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ON PHASE SEPARATION IN SYSTEMS WITH CONTINUOUS SYMMETRY.

The Isotropic D-Vector Model with Kac-Helfand Interactions

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1. INTRODUCTION

Establishing the existence of non-translational-invariant Gibbs states describing sharp interfaces is an interesting and nontrivial problem in the theory of phase transitions. It is known that the ferromagnetic Ising model in two dimensions has no such states [1,2], while for three and more dimensions the contrary is true [3,4]. The absence of sharp interface in the two-dimensional Ising model is due to the existence of large fluctuations in the system, which make the two phases - when brought into contact - to spread one over the other on a thickness $\sim L^{1/2-\epsilon}$ (L is the interface length), resulting in zero magnetization profile [1,5]. On the other hand, the fluctuations could destabilize the interface in the three-dimensional Ising model and a roughening transition at $T_R < T_c(3)$ has been conjectured [6,7]. However, the only models for which a roughening transition has been established rigorously are either SOS models [8] or models with a pinning potential of the sort studied by Abraham [9]. Thus, thermal fluctuations play an extremely important role in the phase separation and it is well known that they are controlled by the symmetry of the Hamiltonian as well as the lattice dimension and the range of the potential. For systems with continuous symmetry, the fluctuations are expected to increase and there is a phenomenological argument [10] according to which the interface should have a diverging width. We adress ourselves in this paper to disproving the existence of a sharp interface for isotropic D-vector models and their spherical limit. In order to suppress the fluctuations and thus favour the localization of the interface, we considered interactions of the Kac-Helfand type [11]. Moreover, we try to pin the interface near one boundary, by lowering there the coupling as was done by Abraham [9] for the two-dimensional Ising model. Despite this, we found that for all temperatures the interface is not localized even nearby the distorted boundary; its width is of an order of the thickness of the sample on the top and bottom of which we imposed "mixed" boundary conditions. In this respect we have explicitly calculated the magnetization profile taking full advantage of the simplification induced by the long range character of the interactions. In turn, the knowledge of the profile allows obtaining the leading asymptotic term, as the number of layers, $M \to \infty$, of the free energy shift induced by the mixed boundary conditions relative to the homogeneous ones. It turns out that this shift behaves as $\theta^2/2\cdot Y\cdot 1/M^2$ (instead of $\sigma\cdot 1/M$ in the presence of a sharp interface), as is phenomenologically expected [12]. Here θ is the angle between the spin on the upper and lower boundaries. The coefficient Y is the so-called helicity modulus. If one accepts the $D=2$ model as describing superfluidity, Y is related to the superfluid density. In the models under consideration here we obtain Y equal to the squared bulk magnetization. We would like to note that the results obtained for the D-vector model hold even in the spherical limit and have been previously announced in the letter [13]. As our method relies on establishing a certain isomorphism (very likely holding only when long range interactions are used) between the magnetization profile of the D-vector model and that of a "D-vectorial spherical model" (which under appropriate limits becomes that considered in [13]), there will in fact be no need to study separately the spherical limit.

The isotropic D-vector model with Kac-Helfand interactions can be described as follows. Consider a slab consisting of M copies of a rectangular array $\Lambda\subset Z^{d-1}$ of "spins"; the energy of a configuration $\{\underset{\sim}{S}_{ir}\in R^D: \|\underset{\sim}{S}_{ir}\|^2=D,\ r\in\Lambda,\ 1\le i\le M\}$ is taken as:

$$\mathcal{H}^{(\gamma)}_{M,\Lambda}(\{\underset{\sim}{S}\}) = -\frac{\gamma^{d-1}}{2}\sum_{r,r'\in\Lambda}\rho(\gamma|r-r'|)\sum_{i,j=1}^{M} J_{ij}\,\underset{\sim}{S}_{ir}\cdot\underset{\sim}{S}_{jr'} - \sum_{r\in\Lambda}\sum_{i=1}^{M} D^{1/2}\underset{\sim}{h}_i\cdot\underset{\sim}{S}_{ir}\ , \tag{1.1}$$

where $\rho: R^{d-1}\to R$ is a positive definite function such that $\int\rho(x)dx=1$, the scaling factor $\gamma>0$ controls the interaction range,

$$J_{ij} = r\,\delta_{ij}+\delta_{|i-j|,1}\ ,\quad i,j=1,\dots,M \quad (r\ge 2) \tag{1.2}$$

and $D^{1/2}\underset{\sim}{h}_i$ is a homogeneous magnetic field acting on the i^{th} layer. To describe the phase separation, we shall eventually take all $\underset{\sim}{h}_i=0$ but $\underset{\sim}{h}_1$ and $\underset{\sim}{h}_M$ in terms of which we describe the boundary conditions. Namely, consider the spins in two extremal layers, $i=0$ and $i=M+1$, fixed along two different directions $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$; moreover, allow a different coupling $J_{0,1}<1$ at one boundary; then $\underset{\sim}{h}_1 = J_{0,1}\underset{\sim}{e}_1$, $\underset{\sim}{h}_M = J_{M,M+1}\underset{\sim}{e}_2$ ($\|\underset{\sim}{h}_1\| = J_{0,1}$, $\|\underset{\sim}{h}_M\|=1$).

