e +

сообщения
объединенного
института
ядерных
исследований
дубна

1928/83

18/IV-83

E17-83-11

N.Angelescu, M.Bundaru*, G.Costache*

ON PHASE SEPARATION IN SYSTEMS WITH CONTINUOUS SYMMETRY.

The Magnetization Profile and Helicity Modulus

* Institute of Physics and Nuclear Engineering, Bucharest, Romania.

1983

This paper accomplishes the study, begun in the first part (ref./1/), of the phase separation phenomenon in the D-vector model with Kac-Helfand interactions and in its spherical limit. Section numbers follow those in/1/, so reference to formulae and results there will be made without further mention, as, e.g., Eq. (1.5) or Proposition 2.1. Section 3 contains the main result disproving the existence of the sharp interface in the D-vector model ($D \geq 2$), while Sec.4 indicates how to use this very proof for the generalized spherical model considered by us/2/.

3. THE PROPERTIES OF THE MAGNETIZATION PROFILE

In Sec.2, the study of the phase separation in our model has been reduced to the study of the properties of the unique solution in $\mathfrak{T}_{\boldsymbol{e}}^{M}$ of the system (1.5) ($\{\boldsymbol{h}\} \in \mathfrak{U}_{\boldsymbol{e}}$), and of its limit when $\{\boldsymbol{h}\}$ approaches in a certain way the boundary of $\mathfrak{U}_{\boldsymbol{e}}$. More precisely, one has to take only $\boldsymbol{h}_1 \equiv \boldsymbol{\xi}_0$ and $\boldsymbol{h}_M \equiv \boldsymbol{\xi}_{M+1}$ different from zero to take into account the boundary conditions as shown in Sec.1; inverting $\mathcal{F}'$ in Eq.(1.5) and introducing the function:

$$F(x) = \beta^{-1} \mathcal{F}'^{-1}(x) - r\,x \tag{3.1}$$

one is left with the following system:

$$\boldsymbol{\xi}_{i+1} + \boldsymbol{\xi}_{i-1} = (\boldsymbol{\xi}_i / \xi_i) F(\xi_i), \quad \boldsymbol{\xi}_i \in \mathfrak{T}_{\boldsymbol{e}}, \quad 1 \leq i \leq M \tag{3.2}$$

with the boundary conditions:

$$\boldsymbol{\xi}_0, \boldsymbol{\xi}_{M+1} \in \bar{\mathfrak{T}}_{\boldsymbol{e}}, \quad \theta \equiv \measuredangle(\boldsymbol{\xi}_0, \boldsymbol{\xi}_{M+1}) < \pi, \quad 0 < \xi_0, \xi_{M+1} \leq 1, \tag{3.3}$$

where $\xi_i \equiv \|\boldsymbol{\xi}_i\|$. From now on we shall study the properties of the unique solution of Eqs. (3.2), (3.3) and of their limit as $\theta \to \pi$ (see Sec.2).

Before proceeding further some simple remarks will be in order. Thus, since $\boldsymbol{\xi}_i \cdot \boldsymbol{e} > 0, 1 \leq i \leq M$ and $\boldsymbol{\xi}_0 \cdot \boldsymbol{e}, \boldsymbol{\xi}_{M+1} \cdot \boldsymbol{e} \geq 0$, Eq. (3.2) implies

$$F(\xi_i) > 0, \quad 1 \leq i \leq M. \tag{3.4}$$

Choosing as positive the sense of rotation of ξ_0 over ξ_{M+1} of angle less than π, all angles $\theta_i = \sphericalangle(\xi_i, \xi_{i+1})$ satisfy:

$$\theta_i \in [0,\pi), \qquad 0 \le \sum_{i=0}^{M} \theta_i = \theta < \pi . \tag{3.5}$$

Now, it is easy to see that (3.2) can be written as ($1 \le i \le M$):

$$\xi_{i+1}\cos\theta_i + \xi_{i-1}\cos\theta_{i-1} = F(\xi_i); \quad \xi_{i+1}\sin\theta_i = \xi_{i-1}\sin\theta_{i-1} \tag{3.6}$$

wherefrom:

$$\xi_{i+1}\xi_i \sin\theta_i = \xi_1 \xi_0 \sin\theta_0 \equiv c, \qquad i = 1,\dots,M . \tag{3.7}$$

The following properties will be needed later:

(i) The quantity c defined in (3.7) satisfies

$$0 < (M+1)\,c \le \theta < \pi . \tag{3.8}$$

(ii) If the solution of (3.2) has $\theta_i \in [0,\pi/2]$, $0 \le i \le M$, then

$$(\xi_{i+1}^2 - c^2/\xi_i^2)^{1/2} + (\xi_{i-1}^2 - c^2/\xi_i^2)^{1/2} = F(\xi_i), \quad i = 1,2,\dots,M. \tag{3.9}$$

(iii) Let $g(c;x)$ be defined on $0 \le \sqrt{c} \le x < 1$ by

$$g(c;x) = F(x) - 2(x^2 - c^2/x^2)^{1/2}. \tag{3.10}$$

The function $g(c;\cdot)$ is convex on its domain. If

$$\beta_c \equiv (\tau + 2)^{-1} , \tag{3.11}$$

then, for $\beta \le \beta_c$, the equation $g(c;\xi)=0$ has only one solution $c = \xi = 0$ and $g(c;\xi)$ is strictly positive otherwise; for $\beta > \beta_c$ the sign of g and its zeros cannot be simply expressed analytically and Figs.1 and 2 provide the missing analysis.

