal{D}a^* \mathcal{D}a \, \exp\left\{ \frac{1}{\hbar} S_0(a^*, a) \right\} \tag{19}$$

with the diagonal action

$$\begin{aligned} S_0(a^*, a) = \beta\hbar \sum_{\nu=-\infty}^{+\infty} \int d^3\vec{k} \Big\{ & [i\hbar\omega_\nu + \eta - E_+(\vec{k})] a_+^*(\vec{k},\nu) a_+(\vec{k},\nu) \\ & + [i\hbar\omega_\nu + \eta - E_-(\vec{k})] a_-^*(\vec{k},\nu) a_-(\vec{k},\nu) \Big\} \end{aligned} \tag{20}$$

allowing for the elementary functional integration of (19). Here $E_+(\vec{k})$ and $E_-(\vec{k})$ are the energy spectra of quasiparticles given by the formulae

$$E_+(\vec{k}) = \frac{\hbar^2}{2m} \left\{ \frac{\vec{q}^2}{4} + \vec{k}^2 - \left[(\lambda \vec{q}\cdot\vec{k} - \gamma_2)^2 + \gamma_1^2 \right]^{1/2} \right\}, \tag{21a}$$

$$E_-(\vec{k}) = \frac{\hbar^2}{2m} \left\{ \frac{\vec{q}^2}{4} + \vec{k}^2 + \left[(\lambda \vec{q}\cdot\vec{k} - \gamma_2)^2 + \gamma_1^2 \right]^{1/2} \right\}. \tag{21b}$$

From (19)-(21) one gets the ordinary Fermi distributions

$$n_\pm(\vec{k}) = \left\{ 1 + \exp\left[\beta E_\pm(\vec{k}) - \beta\eta \right] \right\}^{-1} \tag{22}$$

for the quasiparticle associated with the + and - spinorial components. In the next step we have to calculate the statistical mean value $\langle \psi^*(x) \vec{\sigma} \psi(x) \rangle_0$. By elementary calculations one finds the result

$$4\pi\mu \langle \psi^*(x) \vec{\sigma} \psi(x) \rangle_0 = \frac{1}{\sqrt{2}} \left(\vec{\varepsilon} e^{i\lambda\vec{q}\cdot\vec{x}} + \vec{\varepsilon}^* e^{-i\lambda\vec{q}\cdot\vec{x}} \right) g_1(b_1, b_2) + \frac{\vec{q}}{q} g_2(b_1, b_2), \tag{23}$$

where the following abbreviations are used

$$g_1(b_1, b_2) = 4\pi\mu\gamma_1 \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{n_+(\vec{k}) - n_-(\vec{k})}{\left[(\lambda\vec{q}\cdot\vec{k} - \gamma_2)^2 + \gamma_1^2 \right]^{1/2}},$$

$$g_2(b_1, b_2) = -4\pi\mu \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{\lambda\vec{q}\cdot\vec{k} - \gamma_2}{\left[(\lambda\vec{q}\cdot\vec{k} - \gamma_2)^2 + \gamma_1^2 \right]^{1/2}} \left[n_+(\vec{k}) - n_-(\vec{k}) \right].$$

The statistical mean value (23) is indeed of the same form as the field $\vec{B}_0(x)$ given by (16). The relation (15) requires the following self-consistency conditions

$$\begin{aligned} \frac{\hbar^2}{2m}\gamma_1 &= 4\pi\mu^2\gamma_1 \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{n_+(\vec{k}) - n_-(\vec{k})}{\left[(\lambda\vec{q}\cdot\vec{k} - \gamma_2)^2 + \gamma_1^2 \right]^{1/2}} \\ \frac{\hbar^2}{2m}\gamma_2 &= -4\pi\mu^2 \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{\lambda\vec{q}\cdot\vec{k} - \gamma_2}{\left[(\lambda\vec{q}\cdot\vec{k} - \gamma_2)^2 + \gamma_1^2 \right]^{1/2}} \left[n_+(\vec{k}) - n_-(\vec{k}) \right] \end{aligned} \tag{24}$$

to be satisfied. These two relations determine the parameters γ_1 and γ_2 as certain functions of the temperature T, chemical potential η and the wave vector $\vec{q}$. The free parameter $\vec{q}$ will be determined later on from the minimum of the energy of the system. The self-consistency conditions (24) are very similar to the self-consistency conditions determining the energy gap Δ in the BCS theory of supercoductivity /3/.

