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E17-82-875

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EXACT AND APPROXIMATE GENERALIZED DIFFUSION EQUATION FOR THE LORENTZ GAS

Submitted to "Physica"

1982

1. INTRODUCTION

The models describing irreversible processes help to clarify many qualitative fundamental problems (cf Foch and Ford [1]). In particular, the Lorentz model has been used for reconciliation of the microscopic reversibility with the macroscopic irreversibility (Hauge [2]) and for studying the effects of high density [3,4]. Our paper is devoted to a much more modest end. We shall study the different methods of solving this particular model and compare the obtained results with the exact solution given by Hauge [2,3].

We begin our account in sect. 2 by presenting the exact Hauge results. In sect. 3 we show that the Zwanzig-Mori projection method gives the exact equation for the local density of particles $n(\vec{r}, t)$. Section 5 is devoted to the method proposed by Zubarev and Khonkin [5] for solving the Boltzmann equation. The results obtained in this way correspond to a series expansion of the exact solution. The Appendix describes the familiar Chapman-Enskog method (cf. [6]). This approach yields series expansion of the solution obtained in an elegant way by Hauge [2].

2. EXACT SOLUTION OF THE DETERMINISTIC LORENTZ MODEL

In this model a number of stationary spherical scatterers are distributed randomly over a three-dimensional volume under the restriction that they may not overlap each other. A number of light, mutually noninteracting point particles, having velocities of constant magnitude, are reflected specularly when hitting a scatterer [4]. We shall confine ourselves to the case of small density of scatterers, n_s, and to a rarefied gas of light particles [2,3]. Then, for infinite space with no external forces the one-particle distribution function $f(\vec{r}, \vec{v}, t)$ obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v}\vec{\nabla} f = Bf, \qquad (1)$$

where $\vec{v}$ is the particle velocity. In the case of spherical scatterers of radius a the collision operator B has a simple form

$$B = -\frac{1}{\tau}(1 - \mathcal{P}_0), \qquad (2)$$

where the relaxation time τ is related to the quantities describing the scattering in the following way

$$\tau = (n_s v a^2 \pi)^{-1} .$$

The operator $\mathcal{P}_0$, responsible for averaging over all directions of velocity, e.g.,

$$\mathcal{P}_0 f(\vec{r}, \vec{v}, t) \equiv \frac{1}{4\pi} \int d\Omega_{v'} f(\vec{r}, \vec{v}', t) \equiv \frac{1}{4\pi} n(\vec{r}, t) \qquad (3)$$

defines the only hydrodynamical moment of the distribution function, namely the local density of particles $n(\vec{r}, t)$. Clearly, in view of Eq. (2), the Lorentz model can describe only the relaxation of the angular distribution of particle velocities toward an isotropic distribution, so it does not describe the relaxation toward the complete equilibrium. A natural field of application of the model is the description of motion of neutrons in matter [7].

The operator $\mathcal{P}_0$ is idempotent

$$\mathcal{P}_0^2 = \mathcal{P}_0 . \qquad (4)$$

An arbitrary angle-independent function is the collision integral, e.g.,

$$B\mathcal{P}_0 f(\vec{r}, \vec{v}, t) = 0 .$$

By applying the Fourier-Laplace transform

$$\Phi(\vec{k}, \vec{v}, z) = \int_0^\infty dt\, e^{-zt} \int d^3r\, e^{i\vec{k}\vec{r}} f(\vec{r}, \vec{v}, t)$$

one obtains the following form of the Boltzmann equation

$$(z - i\vec{k}\vec{v})\Phi(\vec{k}, \vec{v}, z) = B\Phi(\vec{k}, \vec{v}, z) + h(\vec{k}, \vec{v}) , \qquad (5a)$$

$h(\vec{k}, \vec{v})$ being the Fourier transform of the initial distribution function. Hauge [2] has found the exact solution of Eq. (5a)

$$\Phi(\vec{k}, \vec{v}, z) = \tau^{-1} \mathcal{R}(\vec{k}, \vec{v}, z)\left[1 - \frac{1}{k v\tau} \arctan\left(\frac{kv\tau}{1 + \tau z}\right)\right] \mathcal{P}_0 (\mathcal{R}(\vec{k}, \vec{v}, z) h(\vec{k}, \vec{v})) + \mathcal{R}(\vec{k}, \vec{v}, z) h(\vec{k}, \vec{v}) , \qquad (6)$$

where $\mathcal{R}(\vec{k}, \vec{v}, z)$ is the resolvent

$$\mathcal{R}(\vec{k}, \vec{v}, z) = (z + \tau^{-1} - i\vec{k}\vec{v})^{-1} .$$

Thus, $\Phi(\vec{k}, \vec{v}, z)$ has a simple pole at $z = ikv - \frac{1}{\tau}$, a cut from $z = -\frac{1}{\tau} - ikv$ to $z = -\frac{1}{\tau} + ikv$ and, for $kv\tau \leq \frac{\pi}{2}$, a simple pole at $z = \frac{1}{\tau}[kv\tau \cot(kv\tau) - 1]$. The first pole reflects one-particle "chaotic" motion, and corresponds to the continuum of eigenvalues