(iv) Let $\{\xi_i; 1 \le i \le M\}$ be such a solution of (3.2) that $\theta_i \in [0,\pi/2]$, $0 \le i \le M$. If for $1 \le i_0 \le M$, ξ_{i_0} is a local minimum (i.e., $\xi_{i_0 \pm 1} \ge \xi_{i_0}$) with $\xi_{i_0} \ge \sqrt{c}$, then either $g(c,\xi_{i_0}) > 0$, or $g(c,\xi_{i_0}) = 0$ in which case $\xi_i = \xi_{i_0}$ for all $0 \le i \le M+1$. If ξ_{i_0} is a local maximum, then either $g(c,\xi_{i_0}) < 0$, or $g(c,\xi_{i_0}) = 0$ and again $\xi_i = \xi_{i_0}$ for all $0 \le i \le M+1$.

(v) Let $\beta > \beta_c$ and define the following continuous function:

$$\underline{\underline{\xi}}(c) = \max\{\sqrt{c},\ \underline{\xi}(c)\}, \qquad c \in [0, a(\beta)], \tag{3.12}$$

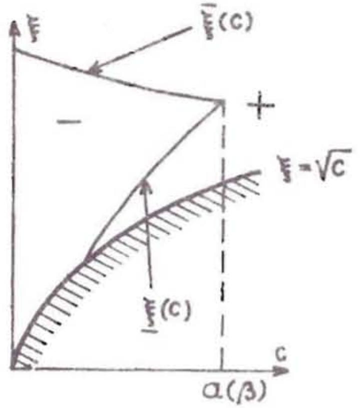

Fig.1. The sign of the function $g(c,\xi)$ for $\beta \in (\beta_c, \tau^{-1}]$.

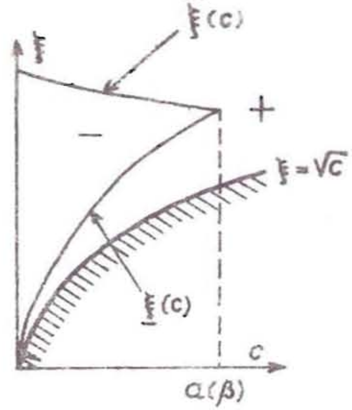

Fig.2. The sign of the function $g(c,\xi)$ for $\beta > \tau^{-1}$.

where $\underline{\xi}(c)$, $\bar{\xi}(c)$ and $a(\beta)$ have been introduced in Figs.1 and 2 ($\underline{\xi}(c)$ is taken 0 when not defined). Let now $\beta > \beta_c$ and ξ_0, ξ_{M+1} be fixed as specified in (3.3); then the solution of (3.2) satisfies for M large enough (depending on β, ξ_0 and ξ_{M+1}) the relations:

$$\xi_i > \underline{\underline{\xi}}(c), \quad i = 1,2,\dots,M; \quad \theta_i \in [0,\pi/2), \quad i = 0,1,\dots,M . \tag{3.13}$$

Indeed, define two continuous functions $\xi_0(t), \xi_{M+1}(t) \in \mathcal{T}_e$, $t \in [0,1]$ of constant moduli $\xi_0(t) = \xi_0$, $\xi_{M+1}(t) = \xi_{M+1}$ and such that $\xi_0(0) = \xi_0 e$, $\xi_{M+1}(0) = \xi_{M+1} e$, while $\xi_0(1) = \xi_0$, $\xi_{M+1}(1) = \xi_{M+1}$. For any $t \in [0,1]$, the system (3.2) with boundary conditions $\xi_0(t)$, $\xi_{M+1}(t)$ has a unique solution $\{\xi_i(t)\}$ which depends continuously on t. Let us choose M so large that $\min\{a(\beta), \xi_0^2, \xi_{M+1}^2\} > \pi/(M+1)$. Recalling (i), one will then have $c(t) < \min\{a(\beta), \xi_0^2, \xi_{M+1}^2\}$. As $\underline{\xi}(c)$ is continuous and $\underline{\xi}(0) = 0$, one can find M_0 such that the inequalities $c(t) < \min\{a(\beta), \xi_0^2, \xi_{M+1}^2\}$, and $\underline{\xi}(c(t)) < \min\{\xi_0, \xi_{M+1}\}$ hold simultaneously for $M \ge M_0$. At $t = 0$ the angle between $\xi_0(0)$ and $\xi_{M+1}(0)$ is zero, so $\theta_i(0) = 0$, $0 \le i \le M$, from Eq. (3.5), hence $c(0) = 0$. As $\underline{\underline{\xi}}(0) = 0$, Eq. (3.13) is fulfilled. From continuity, the set of points $t \in [0,1]$ for which (3.13) is fulfilled is open in [0,1]. If, on the other hand, (3.13) is violated at some point in (0,1], then there exists $t_0 \in (0,1]$ such that for any $t \in [0,t_0)$, (3.13) are satisfied, while for $t = t_0$ either:

(A) $\xi_i(t_0) \ge \xi_{i_0}(t_0) = \underline{\underline{\xi}}(c(t_0))$ and $\theta_i(t_0) \in [0,\pi/2]$ for all i; or

(B) $\xi_i(t_0) > \underline{\underline{\xi}}(c(t_0))$ and $\theta_i(t_0) \le \theta_{i_0}(t_0) = \pi/2$ for all i.

Suppose (A) holds; then, since $c(t_0) < a(\beta)$, $\xi_i(t_0) \geq \xi_{i_0}(t_0) = \underline{\xi}(c(t_0)) \geq \sqrt{c(t_0)}$ and (iv) can be applied providing either

$g(c(t_0), \xi_{i_0}(t_0)) > 0$ and hence $\xi_{i_0}(t_0) > \bar{\xi}(c(t_0)) > \underline{\xi}(c(t_0))$ contradicting (A), or $\xi_i(t_0) = \xi_{i_0}(t_0)$ for all i, in which case

$\underline{\xi}(\alpha(t_0)) = \xi_{i_0}(t_0) = \xi_0(t_0) = \xi_0$, contradicting the choice of M_0.

Suppose now (B) holds; as $M \geq M_0$, ξ_0, $\xi_{M+1} > \sqrt{c}$ and $\xi_i(t_0) > \underline{\xi}(\alpha(t_0)) \geq \sqrt{c(t_0)}$ for all i, wherefrom $\xi_{i_0+1}(t_0) \cdot \xi_{i_0}(t_0) > c(t_0)$, which contradicts (B) $(\alpha(t_0) = \xi_{i_0+1}(t_0)\xi_{i_0}(t_0)$ as $\theta_{i_0}(t_0) = \pi/2)$.

(vi) Let $\beta > 0$ and ξ_0, ξ_{M+1} be fixed. If $\{\xi_i; 1 \leq i \leq M\}$ is the solution of (3.2), then for M large enough (depending on β, ξ_0, ξ_{M+1}) there exists $i_0 = i_0(M) \in \{0, 1, \ldots, M+1\}$ such that ξ_i is a monotone function of i both on $i \geq i_0$ and on $i \leq i_0$.

Indeed, for $\beta \leq \beta_c$, one has $F(\xi) \geq 2\xi$, and (3.6) provides:

$$\xi_{i+1} + \xi_{i-1} \geq 2\xi_i, \qquad i = 1, \ldots, M \tag{3.14}$$

which shows that $\{\xi_i\}$ cannot have a local maximum. Thus, either it is a monotone sequence (in which case $i_0 = 0$ or $M+1$) or it has a unique minimum at some $i_0 \in \{1, \ldots, M\}$. For $\beta > \beta_c$ and $M \geq M_0$, with M_0 as given in (v), one has $\xi_i > \underline{\xi}(c)$ and $\theta_i \in [0, \pi/2)$ for all i. Then, applying (iv), one has that:

$$\xi_{j_0} > \bar{\xi}(c), \text{ if } j_0 \text{ is a local minimum} \tag{3.15a}$$

$$\xi_{j_0} < \bar{\xi}(c), \text{ if } j_0 \text{ is a local maximum.} \tag{3.15b}$$

Thus, the sequence $\{\xi_i; 0 \leq i \leq M+1\}$ cannot have both a local minimum and maximum, implying that it is either monotone (in which case $i_0 = 0$ or $M+1$), or it has a unique local minimum (or maximum) which is attained in $i_0 \in \{1, 2, \ldots, M\}$.

Having established the properties (i)-(vi), we can pass to prove the following.

Proposition 3.1. Let $\beta \neq \beta_c$ and $\{\xi_i; 1 \leq i \leq M\}$ be the solution of (3.2) with boundary conditions ξ_0, ξ_{M+1}. Then there exist $b > 0$, $B > 0$ and M_0 all depending on β, ξ_0, ξ_{M+1}, such that for $M \geq M_0$

$$|\bar{\xi}(c) - \xi_i| < B e^{-bd_i} \quad i = 0, 1, \ldots, M+1 \quad (d_i = \min\{i, M+1-i\}). \tag{3.16}$$