In what follows we will restrict ourselves to the description of the considered system in the mean field $\vec{B}_0(\vec{x})$ approximation thus ignoring effects of quantal fluctuations of the magnetic field $\vec{B}$. The effects of the quantal fluctuations will be considered elsewhere. In this approximation the partition function (11) is evaluated in the form

$$\mathcal{Z} = \exp\left\{ \frac{1}{\hbar} S_{eff}(\vec{B}_0) \right\} = \mathcal{Z}_0 \exp\left\{ -\frac{\beta V}{8\pi} (b_1^2 + b_2^2) \right\}.$$

All thermodynamical properties of the system can be derived from the grand-canonical potential $\Omega = -\beta^{-1} \ln \mathcal{Z}$ which has the explicit form

$$\Omega = \frac{V}{8\pi}(b_1^2 + b_2^2) - \frac{V}{\beta} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left\{ \ln\left[1 + e^{\beta\eta - E_+(\vec{k})} \right] + \ln\left[1 + e^{\beta\eta - E_-(\vec{k})} \right] \right\},$$

where $E_+(\vec{k})$ and $E_-(\vec{k})$ are the energy spectra (21). Since the energy spectra (21) have the anisotropy due to the wave vector $\vec{q}$, the considered system must exhibit totally unique anisotropic properties which will be discussed bellow.

The energy of the system

$$E = \frac{V}{8\pi}\left(b_1^2+b_2^2\right) + V\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left\{E_+(\vec{k})\,n_+(\vec{k}) + E_-(\vec{k})\,n_-(\vec{k})\right\} \tag{25}$$

must have the minimum with respect to any free parameter at the given entropy

$$S = -k_B V\sum_{\sigma}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left\{n_\sigma(\vec{k})\ln\left[n_\sigma(\vec{k})\right] + \left[1-n_\sigma(\vec{k})\right]\ln\left[1-n_\sigma(\vec{k})\right]\right\}.$$

This requirement is however equivalent to the minimalization of (25) with respect to the wave vector $\vec{q}$ at the given occupation numbers $n_\sigma(\vec{k})$. The condition for the energy minimum is gotten with the combinations of (24) in the form

$$\frac{\hbar^2}{2m}\frac{\vec{q}^{\,2}}{2}\,n = \frac{\hbar^2}{2m}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left[(\lambda\vec{q}\cdot\vec{k}-\gamma_2)^2+\gamma_1^2\right]^{1/2}\left[n_+(\vec{k})-n_-(\vec{k})\right]-\frac{1}{4\pi}\left(b_1^2+b_2^2\right), \tag{26}$$

where n is the mean density of the fermions. Thus we have obtained the complete set of the relations (24) and (26) determining the parameters b_1, b_2 and q as certain functions of the temperature T and the chemical potential η. The relation (26) is used to rewrite the energy (25) in the form

$$E = \frac{V\hbar^2}{2m}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\,\vec{k}^{\,2}\left[n_+(\vec{k})+n_-(\vec{k})\right] - \frac{V\hbar^2}{2m}\frac{q^2}{4}\,n - \frac{V}{8\pi}\left(b_1^2+b_2^2\right) \tag{27}$$

which indicates that the macroscopic periodic structures, associated with the parameters b_1, b_2 and q, are energetically preferable whenever external conditions allow for nontrivial solution to the equations (24) and (26).

The detailed analysis of the relations (24) and (26) reveals that the considered fermion system can have three critical temperatures denoted by T_1, T_2, T_3 and satisfying the relations $T_1 > T_2 > T_3$. If the fermion density n is sufficiently high then by decreasing the temperature the spontaneously generated macroscopic field $\vec{B}_0(\vec{x})$ develops to its most general form (16) through four succesive stages. At the first stage, when $T > T_1$ we have no order in the system, i.e., $b_1 = b_2 = q = 0$, and the system is in the disordered paramagnetic state. At the second stage when $T_1 > T > T_2$ we have $b_1 \neq 0$, but $b_2 = q = 0$, i.e.,the macroscopic homogeneous magnetic field is spontaneously generated. The system exists in its ferromagnetic phase. In this phase the parameter b_1 increases by decreasing the temperature. The ferro-

magnetic phase is associated with the simple totally isotropic energy spectra of fermions

$$E_+(\vec{k}) = \frac{\hbar^2}{2m}\left(\vec{k}^2 - \gamma_1\right),$$
$$E_-(\vec{k}) = \frac{\hbar^2}{2m}\left(\vec{k}^2 + \gamma_1\right) \tag{28}$$

which are depicted in Fig. 1a. At the third stage when $T_2 > T > T_3$ we have $b_1 \neq 0$, $q \neq 0$, but $b_2 = 0$, i.e.,the static circularly polarized spin wave is developed. This state will be referred as to the superconducting phase and abbreviately denoted by letter S for reason which will become clear bellow. The S - state is associated with the anisotropic energy spectra