$$\Lambda_\mu = -\frac{1}{\tau}(-1 + ikv\tau\mu), \qquad -1 \leq \mu \leq 1 .$$

The second one, hydrodynamic pole, corresponds to a discrete eigenvalue

$$\Lambda(k) = \frac{1}{\tau}[kv\tau \cot(kv\tau) - 1], \qquad 0 \leq kv\tau \leq \frac{\pi}{2}$$

obtained for the Fourier transform of the Boltzmann equation (1) by Zweifel and Case [7]. For small k, i.e., $kv\tau \ll 1$, $\Lambda(k)$ defines the diffusion constant

$$\Lambda(k) \approx -k^2 D, \qquad D = \frac{1}{3} v^2 \tau .$$

The long-time asymptotics of $n(\vec{k}, t)$ is dominated by the hydrodynamic pole

$$n(k, t) \sim e^{\Lambda(k)t} .$$

This means that after laps of time longer than τ the density of particles obeys the equation

$$\frac{\partial n}{\partial t} + \sum_{p=1}^{\infty} \frac{B_p}{(2p)!} 2^{2p} v^{2p} \tau^{2p-1} (-\nabla^2)^p\, n(\vec{r}, t) = 0 \quad (\lambda \ll L,\ \tau \ll t) , \qquad (7)$$

where $\lambda = v\tau$, $L = (k/2\pi)^{-1}$, B_p are the Bernoulli numbers: $B_1 = 1/6$, $B_2 = 1/30$, etc. The Fourier transform of this equation defines the k-dependent diffusion coefficient $D_{CE}(k)$. It is seen from Eq. (6) that for $\ell > 0$ the ℓ-th moment of the distribution function

$$\int d\Omega_v\, \phi(\vec{k}, \vec{v}, z) P_\ell(\cos\theta), \quad \cos\theta = \frac{\vec{k}\vec{v}}{kv}$$

does not exhibit the pole corresponding to one-particle motion. It can be shown that the contribution of the above-mentioned cut to $f(\vec{k}, \vec{v}, t)$ represents this kind of motion.

3. DYNAMIĆ EQUATION FOR THE LOCAL DENSITY OF PARTICLES

As the considered system is isotropic, the distribution functions depend on one angle θ only. For any function of θ one can write

$$\mathcal{P}_0 \, A(\theta) = P_0 \, (P_0 , P_0)^{-1} \, (P_0 \, , A),$$

where $P_0(\mu)$ is the zeroth Legendre polynomial. We introduced also the scalar product

$$(A, B) = \int_{-1}^{1} d\mu \, A^*(\mu) \, B(\mu) \, ,$$

where

$$\mu = \cos\theta .$$

Introducing the operator $\mathcal{Q}_0 = 1 - \mathcal{P}_0$, which obeys the obvious relations

$$\mathcal{Q}_0 + \mathcal{P}_0 = 1 \, , \quad \mathcal{Q}_0 \mathcal{P}_0 = \mathcal{P}_0 \mathcal{Q}_0 = 0 \, , \quad \mathcal{Q}_0^2 = \mathcal{Q}_0 \, ,$$

we can rewrite the Boltzmann equation (5a) in the following form:

$$(z - ikv\mu) \, \Phi(\vec{k}, \vec{v}, z) = -\frac{1}{\tau} \, \mathcal{Q}_0 \Phi(\vec{k}, \vec{v}, z) + h(\vec{k}, \vec{v}) \, . \tag{5b}$$

Let us introduce the projection operator onto ℓ-th Legendre polynomial $P_\ell(\mu)$

$$\mathcal{P}_\ell \, A(\mu) = P_\ell(\mu) \, (P_\ell \, , P_\ell \,)^{-1} \, (P_\ell \, , A) \, . \tag{8}$$

This means that the multiplicity of degeneration of the spectrum of collision operator **B** is infinite. One of the eigenvalues is equal to zero and corresponds to the eigenfunction P_0. The other eigenvalues are equal to $-\frac{1}{\tau}$ and correspond to P_1, P_2, $P_3 \ldots$.
The Legendre polynomials form the orthogonal set

$$(P_\ell \, , P_m \,) = \frac{2}{2\ell + 1} \, \delta_{\ell, m} \, . \tag{9}$$

Our goal is to derive the equation for $\mathcal{P}_0 \Phi$ using the Zwanzig-Mori projection technique [8,9]. Projecting both sides of Eq. (5b) onto $P_0(\mu)$ and eliminating $\mathcal{Q}_0 \Phi$ we obtain an exact equation for $(P_0 \, , \Phi)$

$$[\, z + \frac{(kv)^2}{(P_0 , P_0)} (P_1 \, , \mathcal{R}_{\mathcal{Q}_0}(k, v, z) P_1)] \, (P_0 \, , \Phi(k, v, z)) = \tag{10}$$

$$= ikv(P_1 \, , \mathcal{R}_{\mathcal{Q}}(k, v, z) \, \mathcal{Q}_0 \, h(k, v)) + (P_0 \, , h(k, v)) \, .$$