The model under consideration is the limit as $\gamma\downarrow 0$ of the model defined by the Hamiltonian (1.1) in the thermodynamic limit $\Lambda\to\infty$, and it is an inhomogeneous mean field model with M

D-vector order parameters. In particular, the $\gamma\downarrow 0$ limit of the free energy per spin and per spin-component exists by standard arguments [14] and is given by the absolute minimum with respect to $\{\underset{\sim}{\xi}\} = \{\underset{\sim}{\xi}_i : \underset{\sim}{\xi}_i\in R^D,\ 1\le i\le M\}$ of the function:

$$f(\beta,\{\underset{\sim}{h}\},\{\underset{\sim}{\xi}\}) = \frac{1}{2M}\sum_{i,j=1}^{M} J_{ij}\,\underset{\sim}{\xi}_i\cdot\underset{\sim}{\xi}_j - (\beta M)^{-1}\sum_{i=1}^{M}\mathcal{F}(\beta\|\sum_{j=1}^{M}J_{ij}\underset{\sim}{\xi}_j + \underset{\sim}{h}_i\|)\ , \tag{1.3}$$

where

$$\mathcal{F}(\|\underset{\sim}{x}\|) = D^{-1}\log\int_{\|\underset{\sim}{S}\|^2=D} dS\,\exp[D^{1/2}\underset{\sim}{x}\cdot\underset{\sim}{S}] \tag{1.4}$$

is the free energy of one spin in the external field $D^{1/2}x$, and has the properties (i)-(iv) listed in App.A. Taking into account that $\mathcal{F}$ has a linear behaviour at infinity ($|\mathcal{F}'|<1$) and that the matrix J, Eq. (1.2), is strictly positive definite, one concludes that $f(\beta,\{\underset{\sim}{h}\},\cdot)$ attains its minimum at a finite distance. Since $\mathcal{F}$ is an even function, $f(\beta,\{\underset{\sim}{h}\},\cdot)$ is differentiable on R^{DM}, and hence its minimum points are among its stationary points, i.e., among the solutions of the system:

$$\underset{\sim}{\xi}_i = \mathcal{F}'(\beta\|\sum_{j=1}^{M}J_{ij}\underset{\sim}{\xi}_j+\underset{\sim}{h}_i\|)\cdot(\sum_{j=1}^{M}J_{ij}\underset{\sim}{\xi}_j+\underset{\sim}{h}_i)/\|\sum_{j=1}^{M}J_{ij}\underset{\sim}{\xi}_j+\underset{\sim}{h}_i\|\ ,\quad 1\le i,j\le M. \tag{1.5}$$

The minimum point $\{\underset{\sim}{\xi}\}$ is intimately related to the magnetization profile. This first part of the paper (together with the lemma on convex function in Appendix B, Part II, which seems to be new, and therefore of independent interest) develops the techniques required for solving Eq. (1.5).

2. THE LAYER MAGNETIZATIONS AND THE MINIMUM POINT

We have seen in Sec. 1 that the model under consideration has a mean-field character and thus solving it requires finding the absolute minimum of the function $f(\beta,\{\underset{\sim}{h}\},\cdot)$ defined by Eq. (1.3). We are however interested in the phase separation phenomenon, what requires studying the magnetization profile across the slab thickness. This is equivalent to the detailed characterization of the point at which the absolute minimum of f is attained. To be more precise, suppose $f(\beta,\{\underset{\sim}{h}\},\cdot)$ attains the absolute minimum at a unique point $\xi(\{\mathbf{h}\})$, where moreover the Hessian matrix $\partial^2 f/\partial\xi_{i\alpha}\partial\xi_{j\beta}$ is nonsingular; then the layer magnetizations, at the given β and $\mathbf{h}$:

$$\mathbf{m}_i = \lim_{\gamma\downarrow 0}\ \lim_{\Lambda\to\infty} \langle D^{-1/2}|\Lambda|^{-1}\sum_{\mu\in\Lambda} \mathbf{S}_{\mu,i}\rangle^{(M)}_{\gamma,\Lambda} = \tag{2.1}$$