Proof. Let M be so large that (vi) holds. Let $\beta < \beta_c$. Then

$$x^{-1}(F(x) - 2x) \geq F'(0) - 2 \equiv \omega^{-1} > 0 \quad \text{for } x \geq 0 \tag{3.17}$$

and recalling (3.6), $F(\xi_i) \leq \xi_{i+1} + \xi_{i-1}$, wherefrom

$$\xi_i \leq \omega(\xi_{i+1} + \xi_{i-1} - 2\xi_i), \quad i = 1, \ldots, M. \tag{3.18}$$

For $i \neq i_0 = i_0(M)$ given by (vi), (3.18) leads to

$$\xi_i \leq [\omega/(1+\omega)]^i \xi_0 \; (i \leq i_0 - 1); \; \xi_i \leq [\omega/(1+\omega)]^{M+1-i} \xi_{M+1} \; (i \geq i_0 + 1) \tag{3.19}$$

while, for $i = i_0$, it implies $\xi_{i_0} \leq [\omega/(1+2\omega)](\xi_{i_0+1} + \xi_{i_0-1})$. Hence:

$$\xi_{i_0} \leq [1 + \omega/(1+2\omega)]\{[\omega/(1+\omega)]^{i_0} \xi_0 + [\omega/(1+\omega)]^{M+1-i} \xi_{M+1}\}. \tag{3.20}$$

Taking $b = \ln(1 + 1/\omega)$, $B = 2\max\{\xi_0, \xi_{M+1}\}$ and accounting that $\bar{\xi}(c) = 0$, one obtains (3.16).

Let us suppose now $\beta > \beta_c$. Then, by (vi), the sequence $\{\xi_i; 0 \leq i \leq M+1\}$ either has only one local extremum in $i_0 \in \{1, \ldots, M\}$, or it is monotone.

a) If $\{\xi_i\}$ has a local minimum at i_0, it is monotonically increasing of $|i - i_0|$ both for $i \leq i_0$ and for $i \geq i_0$. Besides, (3.15a) will provide

$$\max\{\xi_0, \xi_{M+1}\} \equiv \xi^* \geq \xi_i > \bar{\xi}(c), \quad 0 \leq i \leq M+1. \tag{3.21}$$

Remembering that $\bar{\xi}(c) > 0$ and $g(c, x)$ is convex on $x \geq \sqrt{c}$, it is easy to see that, for $\xi \geq \bar{\xi}(c)$:

$$\xi - \bar{\xi}(c) \leq \tilde{\omega}(c) g(c, \xi), \text{ where } \tilde{\omega}(c) = 1/g'(c, \bar{\xi}(c)) < \infty. \tag{3.22}$$

We apply this inequality to ξ_i. On the other hand, operating on Eq. (3.9) (for, e.g., $i < i_0$), one obtains $g(c, \xi_i) \leq (\bar{\xi}(c)^4 - c^2)^{-1/2} (\xi_{i-1} + \xi_{i+1} - 2\xi_i) \leq (\bar{\xi}(c)^4 - c^2)^{-1/2} [(\xi_{i-1} - \bar{\xi}(c)) - (\xi_i - \bar{\xi}(c))]$, which

gives, in conjunction with Eq. (3.22), $\xi_i - \bar{\xi}(c) \leq \frac{K}{1+K}(\xi_{i-1} - \bar{\xi}(c))$

with $K = \tilde{\omega}(c) \cdot (\bar{\xi}(c)^4 - c^2)^{-1/2}$, wherefrom Eq. (3.16) immediately follows.

b) If $\{\xi_i; 0 \leq i \leq M+1\}$ has a local maximum in i_0, again by monotonity and (3.15b):

$$\min\{\xi_0, \xi_{M+1}\} \equiv \xi_* \leq \xi_i < \bar{\xi}(c), \quad 0 \leq i \leq M+1. \tag{3.23}$$

By the choice of M in (vi), $\xi_* > \sqrt{c}$ and the convexity of g provides:

$$\bar{\xi}(c) - \xi \leq -\tilde{\omega}(c)\, g(c,\xi), \text{where} \quad \tilde{\omega}(c) = -g(c,\xi_*)(\bar{\xi}(c) - \xi_*)^{-1} \tag{3.24}$$

valid for $\xi < \bar{\xi}(c)$. Further one proceeds as above.

c) The case of monotone $\{\xi_i ; 0 \leq i \leq M+1\}$ can be easily reduced to a) or b) before.