$$E_+(\vec{k}) = \frac{\hbar^2}{2m}\left\{\frac{\vec{q}^{\,2}}{4}+\vec{k}^2-\left[(\vec{q}\cdot\vec{k})^2+\gamma_1^2\right]^{1/2}\right\}$$
$$E_-(\vec{k}) = \frac{\hbar^2}{2m}\left\{\frac{\vec{q}^{\,2}}{4}+\vec{k}^2+\left[(\vec{q}\cdot\vec{k})^2+\gamma_1^2\right]^{1/2}\right\} \tag{29}$$

which are depicted in Fig. 1b. The spectra (29) are symmetric with respect to the reflection $\vec{k} \longrightarrow -\vec{k}$. In the S - state there is the non-vanishing orbital electromagnetic supercurrent with the density $\vec{j}(x)$ given by the relation

$$\vec{j}(\vec{x}) = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}_0 = \frac{c}{4\pi}\frac{i\lambda b_1}{\sqrt{2}}\left\{(\vec{q}\times\vec{\varepsilon})e^{i\lambda\vec{q}\cdot\vec{x}} - (\vec{q}\times\vec{\varepsilon}^{\,*})e^{-i\lambda\vec{q}\cdot\vec{x}}\right\}.$$

The supercurrent $\vec{j}(\vec{x})$ must, of course, satisfy the boundary condition

$$\vec{s}\cdot\vec{j} = 0 \tag{30a}$$

at the container surface, where $\vec{s}$ is a unit vector normal to the surface. In addition to the boundary condition (30a), we must make sure that the current $\vec{j}$ into the wall of a container vanishes. This requires that

$$(\vec{s}\cdot\vec{\nabla})\vec{j} = 0 \tag{30b}$$

for those components not going to zero by the condition (30a). The explicit form of (30.b) is

$$(\vec{s}\cdot\vec{q})\left\{(\vec{q}\times\vec{\varepsilon})\,e^{i\lambda\vec{q}\cdot\vec{x}} + (\vec{q}\times\vec{\varepsilon}^{\,*})\,e^{-i\lambda\vec{q}\cdot\vec{x}}\right\} = 0\,.$$

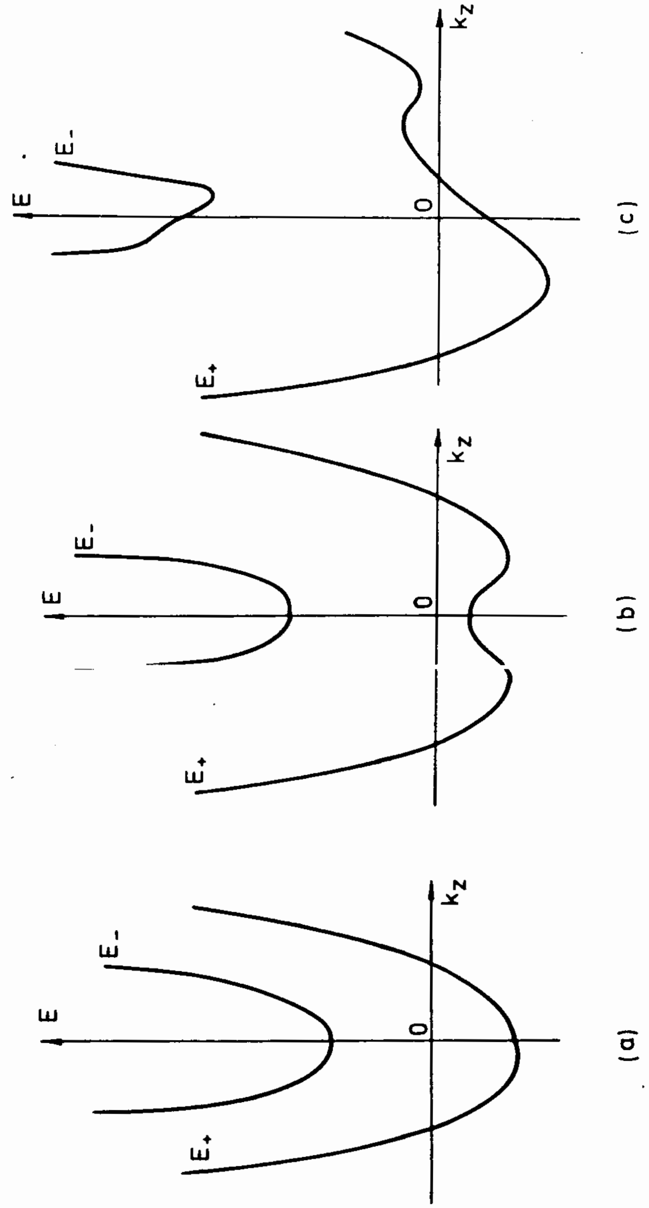

Fig. 1. The energy spectra of the quasiparticles associated with (a) the ferromagnetic phase, (b) S-phase and (c) SF-phase.