$$= -\lim_{\gamma\downarrow 0}\ \lim_{\Lambda\to\infty} M\nabla_{\mathbf{h}}\, f^{(M)}_{\gamma,\Lambda}(\beta,\{\mathbf{h}\})$$

are nothing but $\mathbf{m}_i=\xi_i(\{\mathbf{h}\})$. ($\langle\,\rangle^{(M)}_{\gamma,\Lambda}$ and $f^{(M)}_{\gamma,\Lambda}$ denote respectively the Gibbs state and free energy defined by the Hamiltonian (1.1)). Indeed, the minimum is attained on a solution of the system (1.5). Since the Hessian matrix is nonsingular, for all $\{\mathbf{h}'\}$ in a neighbourhood of $\{\mathbf{h}\}$, the system (1.5) has a unique solution $\xi(\{\mathbf{h}'\})$ in the neighbourhood of $\xi(\{\mathbf{h}\})$, which depends differentiably on $\{\mathbf{h}'\}$ and is the unique point of absolute minimum of $f(\beta,\{\mathbf{h}'\},\cdot)$. (For the latter fact, remark that the minimum point is always in the compact $\|\xi_i\|<1$, $i=1,\dots,M$, as is seen from Eq. (1.5)). Thus $f(\beta,\{\mathbf{h}'\},\xi(\{\mathbf{h}'\}))$ is differentiable at $\{\mathbf{h}'\}=\{\mathbf{h}\}$. Remembering that $f^{(M)}_{\gamma,\Lambda}(\beta,\{\mathbf{h}'\})$ are convex of $\{\mathbf{h}'\}$ and converge for $\Lambda\to\infty$, $\gamma\downarrow 0$ to $f(\beta,\{\mathbf{h}'\},\xi(\{\mathbf{h}'\}))$, the assertion follows from Griffith's theorem /15/.

In the next proposition we shall exhibit a convenient domain for $\{\mathbf{h}\}$ on which the situation above takes place and suited for describing phase separation. We start with a few definitions. Let us fix $\mathbf{e}\in R^D$ and define:

$$\mathfrak{D}_{\mathbf{e}} = \{\mathbf{x}\in R^D:\ \mathbf{x}\cdot\mathbf{e}>0\},$$
$$\mathfrak{U}_{\mathbf{e}} = \{\{\mathbf{x}\}=(\mathbf{x}_1,\dots,\mathbf{x}_M)\in\bar{\mathfrak{D}}^M_{\mathbf{e}}:\ \sum_{i=1}^M \mathbf{x}_i\cdot\mathbf{e}>0\}, \tag{2.2}$$

where $\bar{\mathfrak{D}}$ stands for the closure of $\mathfrak{D}$. For $\{\mathbf{h}^*\}=\{\mathbf{h}^*_1,\dots,\mathbf{h}^*_M\}\in\bar{\mathfrak{U}}_{\mathbf{e}}$ with $\mathbf{h}^*_i\mathbf{e}=0$, we define

$$\mathcal{E}_{\mathbf{h}^*} = \{\{\mathbf{h}\}:\ \mathbf{h}_i = \sum_{j=1}^M a_{ij}\mathbf{h}^*_j + a_i\mathbf{e},\ 1\le i\le M,\ a_{ij}\in R,\ a_i\ge 0\}, \tag{2.3}$$

Proposition 2.1. Let $\{\mathbf{h}\}\in\mathfrak{U}_{\mathbf{e}}$ and $\beta>0$ be given. Then, the absolute minimum of $f(\beta,\{\mathbf{h}\},\cdot)$, Eq. (1.3), is attained at one and only one point, $\xi(\{\mathbf{h}\})$. Moreover:

(i) $\xi(\{\mathbf{h}\})\in\mathfrak{D}^M_{\mathbf{e}}$ and is the unique solution in $\mathfrak{D}^M_{\mathbf{e}}$ of Eq. (1.5);
(ii) $\xi(\{\mathbf{h}\})$ is differentiable on $\mathfrak{U}_{\mathbf{e}}$;
(iii) there exists $\lim\xi(\{\mathbf{h}\})$ for $\{\mathbf{h}\}\to\{\mathbf{h}^*\}$, $\{\mathbf{h}\}\in\mathcal{E}_{\mathbf{h}^*}\cap\mathfrak{U}_{\mathbf{e}}$.