Proposition 3.2. Let $\{\xi_i ; 1 \leq i \leq M\}$ be the solution of the system (3.2) with boundary conditions ξ_0, ξ_{M+1}. Let $\beta > \beta_c$, $\theta_i = \sphericalangle(\xi_i, \xi_{i+1})$, $\theta = \sphericalangle(\xi_0, \xi_{M+1})$, and d_i, b, B be the quantities introduced in Prop.3.1. Then, there exist $C>0$, M_0 depending on β, ξ_0, ξ_{M+1} only such that, for $M \geq M_0$,

$$|\theta_i - \theta/(M+1)| < CM^{-2} \ln M \tag{3.25}$$

whenever i is such that $d_i > (1/b) \ln BM$, while otherwise:

$$\theta_i = O(1/M). \tag{3.26}$$

Before proving this we give without proof the following

Lemma 3.3. Let $\phi_i \in (0, \pi/2)$ and $\sum_{i=1}^{N} \phi_i = \phi$. If there exist $A > 0$, $a > 0$ and $i_0 \in \{1,\dots,N\}$ such that:

$$a \sin\phi_{i_0} \leq \sin\phi_i \leq A \sin\phi_{i_0} \quad i = 1,\dots,N, \tag{3.27}$$

then

$$(a\phi/AN)(1-(A\phi/aN)^2) \leq \sin\phi_i \leq A\phi/aN, \quad i = 1,\dots,N. \tag{3.28}$$

Proof. If $\theta = 0$, (3.25) is trivially satisfied. So, let $\theta \in (0,\pi)$. Let us choose M_1 such that, for $M \geq M_1$, (v), (vi) and Prop.3.1 hold true. Property (vi) and (3.15) imply:

$$\begin{aligned} \xi_i &\geq \min\{\xi_0, \xi_{M+1}, \bar{\xi}(c)\} = \xi_*(c) \\ \xi_i &\leq \max\{\xi_0, \xi_{M+1}, \bar{\xi}(c)\} = \xi^*(c) \end{aligned} \qquad i = 0,1,\dots,M+1, \tag{3.29}$$

while Eq. (3.7) leads to

$$\sin\theta_i = (\xi_{i_0+1}\, \xi_{i_0} / \xi_{i+1}\xi_i) \sin\theta_{i_0}, \quad i, i_0 \in \{0,1,\dots,M\} \tag{3.30}$$

providing

$$(\xi_*(c)/\xi^*(c))^2 \sin\theta_{i_0} \leq \sin\theta_i \leq (\xi^*(c)/\xi_*(c))^2 \sin\theta_{i_0}.$$

In virtue of the lemma there exist two positive constants C^* and C_*, both independent of M, such that

$$C_*(\theta/M+1) \leq \theta_i \leq C^*(\theta/M+1), \quad i = 0,1,\dots,M. \tag{3.31}$$

Let now $S_M = \{i : d_i \geq (1/b)\ln BM,\ 0 \leq i \leq M+1\}$ and let $\tilde{M}$ be its cardinal. S_M is nonvoid for M large enough, and

$$(M - \tilde{M})/M = O(M^{-1} \ln M). \tag{3.32}$$

Besides, recalling Prop.3.1, we have:

$$|\xi_i - \bar{\xi}(c)| < \frac{1}{M}, \quad i \in S_M. \tag{3.33}$$

The Eqs. (3.30) and (3.33) imply for $i, i_0 \in S_M$:

$$A^{-1} \sin\theta_{i_0} \leq \sin\theta_i \leq A \sin\theta_{i_0}, \text{ with } A = (\bar{\xi}(c) + \tfrac{1}{M})^2 (\bar{\xi}(c) - \tfrac{1}{M})^{-2} \tag{3.34}$$

But, since $\bar{\xi}(c) - \bar{\xi}(0) = O(c)$ and $c \leq \pi/M+1$, one gets

$$A = 1 + O(1/M). \tag{3.35}$$

For θ_i, $i \in S_M$, the Lemma 3.3 provides, with $\tilde{\theta} = \sum_{i \in S_M} \theta_i$:

$$A^{-2} \tilde{M}^{-1} \tilde{\theta}(1 - (A^2 \tilde{\theta}/\tilde{M})^2) \leq \sin\theta_i \leq \tilde{\theta} A^2 M^{-1} \tag{3.36}$$

As $\theta = \tilde{\theta} + \sum_{i \notin S_M} \theta_i \leq \tilde{\theta} + C^*(M - M^*)/(M+1) = \tilde{\theta} + O(M^{-1} \ln M)$,

$$0 < \theta - \tilde{\theta} = O(M^{-1} \ln M). \tag{3.37}$$

Collecting (3.32), (3.35), (3.37) and using (3.36), one gets:

$$\theta_i = \theta/M + O(M^{-2} \ln M), \quad i \in S_M. \tag{3.38}$$

Proposition 3.4. Let $\{\xi_i ; 1 \leq i \leq M\}$ be the solution of (3.2) with boundary conditions $\underset{\sim}{\xi}_0$, $\underset{\sim}{\xi}_{M+1} \in \mathfrak{T}_{\underset{\sim}{e}}$ such that $\underset{\sim}{\xi}_0 \cdot \underset{\sim}{e} = 0$, $\underset{\sim}{\xi}_{M+1} \cdot \underset{\sim}{e} > 0$. Then for $\beta > \beta_c$, and arbitrary $x \in (0,1)$:

$$\lim_{M \to \infty,\ \frac{k}{M} \to x} \underset{\sim}{\xi}_k = \xi_B(\beta)[(\underset{\sim}{\xi}_0/\xi_0)\cos\theta x + \underset{\sim}{e} \sin\theta x], \tag{3.39}$$

where $\theta = \sphericalangle(\underset{\sim}{\xi}_0, \underset{\sim}{\xi}_{M+1})$ and $\xi_B(\beta) = \bar{\xi}(0)$ is the bulk spontaneous magnetization.