The boundary conditions (30) show that the shape of the macroscopic field $\vec{B}_0(x)$ is distorted from its original form (16) by the presence of boundaries, namely the wave vector $\vec{q}$ becomes $\vec{x}$ dependent. In the physics of the superfluid He^3 such distortions of vector fields are called textures /1,2/. The relations (30) require two vectors $\vec{j}(\vec{x})$ and $\vec{q}(\vec{x})$ to form the tangent vector fields with respect to the container surface. However, tangent vector fields at a closed surface topologically equivalent to a sphere must exhibit singularities /2/. Therefore the considered system of fermions confined in a sphere will exhibit singularities in the distributions of the vector fields $\vec{j}(\vec{x})$ and $\vec{q}(\vec{x})$. These singularities are exactly of the same origin as the singularities of the order parameters of the superfluid He^3 /1,2/. The vector

$$\vec{l} = \vec{j} \times \vec{q} ,$$

which is the normal vector at the surface will exhibit singularities which are called disgyrations of de Gennes /2,7/. The analysis of Mermin /2/ applied to our case shows that the fermion system confined in a torus may have the textures free of singularities.

In the S-phase the Fermi surface associated with the energy spectrum (29a) may change its topological structure. As long as

$$\frac{\hbar^2}{2m}\left(\frac{q^2}{4} - \nu_0\right) < \lambda \tag{31}$$

the Fermi surface is topologically equivalent to a sphere in the momentum space. If q^2 becomes so large that the opposite inequality to that of (31) holds, then the Fermi surface becomes topologically equivalent to a torus in the momentum space.

At the fourth stage when $T < T_3$ we have also $b_2 \neq 0$ and the homogeneous magnetic field $\vec{b}_2$ is developed parallel or antiparallel with $\vec{q}$ to accompany the spin wave. This state of the system is referred to as the superconducting and superfluid phase and is denoted by abbreviation SF. The SF-phase is associated with the anisotropic energy spectra (21) which are depicted in Fig. 1.c for $\lambda = +1$ and $\vec{b}_2$ parallel with $\vec{q}$. The energy spectra (21) as it is shown in Fig. 1.c are asymmetric with respect to the reflection $\vec{k} \to -\vec{k}$. Therefore there exists an excess of fermions with momenta oriented in a certain direction specified by the vectors $\vec{b}_2$ and $\vec{q}$, and the value of helicity $\lambda = \pm 1$. This phase is peculiar for a spontaneous self-ordering in the momentum space of the fermions and for giving rise to a macroscopic flow of the mass. Thus the system exhibits, in addition to the superconducting properties of the S-phase, the properties of superfluidity. The velocity field of the mass will exhibit the same sorts of texture as the supercurrent $\vec{j}(\vec{x})$ in the S-phase. In the SF-phase the fermion system confined in a torus may exhibit the superfluid motion of the fermions.

What concerns the critical temperatures T_1, T_2 and T_3, they must be calculated numerically. In the approximation associated with the non-degenerate fermion gas $T \gg T_F$ one gets, e.g., T_1 in the analytic form

$$k_B T_1 = 4\pi \mu^2 n \, , \tag{32}$$

where T_F is the Fermi temperature.

When the formula (32) is applied to the condensed He^3, one gets $T_1 \approx 10^{-7}$ K. This critical temperature violates the condition $T_1 \gg T_F$ and the non-degenerate fermion gas approximation is not admissible. In the degenerate fermion gas approximation the calculation of the critical temperatures cannot be done analytically, because at T = 0 the system gets the most complicated structure.

We have demostrated by using merely basic principles that the simplest physical system exhibits so many different phases with properties which are as complex as those of any inorganic system.

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Нога М., Надь М. E17-83-674
О новых фазах в ферми-системах с дипольным взаимодействием

Исследована самоорганизация макроскопических периодических структур в системе взаимодействующих фермионов. Показано, что система незаряженных фермионов с магнитными моментами может иметь по меньшей мере четыре различных фазы. Две из них обладают свойствами, аналогичными свойствам сверхпроводящего и сверхтекучего состояний. В рассматриваемой системе имеют место структурные особенности, подобные дисгирациям в сверхтекучем He^3.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1983

Noga M., Nagy M. E17-83-674
On New Phases in Fermi-Systems with Dipole Interaction

Formations of self-organized macroscopic and periodic structures in a physical system of interacting fermions are derived from first principles. The system of structureless and chargeless fermions with magnetic moments can have at least four different phases. Two of these phases exhibit the totally unique phenomena typical for superconductivity and superfluidity as well as various kinds of structural singularities such as disgyrations in the superfluid He^3.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983