Proof. We proceed in several steps:
a) The points of absolute minimum of f are in $\mathfrak{D}^M_{\mathbf{e}}$.
b) Eq. (1.5) has in $\mathfrak{D}^M_{\mathbf{e}}$ one and only one solution.
c) The Hessian matrix of f is nondegenerate at $\xi=\xi(\{\mathbf{h}\})$.
d) The existence of the limit in (iii).
a) For any $\{\mathbf{h}\}\in\mathfrak{U}_{\mathbf{e}}$ and $\{\xi\}\in R^{DM}$ one can write

$$\mathbf{h}_i=\mathbf{h}'_i+a_i\mathbf{e},\quad \xi_i=\xi'_i+\alpha_i\mathbf{e},\qquad i=1,\dots,M, \tag{2.4}$$

where $\mathbf{h}'_i\cdot\mathbf{e}=\xi'_i\cdot\mathbf{e}=0$; obviously $a_i\ge 0$, $1\le i\le M$ and $\sum_{i=1}^M a_i>0$.

Let us denote by $\mathbf{a}$, $\alpha\in R^M$ the vectors of components a_i, α_i respectively. Accordingly, $f(\beta,\{\mathbf{h}\},\{\xi\})$ can be written as:

$$f(\beta,\{\mathbf{h}\},\{\xi\}) = K(\{\xi'\}) + g(\{\mathbf{h}'\},\{\xi'\};\mathbf{a},\alpha),$$

where

$$K(\{\xi'\}) = (1/2M)\sum_{i,j=1}^M J_{ij}\,\xi'_i\xi'_j\ ,$$

$$g(\{\mathbf{h}'\},\{\xi'\};\mathbf{a},\alpha) = (1/2M)\sum_{i,j=1}^M J_{ij}\,\alpha_i\alpha_j - (1/M)\sum_{i=1}^M \Phi_i((J\alpha+\mathbf{a})_i)$$

the functions $\Phi_i(x)$ being defined by:

$$\Phi_i(x) = \beta^{-1}\mathcal{F}(\beta\sqrt{k_i^2+x^2}),\qquad k_i=\|(J\xi'+\mathbf{h}')_i\|,\ 1\le i\le M. \tag{2.5}$$

The set $\{\Phi_i\}_{1\le i\le M}$ satisfies the properties (i)-(iv) in App. A and hence the Lemma stated there can be applied to see that $\inf_{\alpha} g(\{\mathbf{h}'\},\{\xi'\};\mathbf{a},\alpha)$ is realized at one and only one point $\alpha(\{\xi'\})>0$. Considering now a point $\{\xi^*\}$ at which $f(\beta,\{\mathbf{h}\},\cdot)$ attains the absolute minimum, it is obvious that $\inf_{\alpha} g(\{\mathbf{h}'\},\{\xi^{*\prime}\};\mathbf{a},\alpha)$ is attained at α^*, where $\{\xi^{*\prime}\}$, $\{\alpha^*\}$ represent the decomposition of $\{\xi^*\}$, cf. Eq. (2.4). It follows that $\alpha^*>0$, i.e. $\{\xi^*\}\in\mathfrak{D}^M_{\mathbf{e}}$.

b) We can restrict from now on the domain of all the functions entering into Eq. (1.5) to $\mathfrak{L}=\{\xi\in\mathfrak{D}^M_{\mathbf{e}}:\ \xi_i\equiv\|\xi_i\|<1,\ 1\le i\le M\}$ and define $\psi:\mathfrak{L}\to R^M$ by:

$$\psi_i(\{\xi\}) = \beta^{-1}\mathcal{F}'^{-1}(\xi_i)/\xi_i\ ,\qquad i=1,\dots,M. \tag{2.6a}$$

As for any solution ξ_i of (1.5), $\xi_i=\mathcal{F}'(\beta\|(J\xi+\mathbf{h})_i\|)$, the system (1.5) is equivalent on $\mathfrak{L}$ to:

$$[\mathrm{diag}\,\psi(\{\xi\}) - J]\,\xi = \mathbf{h}\,. \tag{2.6b}$$

where $\mathrm{diag}\,\gamma$ denotes the M×M diagonal matrix $\delta_{ij}\gamma_i$. Now, if $\{\xi^*\}\in\mathcal{L}$ satisfies (2.6b), the matrix $[\mathrm{diag}\psi(\{\xi^*\}) - J]$ transforms the strictly positive vector α^* into the positive vector $a\neq 0$, so it is a strictly positive definite matrix (see, e.g., ref. /18/). Then, Eq. (2.6b) implies:

$$\xi_i^{*2} = [(\mathrm{diag}\psi(\{\xi^*\}) - J)^{-1} h]_i^2, \quad i = 1,\dots,M. \tag{2.7}$$