Proof. Take into account that $\underset{\sim}{\xi}_k = \xi_k[(\underset{\sim}{\xi}_0/\xi_0)\cos(\sum_{i=0}^{k}\theta_i) + \underset{\sim}{e}\sin(\sum_{i=0}^{k}\theta_i)]$. Prop. 3.1 ensures that $\xi_k \to \bar{\xi}(0) = \xi_B(\beta)$, whenever $x \in (0,1)$, while Prop. 3.2 enables us to assert that

$$\lim_{\substack{M\to\infty,\\ k/M\to x}} \sum_{i=0}^{k} \theta_i = \theta \cdot x$$

wherefrom (3.39) results.

Proposition 3.5. Let $f_{\theta,M}(\beta)$ be the free energy of the model with M layers under boundary conditions as in Prop.3.4 (obtained by substituting the solution $\{\underset{\sim}{\xi}_i\}$ of Eq. (3.2) into $f(\beta,\{\underset{\sim}{h}\},\{\underset{\sim}{\xi}\})$ defined by Eq. (1.3)). Then:

$$Y(\beta) \equiv \lim_{M\to\infty} \frac{2M^2}{\theta^2}[f_{\theta,M}(\beta) - f_{0,M}(\beta)] = \xi_B(\beta)^2. \qquad (3.40)$$

Proof. Remark that $c = \theta\,\bar{\xi}(c)^2/(M+1) + O((M+1)^{-2}\ln M)$ wherefrom:

$$\xi_B(\beta) - \bar{\xi}(c) = \frac{\theta^2 \xi_B(\beta)}{(M+1)^2}[r + 2 - 1/\beta\mathcal{F}''(\mathcal{F}'^{-1}(\xi_B(\beta)))] + O((M+1)^{-3}\ln M)$$

Using this, Eq. (3.40) can be checked straightforwardly.

4. CONCLUDING REMARKS

Prop.3.3 shows that looking at a region far away from both boundaries, the state is translationally invariant. The direction of the local order parameter is intermediate between the directions of the boundary fields and depends on how the thermodynamic limit is taken. Also, Prop.3.2 provides the following information on the behaviour near boundaries:

$$\lim_{M\to\infty} \xi_k = (\underset{\sim}{\xi}_0/\xi_0)\, m_k(\beta,\xi_0).$$

Here $m_k(\beta,\xi_0) > 0$ are the layer magnetizations of a semi-infinite system with boundary field ξ_0 (as defined in [3]) which approach exponentially fast the bulk spontaneous magnetization. In other words, the layers at finite distance from one boundary, however small the coupling to it, do not feel the phase at the

other boundary. Then we can conclude that an interface can be localized neither deep in the bulk, nor near the boundary. The same conclusion holds also in the spherical limit of the model we considered here, as announced in [2]. To see this, it is sufficient to remark that the function the minimum of which is looked for in the spherical model is nothing but the limit, when $|\underset{\sim}{h}|$ converges to the boundary of $\mathcal{U}_{\underset{\sim}{e}}$ and $D \to \infty$, of the function $\tilde{T}_h(\underset{\sim}{y})$ appearing in Eq. (2.11) as an artifact of the proof of Prop.2.1, and to apply the lemma in Appendix B. Thus the D-vector model and the spherical one can both be solved in one stroke.

The interface problem has been recently solved for short range interactions in Ref.[4] within the spherical model of Berlin and Kac[5]. They found as well that the interface is diffuse at all temperatures. However, the magnetization profile there has some unphysical features, which led them to suggest that the model itself is inadequate for considering such "non-translationally invariant" problems. Our generalized spherical model is free of this objection and indeed the profile we obtain is physically sound.

Incidentally, Eq. (3.40) also holds for the spherical model of Berlin and Kac[5], where $Y(\beta)$ is defined by the difference of the free energies with antiperiodic and periodic boundary conditions[6]. We propose to study its status in the generalized spherical model with short range interactions in a future publication.