This equation contains the memory kernel $(P_1 , \mathcal{R}_{\mathcal{Q}_0}(k, v, z) P_1)$ which is essentially the Fourier-Laplace transform of the current-current correlation function. The resolvent, $\mathcal{R}_{\mathcal{Q}_0}$, is defined as

$$\mathcal{R}_{\mathcal{Q}_0}(k, v, \mu ; t) = [\, z + \frac{1}{\tau} - ikv \, \mathcal{Q}_0 P_1(\mu) \mathcal{Q}_0 \,]^{-1} \, . \tag{11}$$

Our task is thus reduced to the calculation of the memory kernel $(P_1 , \mathcal{R}_{\mathcal{Q}_0} P_1)$. For models more realistic than the Lorentz gas model it is a nontrivial task [10,11]. For the collision integrals with a gap in spectrum, one can find the asymptotic behaviour of the memory kernel for small **k**.
The standard manipulations (cf. [12]), together with the identity relation

$$\mu P_n(\mu) = \frac{n+1}{2n+1} P_{n+1}(\mu) + \frac{n}{2n+1} P_{n-1}(\mu) \, , \tag{12}$$

yield the following expression:

$$(P_1 \, , \mathcal{R}_{\mathcal{Q}_0} P_1 \,) = \frac{(P_1 \, , P_1)}{z + \tau^{-1} + (\frac{2}{3} kv)^2 (P_1 , P_1)^{-1} \, (P_2 \, , \mathcal{R}_{\mathcal{Q}_0 \mathcal{Q}_1} P_2 \,)} \, , \tag{13}$$

where

$$\mathcal{R}_{\mathcal{Q}_0 \mathcal{Q}_1}(k, v, \mu, t) = [\, z + \tau^{-1} - ikv \mathcal{Q}_1 \mathcal{Q}_0 \, P_1(\mu) \mathcal{Q}_0 \mathcal{Q}_1]^{-1} .$$

and

$$\mathcal{Q}_1 = 1 - \mathcal{P}_1 .$$

Such a simple relation connecting the diagonal matrix elements of consequtive resolvents suggests the existence of a general relation between $(P_n \, , \mathcal{R}_{\mathcal{Q}_0 \mathcal{Q}_1 \ldots \mathcal{Q}_{n-1}} P_n)$ and $(P_{n+1} \, , \mathcal{R}_{\mathcal{Q}_0 \mathcal{Q}_1 \ldots \mathcal{Q}_n} P_{n+1})$. Here, we have introduced the operators $\mathcal{Q}_i = 1 - \mathcal{P}_i$ $(i = 1,2,\ldots,n)$. Indeed, acting with P_n on both sides of the identity:

$$[z + \tau^{-1} - ikv\mathcal{Q}_{n-1}\ldots\mathcal{Q}_0 P_1(\mu)\mathcal{Q}_0\ldots\mathcal{Q}_{n-1}]\,\mathcal{R}_{\mathcal{Q}_0\ldots\mathcal{Q}_{n-1}} = 1\,,$$

excluding the off-diagonal part $\mathcal{Q}_n\mathcal{R}_{\mathcal{Q}_0\ldots\mathcal{Q}_{n-1}}\mathcal{P}_n$, and using Eq.(12), we obtain the desired relation

$$(P_n, \mathcal{R}_{\mathcal{Q}_0\ldots\mathcal{Q}_{n-1}} P_n) = \frac{(P_n, P_n)}{z + \tau^{-1} + \dfrac{(kv)^2}{(P_n,P_n)}\left(\dfrac{n+1}{2n+1}\right)^2 (P_{n+1}, \mathcal{R}_{\mathcal{Q}_0\ldots\mathcal{Q}_n} P_{n+1})}, \qquad (n = 1,2,\ldots). \tag{14}$$