In order to prove that Eq. (2.6b) has exactly one solution in $\mathcal{L}$, we shall try to find a change of variables under which (2.6b) transforms into the extremum condition for a certain strictly convex differentiable function. For this aim, define $G:[0,1]\to R$ by:

$$G(x^2) = \beta^{-1}\mathcal{F}'^{-1}(x)/x \tag{2.8}$$

and remark that

$$\psi_i(\{\xi\}) = G(\xi_i^2), \quad i = 1,\dots,M. \tag{2.9}$$

Keeping in mind the properties of $\mathcal{F}$ (see (i)-(iv) in App. A). one can see that G is strictly increasing, continuous and transforms $[0,1)$ onto $[\beta^{-1},\infty)$, besides, G is differentiable and $(G^{-1})'>0$ on (β^{-1},∞). If H is a primitive of G^{-1}, it can be defined on $[\beta^{-1},\infty)$, where moreover it is strictly convex. Let $\mathfrak{D}$ be the open convex set:

$$\mathfrak{D} = \{\gamma\in R^M:\ \mathrm{diag}\,\gamma - J > 0,\ \gamma_i > \beta^{-1}\} \tag{2.10}$$

and $T_h:\mathfrak{D}\to R$ be the following function:

$$T_h(\gamma) = \sum_{i,j=1}^{M} (\mathrm{diag}\gamma - J)^{-1}_{ij}\, h_i h_j + \sum_{i=1}^{M} H(\gamma_i), \tag{2.11}$$

where $\{h\}\in\mathcal{U}_e$. As the mapping $X\to X^{-1}$ is convex on the set of strictly positive definite matrices /17/ and H is strictly convex, T_h is strictly convex on $\mathfrak{D}$. Hence the system:

$$\frac{\partial T_h}{\partial\gamma_i} = G^{-1}(\gamma_i) - [(\mathrm{diag}\gamma - J)^{-1} h]_i^2 = 0, \quad i = 1,\dots,M \tag{2.12}$$

has at most one solution on $\mathfrak{D}$.

Let $\{\xi^*\}$ be a solution of (2.6b) and let $\gamma^*\equiv\psi(\{\xi^*\})\in\mathfrak{D}$. Then γ^* is a stationary point of T_h. Indeed, we have already seen that $[\mathrm{diag}\psi(\{\xi^*\}) - J] > 0$. On the other hand $\psi_i(\{\xi^*\}) = G(\xi_i^{*2}) \geq G(\alpha_i^*) > \beta^{-1}$ and thus $\gamma^*\in\mathfrak{D}$. Accounting that by the definition of γ^*, $\xi_i^{*2} = G^{-1}(\gamma_i^*)$ and invoking (2.7), it follows that γ^* satisfies (2.12). But T_h has only one stationary point, and therefore $\{\xi^*\}\in\psi^{-1}(\gamma^*)$. Further we shall consider another solution $\{\xi^{**}\}$ and note that necessarily $\psi(\{\xi^{**}\}) = \gamma^*$. Then Eq. (2.6b) will provide $\xi^{**} = (\mathrm{diag}\gamma^* - J)^{-1}h = \xi^*$.

c) The Hessian matrix of f at the point $\xi = \xi(\{h\})$ is:

$$M\partial^2 f/\partial\xi_{i\mu}\partial\xi_{j\nu} = J_{ij} - \sum_{p=1}^{M} J_{ip}\eta_p^{-1}\mathcal{F}'(\beta\eta_p)J_{pi} + \beta\sum_{p=1}^{M} J_{ip}\eta_p^{-2}\xi_{p\mu}\xi_{p\nu}[\beta^{-1}\eta_p^{-1}\mathcal{F}'(\beta\eta_p) - \mathcal{F}''(\beta\eta_p)]J_{pj}, \tag{2.13}$$

where $\eta = (J\xi+h)_p$ and $\eta_p = \|\eta_p\|$. The last term in the r.h.s. of Eq. (2.13) defines a matrix of the form JAJ, and since $\mathcal{F}'(x)/x > \mathcal{F}''(x)$ for $x\geq 0$ it can be seen that A is positive definite. It remains to check that the remaining part is strictly positive definite. But $\mathcal{F}'(\beta\eta_p)/\eta_p = 1/\psi_p(\xi)$ when $\xi = \xi(\{h\})$ and, accounting that $\psi(\xi(\{h\})) > J > 0$, it can be easily seen that indeed the first two terms in (2.13) define a strictly positive matrix. The proof of statement (ii) in Prop.2.1 is thus completed.

c) We shall begin by noting that T_h introduced in Eq.(2.11) is well defined on $\mathfrak{D}$ for all $\{h\}\in R^{MD}$. We shall denote by T_h its lower semicontinuous extension to $\bar{\mathfrak{D}}$ which is strictly convex on $\bar{\mathfrak{D}}$ (i.e., strictly convex on the set on which T_h is finite) and consequently it has only one point, $\gamma(\{h\})$, at which its absolute minimum is attained. When $\{h\}\to\{h^*\}$, $T_h\to T_{h^*}$ uniformly on compacts in $\mathfrak{D}$ and we can apply the lemma in App.B (see Part II), to see that $\gamma(\{h\})\to\gamma(\{h^*\})$.