APPENDIX B

Let $D \subset R^n$ be an open convex set and $f: D \to \underset{\sim}{R}$ a continuous convex function. Let $\underline{f}$ be the extension of f to $\bar{D}$ defined by:

$$\underline{f}(x) = \liminf_{y\to x;\, y\in D} f(y), \qquad x \in \bar{D}. \qquad (B1)$$

Evidently f and $\underline{f}$ have the same lower bound and $\underline{f}$ is lower semicontinuous on $\bar{D}$. For every $a > \inf f$, the set $Q(f,a) = \{x \in \bar{D}: \underline{f}(x) \le a\}$ is convex and closed[7], and $\underline{f}$ attains its absolute minimum on the set $Q(f) = \bigcap_{a>\inf f} Q(f,a) = \{x \in \bar{D}: \underline{f}(x) = \inf f\}$. Let $\mathcal{C}$ be the set of all continuous convex functions on D, and $\mathcal{C}_0$ the set of all $f \in \mathcal{C}$ for which Q(f) is nonvoid and bounded. Then, the following continuity property holds:

Lemma. Let $f \in \mathcal{C}_0$. For every $\epsilon > 0$, there exist $\eta > 0$ and a compact $K \subset D$ such that for every $g \in \mathcal{C}$ satisfying

$$\sup_{x\in K} |f(x) - g(x)| < \eta$$

one has $g\in \mathcal{C}$ and

$$Q(g)\subset\{x:\ d(x,Q(f))<\epsilon\};\qquad |\inf f-\inf g|<\epsilon\ , \tag{B2}$$

where $d(x,Q(f))$ is the distance between x and $Q(f)$.

Proof. Without loss of generality one can suppose $\inf f=0$. Let V be the set

$$V=\{x:\quad d(x,Q(f))<\epsilon\}. \tag{B3}$$

Since $Q(f)$ is nonvoid, convex and compact, V is convex, open and with compact closure, $\bar{V}$. As f is lower semicontinuous on $\bar{D}$, it can be asserted/7/ that there is a $\delta>0$ such that $Q(f,\delta)\subset V$. Let us choose $\delta'\in(0,\delta)$, $\delta'<\epsilon/4$ and $x_0\in D$ such that

$$f(x_0)=\delta'. \tag{B4}$$

Let us now consider the following set:

$$K_\delta=\{z:\ \underline{f}(z)=\delta,\quad z\in[x_0,y],\quad y\in(\bar{D}\setminus V)\cap\bar{V}. \tag{B5}$$

Then the following properties hold:

(i) $K_\delta\subset V\cap D$;
(ii) for every $y\in\bar{D}\setminus V$, $[x_0,y]\cap K_\delta\neq\emptyset$;
(iii) K_δ is a compact set.

Indeed

i) Obviously $K_\delta\subset Q(f,\delta)\subset V$. If $z\in K_\delta$, then $z\in[x_0,y]$ with $y\in\bar{D}\setminus V$; but $y\in\bar{D}\setminus V$ implies $\underline{f}(y)>\delta$ and, since $x_0\in D$ then $[x_0,y]\subset D$ and therefore $K_\delta\subset D$.

ii) Let $y\in\bar{D}\setminus V$; as $x_0\in V\cap D$ and V is open and convex, one has $[x_0,y]\cap V=[x_0,y_1)$, $[x_0,y]\cap(\bar{D}\setminus V)=[y_1,y]$, where $y_1\in\bar{V}\cap(\bar{D}\setminus V)$. Evidently, $f|_{[y_1,y]}>\delta$ and therefore supposing that $[x_0,y]\cap K_\delta=\emptyset$ implies $\delta\notin f([x_0,y_1))$. But since f is continuous on D and $[x_0,y_1)\subset D$, $f([x_0,y_1))$ is an interval and, accounting that $f(x_0)=\delta'<\delta$ it follows $f|_{[x_0,y_1)}<\delta$. Recalling that $\underline{f}$ is lower semicontinuous, we shall have

$$\underline{f}(y_1)=\liminf_{z\to y_1;\, z\in[x_0,y_1)} f(z)\le\delta$$

which contradicts $f(y_1)>\delta$. Hence $[x_0,y]\cap K_\delta\neq\emptyset$.

iii) We know that K_δ is bounded. We shall show that K_δ is closed. Let us consider a convergent sequence $\{z_n\}\subset K_\delta$, $z_n\to z$. There exist $y_n\in(\bar{D}\setminus V)\cap\bar{V}$ and $\lambda_n\in[0,1]$ such that

$$z_n=(1-\lambda_n)x_0+\lambda_n y_n. \tag{B6}$$

But $(\bar{D}\setminus V)\cap\bar{V}$ is compact, and hence there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to $y\in(\bar{D}\setminus V)\cap\bar{V}$. Eq. (B6) provides:

$$|z_{n_k}-x_0|=\lambda_{n_k}|y_{n_k}-x_0|. \tag{B7}$$

Let us remark that $\lim_k|y_{n_k}-x_0|>0$ (because, otherwise, $x_0=y$ which is impossible since $x_0\in V$ but $y\in\bar{D}\setminus V$), which in turn implies that λ_{n_k} is convergent. Let $\lambda=\lim\lambda_{n_k}$. Eq. (B6) provides then $z=(1-\lambda)x_0+\lambda y$ with $\lambda\in[0,1]$, $y\in(\bar{D}\setminus V)\cap\bar{V}$. But, since $\{z_n\}\subset K_\delta\subset Q(f,\delta)$, $z\in Q(f,\delta)\cap V$. But $z\in[x_0,y]$, so $z\in[x_0,y]\cap V=[x_0,y)\subset D$. Thus $z\in D$ and therefore is a continuity point of f which leads to

$$f(z)=\lim_k f(z_{n_k})=\delta$$

implying $z\in K_\delta$. Hence K_δ is closed, establishing iii).

Let further consider for $\alpha>1$ the following set

$$K=\{z:\ z=x_0+\alpha^{-1}(x-x_0),\quad x\in\bar{V}\cap\bar{D}\}. \tag{B8}$$

Obviously $K\subset V\subset D$ and, for $\alpha\to1$, K gets closer to $V\cap D$; since the compact $K_\delta\subset V\cap D$, there exists $\alpha>1$ which can be safely taken less than 2, such that $K_\delta\subset K\subset V\cap D$. Consider now $g\in\mathcal{C}$ such that $\sup_{x\in K}|f(x)-g(x)|<\eta$, with $\eta<\epsilon/4$, $(\delta-\delta')/2$, and remark that for $x\in\bar{D}\setminus V$, $x=\lambda x_1-(\lambda-1)x_0$ with $x_1\in K_\delta\subset K$ and $\lambda>1$. Therefore

$$g(x)\ge\lambda g(x_1)-(\lambda-1)g(x_0)\ge\lambda(f(x_1)-\eta)+(\lambda-1)(f(x_0)+\eta)=$$

$$=\delta-\eta+(\lambda-1)(\delta-\delta'-2\eta).$$

Since $\delta-\delta'>2\eta$, we have

$$g(x)>\delta-\eta>\frac{1}{2}(\delta+\delta'). \tag{B9}$$

On the other hand

$$g(x_0)\le f(x\ )+\eta=\delta'+\eta<\frac{1}{2}(\delta+\delta') \tag{B10}$$

which together with (B9) implies that $\inf g$ is attained on $\bar{D}\cap V$ and therefore $g\in\mathcal{C}_0$ and $Q(g)\subset V\cap\bar{D}$.

Let now $x\in\bar{V}\cap\bar{D}$. With (B8) one can write $x=\alpha z-(\alpha-1)x_0$, with z, $x_0\in K$ and $\alpha\in(1,2)$ as previously fixed. Using the convexity of g, we obtain

$$g(x) \geq a g(z) - (a-1) g(x_0) \geq a (f(z) - \eta) - (a-1)(f(x_0) + \eta),$$

and since $f(z) \geq \inf f = 0$ and $a \in (1,2)$ we get

$$g(x) \geq -3\eta - \delta' \geq -\epsilon .$$

As $Q(g) \subset \bar{V} \cap \bar{D}$, it follows

$$\inf g \geq -\epsilon . \qquad (B11)$$

On the other hand, $g(x_0) \leq f(x_0) + \eta = \delta' + \eta \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon$ wherefrom

$$\inf g \leq \epsilon . \qquad (B12)$$

Eq. (B11) and (B12) read together $|\inf g| \leq \epsilon$, providing the last part of the lemma.

ACKNOWLEDGEMENTS

The authors would like to thank Profs. H.Wagner and C.J.Thompson for many interesting discussions on phase separation in magnets.

REFERENCES

1. Angelescu N. et al. JINR, E17-83-10, Dubna, 1983.
2. Angelescu N. et al. J.Phys.A: Math.Gen., 1981, 14, p.L533.
3. Angelescu N. et al. J.Stat.Phys., 1981, 24, p.529.
4. Abraham D.B., Robert M.A. J.Phys.A: Math.Gen., 1980, 13, p.2229.
5. Berlin T.H., Kac M. Phys.Rev., 1952, 86, p.821.
6. Barber N.N., Fisher M.E. Ann.Phys., 1973, 77, p.1.
7. Rockafellar R.T. Convex Analysis. Princeton Univ.Press, 1970.

Received by Publishing Department
on January 7, 1983.

Ангелеску Н., Бундару М., Костаке Г. E17-83-11

О разделении фаз в системах с непрерывной симметрией. Профиль намагниченности и модуль спиральности

Доказано, что граница раздела фаз в изотропных D-векторных моделях с взаимодействием Каца-Гельфанда является диффузной при всех температурах как для конечных значений D, так и в пределе $D\to\infty$. Показано, что граница раздела фаз остается неустойчивой даже в случае включения "прикалывающего" потенциала типа Абрахама. Найдено точное выражение для профиля намагниченности. Модуль спиральности оказывается равным квадрату объемной спонтанной намагниченности.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1983

Angelescu N., Bundaru M., Costache G. E17-83-11

On Phase Separation in Systems with Continuous Symmetry. The Magnetization Profile and the Helicity Modulus

It is proved that the interface in the isotropic D-vector model and its $D\to\infty$ limit, both with Kac-Helfand interactions, is diffuse at all temperatures. The interface does not stabilize even when a pinning potential of the Abraham type is accomodated. The magnetization profile is explicitly calculated. The helicity modulus equals the squared bulk spontaneous magnetization.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983