Let us consider now $\{h^*\}\in\mathcal{U}_e$, $h_i^*\cdot e = 0$, $1\leq i\leq M$ and let $e_1,\dots,e_p$ be an orthonormal basis in the subspace of R^D generated by h_i^*, $1\leq i\leq M$. If $\{h\}\in\mathcal{E}_{h^*}$:

$$h_i = h_i' + a_i e_i, \quad h_i' = \sum_{\mu=1}^{p} h'_{i\mu} e_\mu, \quad a_i\geq 0, \quad 1\leq i\leq M. \tag{2.14}$$

If moreover $\{h\}\in\mathcal{E}_{h^*}\cap\mathcal{U}_e$, $\sum_{i=1}^{M} a_i > 0$. Note also that $p\leq M$. As $\gamma(\{h\})$ converges when $\{h\}\to\{h^*\}$, $\{h\}\in\mathcal{E}_{h^*}\cap\mathcal{U}_e$, every limit point ξ^* of $\xi(h)$ will satisfy:

$$[\mathrm{diag}\gamma(\{h^*\}) - J]\,\alpha^* = 0, \tag{2.15}$$

$$[\mathrm{diag}\gamma(\{h^*\}) - J]\,\xi_\mu^{*\prime} = h_\mu^*, \quad 1\leq\mu\leq p$$

and:

$$\gamma_i(\{h^*\}) = G(\alpha_i^{*2}+\xi_i^{*\prime 2}) , \quad 1\le i\le M, \tag{2.16}$$

where $h^*_\mu = (h^*_{i\mu} : 1\le i\le M)$; $\xi^{*\prime}_i = \sum_{\mu=1}^{p} \xi^{*\prime}_{i\mu} e_\mu$, α^* realise the decomposition (2.14) of ξ^*, while $\xi^{*\prime}_\mu = (\xi^{*\prime}_{i\mu} : 1\le i\le M)$.

The proof will be completed by showing that Eqs. (2.15), (2.16) determine uniquely ξ^* in terms of h^*. If $[\mathrm{diag}\,\gamma(\{h^*\}) - J]>0$, then Eq. (2.15) provides uniquely ξ^* . If however this matix has the zero eigenvalue (necessarily simple with normalized eigenvector $v>0$), then $p<M$, and $h^*_\mu\cdot v=0$, $1\le\mu\le p$. Under these conditions, Eq. (2.15) shows that $\alpha^* = \eta v (\eta\ge 0)$ and $\xi^{*\prime}_\mu = \lambda_\mu v + u_\mu$ with u_μ $(u_\mu\cdot v=0)$ uniquely determined and linearly independent. To compute η and λ_μ, use is made of Eq. (2.16) written in the form:

$$G^{-1}(\gamma_i(\{h^*\})) = (\eta^2 + \sum_{\mu=1}^{p}\lambda_\mu^2) v_i^2 + 2\sum_{\mu=1}^{p}\lambda_\mu u_{i\mu} v_i + \sum_{\mu=1}^{p} u_{i\mu}^2, \; 1\le i\le M . \tag{2.17}$$

Summing over i and using $u_\mu\cdot v = 0$, one gets $\eta^2 + \sum_{\mu=1}^{p}\lambda_\mu^2$; then Eq. (2.17) becomes a linear system of rank p, which determines λ_μ , $1\le\mu\le p$.

This completes the proof of Proposition 2.1.

In conclusion, it has been shown that whenever the layer magnetic fields h_i are all lying in a half-space (fixed by e) the Gibbs state could be essentially determined. Calculating the layer magnetizations m_i when $\{h\}\in\mathfrak{U}_e$ is equivalent to finding the unique solution in $\mathfrak{D}^M_e$ of the system (1.5). Moreover, it has been shown that whenever a certain limiting procedure (closely resembling that through which the usual spontaneous magnetization is defined) is adopted, one can determine the magnetizations m_i even when $\{h\}$ lies on the boundary of $\mathfrak{U}_e$. The importance of this point stems from the fact that in the phase separation problem the case when $\{h\}\in\partial\mathfrak{U}_e$, more specifically when h_1 and h_M have opposite directions, while the other magnetic fields are zero, has to be considered.

APPENDIX A.

Let $\{\Phi_i : R\to R;\ i = 1,\dots,M\}$ be a set of functions, such that for all $i = 1,\dots,M$ the following conditions are fulfilled:

(i) $\Phi_i(x) = \Phi_i(-x)$, $\Phi_i(R)\subset R_+$, $\Phi_i\in\mathcal{C}^3(R)$;

(ii) $\Phi_i'(x)>0$ for $x>0$ and $\lim_{x\to\infty}\Phi_i'(x) = 1$;

(iii) $\Phi_i''(x)>0$ for every $x\in R$;

(iv) $\Phi_i'''(x)<0$ for $x>0$.

Let us now define $\mathcal{F}_\Phi : R^M \to R$ by:

$$\mathcal{F}_\Phi(x) = \frac{1}{2}(Jx, x) - \beta^{-1}\sum_{i=1}^{M}\Phi_i((Jx+h)_i) , \tag{A.1}$$

where J is an $M\times M$ strictly positive definite matrix, positive with respect to the componentwise order in R^M (i.e., with positive entries) and irreducible, while $h\in R^M$, $h\ne 0$ and $h\ge 0$. Then:

Lemma. The absolute minimum of $\mathcal{F}_\Phi$ on R^M is attained at one and only one point $x^{(\Phi)}$ satisfying $x^{(\Phi)}>0$. Moreover, if $\{\tilde\Phi_i : i=1,\dots,M\}$ is another set of functions satisfying the conditions (i)-(iv) and $\Phi_i'\ge\tilde\Phi_i'$, $i=1,\dots,M$, then $x^{(\Phi)} > x^{(\tilde\Phi)}$.

Proof. Let $\phi : R^M \to R^M$ be defined by:

$$\phi_i(x) = \Phi_i'(\beta(Jx+h)_i), \quad i = 1,\dots,M. \tag{A.2}$$

Now, since J is strictly positive definite and Φ_i have linear behaviour at infinity (see (ii)), $\mathcal{F}_\Phi$ attains its absolute minimum at least at one point, which should be among the solutions of the system: $\nabla_x\mathcal{F}_\Phi = Jx - J^T\phi(x)=0$. But $J = J^T$, and J^{-1} exists, hence the system can be brought to the form:

$$x = \phi(x) \tag{A.3}$$

showing that the stationary points of $\mathcal{F}_\Phi$ are fixed points of ϕ. The following properties of ϕ will be needed:

a) ϕ is increasing on R^M (with respect to the order introduced above); its fixed points x satisfy $|x_i|<1$.

b) If $x\le\phi(x)$ (or $x\ge\phi(x)$) the sequence $\phi^{\circ n}$ converges when $n\to\infty$ to a fixed point of ϕ.

c) There exists n such that $\phi^{\circ n}(\{x: x\ge 0,\ x\ne 0\})\subset\{x : x>0\}$. In particular, if $x>0, x\ne 0$ is a fixed point of ϕ, $x>0$.

d) ϕ has one and only one fixed point in the set $\{x: x>0, x\ne 0\}$, namely $\xi = \lim_{n\to\infty}\phi^{\circ n}(0)$.

Properties a),b),c) follow easily by inspection. For d), account for Φ_i' being strictly concave on $x\ge 0$ to arrive at:

$$\phi_i(\lambda x + (1-\lambda)y) \ge \lambda\phi_i(x) + (1-\lambda)\phi_i(y), \quad i = 1,\dots,M \tag{A.4}$$

with at least one strict inequality; here $x, y>0$, $x\ne y$, and $\lambda\in(0,1)$. If x, y are moreover fixed points of ϕ, we know by c) that $x, y>0$. As $x\ne y$, one can find $\lambda_0\notin[0,1]$ and an index

$i_0 \in \{1,\dots,M\}$ such that

$$\underset{\sim}{z} = \lambda_0 \underset{\sim}{x} + (1-\lambda_0)\underset{\sim}{y} \geq 0 \text{ and } z_{i_0} = \lambda_0 x_{i_0} + (1-\lambda_0) y_{i_0} = 0.$$

With (A.4) we shall have $\underset{\sim}{\phi}(z) \leq z$ and $\underset{\sim}{\phi}(z) \neq z$. Property c) enables us to find n such that $\tilde{\phi}^{on}(\tilde{z}) > 0$, while a) leads to $z \geq \geq \phi^{on}(z) > 0$, which contradicts $z_{i_0} = 0$.

We are now prepared to prove the lemma. We begin by noting that if $\underset{\sim}{y}$ is an arbitrary fixed point of $\underset{\sim}{\phi}$, then there exists a fixed point $\xi > 0$ such that $|y_i| \leq \xi_i$, $1 < i < M$. Indeed, let y^* be a vector of components $y_i^* = |y_i|$. Then one has $0 \leq \underset{\sim}{y}^* \leq \underset{\sim}{\phi}(\underset{\sim}{y}^*)$ (account that $\underset{\sim}{h} \geq 0$). Hence by b), $\phi^{on}(y^*) \to \xi$ monotonously increasing, $\underset{\sim}{\xi}$ being a fixed point of $\underset{\sim}{\phi}$; thus $\tilde{\phi}^{on}(y^*) < \underset{\sim}{\xi}$. Further let y be an arbitrary stationary point of $\mathcal{F}_{\Phi}$. Then:

$$\mathcal{F}_{\Phi}(\underset{\sim}{y}) = \beta^{-1} \sum_{i=1}^{M} G_i(y_i). \qquad \text{(A.5)}$$

where

$$G_i(y) = (y/2)\,\Phi_i'^{-1}(y) - \Phi_i \circ \Phi_i'^{-1}(y) - \frac{\beta}{2} h_i\, y, \qquad \text{(A.6)}$$

which is strictly decreasing for $y > 0$ and has the property $G_i(y) \geq G_i(|y|)$. Using now (A.5) we have $\mathcal{F}_{\Phi}(\underset{\sim}{y}) > \mathcal{F}_{\Phi}(\underset{\sim}{y}^*) > \mathcal{F}_{\Phi}(\underset{\sim}{\xi})$, where $\underset{\sim}{\xi} = \lim_{n\to\infty} \underset{\sim}{\phi}^{on}(\underset{\sim}{y}^*)$. Thus, the absolute minimum of $\mathcal{F}_{\Phi}$ is attained at the only positive fixed point of $\underset{\sim}{\phi}$, $\underset{\sim}{\xi}$.

REFERENCES

1. Gallavotti G. Commun. Math.Phys., 1972, 27, p.103.
2. Aizenman M. Commun.Math.Phys., 1980, 73, p.83.
3. Dobrushin R.L. Theory Probability Appl., 1972, 17, p.582.
4. van Beijeren H. Commun.Math.Phys., 1975, 40, p.1.
5. Abraham D.B., Reed P., Commun.Math.Phys., 1976, 49, p.35.
6. Burton J.A. et al. Phil.Trans.Roy.Soc.Ser.A, 1951, 243, p.299.
7. Weeks J.D. et al. Phys.Rev.Lett., 1973, 31, p.549.
8. Frohlich J., Spencer F. Commun.Math.Phys., 1981, 81, p.527.
9. Abraham D.B. Phys.Rev.Lett., 1980, 44, p.1165.
10. Kittel C. Introduction to Solid State Physics, New York, Wiley, 1971.
11. Kac M., Helfand E. J.Math.Phys., 1963, 4, p.1078.
12. Fisher M.E. et al. Phys.Rev.A, 1973, 8, p.1111.
13. Angelescu N. et al. J.Phys.A: Math.Gen., 1981, 14, p.L533.

14. Thompson C.J., Silver H. Commun.Math.Phys., 1973, 33, p.53.
15. Griffiths R.B. J.Math.Phys., 1964, 5, p.1215.
16. Angelescu N. et al. J.Phys.A: Math.Gen., 1979, 12, p.2457.
17. Lieb E.H., Ruskai M.B. Adv.Math., 1974, 12, p.269.

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Ангелеску Н., Бундару М., Костаке Г. E17-83-10

О разделении фаз в системах с непрерывной симметрией.
Изотропная D-векторная модель с взаимодействиями Каца-Гельфанда

Исследуется граница раздела фаз в изотропных D-векторных моделях с взаимодействием Каца-Гельфанда при конечных D и в пределе D→∞. Детально изучается проблема минимизации, возникающая в решении модели с граничными условиями, обеспечивающими разделение фаз.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Angelescu N., Bundaru M., Costache G. E17-83-10

On Phase Separation in Systems with Continuous Symmetry.
The Isotropic D-Vector Model with Kac-Helfand Interactions

The interface in the isotropic D-vector model and its D→∞ limit, both with Kac-Helfand interactions, is studied. Besides the general presentation of the problem and results, the first part contains a detailed study of the minimum problem one is faced with when solving the model under the boundary conditions needed to study the phase separation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983