It defines a set of linear transformations

$$t_n(w) = \frac{(P_n,P_n)}{(z + \tau^{-1}) + \dfrac{(kv)^2}{(P_n,P_n)}\left(\dfrac{n+1}{2n+1}\right)^2 w} \qquad (n = 1,2,3,\ldots) \tag{15}$$

generating the continued fraction [13] which in the standard notation[14] is given by

$$(P_1, \mathcal{R}_{\mathcal{Q}_0} P_1) = \frac{(P_1,P_1)}{(z+\tau^{-1})+}\;\frac{\left(\frac{2}{3}\right)^2\frac{(P_2,P_2)}{(P_1,P_1)}(kv)^2}{(z+\tau^{-1})+}\;\frac{\left(\frac{3}{5}\right)^2\frac{(P_3,P_3)}{(P_2,P_2)}(kv)^2}{(z+\tau^{-1})+}\cdots =$$

$$= \frac{\frac{(P_1,P_1)}{z+\tau^{-1}}}{1+}\;\frac{\left(\frac{2}{3}\right)^2\frac{(P_2,P_2)}{(P_1,P_1)}\left(\frac{kv\tau}{1+\tau z}\right)^2}{1+}\;\frac{\left(\frac{3}{5}\right)^2\frac{(P_3,P_3)}{(P_2,P_2)}\left(\frac{kv\tau}{1+z\tau}\right)^2}{1+}\cdots\frac{\left(\frac{n+1}{2n+1}\right)^2\frac{(P_{n+1},P_{n+1})}{(P_n,P_n)}\left(\frac{kv\tau}{1+\tau z}\right)^2}{1+}\cdots \tag{16}$$

It is possible to relate the continued fraction (16) for $(P_1, \mathcal{R}_{\mathcal{Q}_0} P_1)$ to the continued fraction $\arctan\left(\frac{kv\tau}{1+\tau z}\right)$[14]. One can easily check that in the latter case the continued fraction can be written in the following nice form:

$$\arctan x = \frac{x}{1+}\;\frac{\left(\frac{0+1}{2\cdot 0+1}\right)^2\frac{(P_1,P_1)}{(P_0,P_0)}x^2}{1+}\;\frac{\left(\frac{1+1}{2\cdot 1+1}\right)^2\frac{(P_2,P_2)}{(P_1,P_1)}x^2}{1+}\cdots\frac{\left(\frac{n+1}{2n+1}\right)^2\frac{(P_{n+1},P_{n+1})}{(P_n,P_n)}x^2}{1+}\cdots \tag{17}$$

This expansion is valid exterior to the cut along imaginary axis from i to $+i\infty$, and from $-i$ to $-i\infty$, or equivalently from $-i$ to i [13]. Using (17) we obtain an exact relation

$$(P_1, \mathcal{R}_{\mathcal{Q}_0} P_1) = \left[\frac{\left(\frac{kv\tau}{1+z\tau}\right)}{\arctan\left(\frac{kv\tau}{1+z\tau}\right)} - 1\right]\frac{(z+\tau^{-1})}{(kv)^2}(P_0,P_0)\,. \tag{18}$$

Taking into account Eq. (18) the dynamic equation (10) for $\mathcal{P}_0\Phi$ reads

$$\left[1 - \frac{1}{kv\tau}\arctan\left(\frac{kv\tau}{1+\tau z}\right)\right](P_0,\Phi) = i\arctan\left(\frac{kv\tau}{1+\tau z}\right)(P_1, \mathcal{R}_{\mathcal{Q}_0}\mathcal{Q}_0 h) + (kv)^{-1}\arctan\left(\frac{kv\tau}{1+\tau z}\right)(P_0, h)\,. \tag{19}$$

Below we prove the equivalence of formula (19) and the equation

$$\left[1 - \frac{1}{kv\tau}\arctan\left(\frac{kv\tau}{1+\tau z}\right)\right](P_0,\Phi) = \frac{1}{kv}\arctan\left(\frac{kv\tau}{1+\tau z}\right)(P_0,h) + (P_0, \mathcal{R}\,\mathcal{Q}_0 h)\,, \tag{20a}$$

which follows from the solution (6) obtained by Hauge [2]. The well known operator identity

$$(A+B)^{-1} = A^{-1} - (A+B)^{-1}BA^{-1}$$

yields the following identity for the resolvents $\mathcal{R}$, $\mathcal{R}_{\mathcal{Q}_0}$ and $P_1(\mu)$

$$\mathcal{R} = \mathcal{R}_{\mathcal{Q}_0} + ikv\mathcal{R}\mathcal{Q}_0 P_1\mathcal{P}_0\mathcal{R}_{\mathcal{Q}_0} + ikv\mathcal{R}\mathcal{P}_0 P_1\mathcal{Q}_0\mathcal{R}_{\mathcal{Q}_0}\,.$$

Since, as it can be easily shown, the matrix elements $(P_0, \mathcal{R}_{\mathcal{Q}_0}\mathcal{Q}_0 h)$ and $(P_0, \mathcal{R}\mathcal{Q}_0 P_1\mathcal{P}_0\mathcal{R}_{\mathcal{Q}_0}\mathcal{Q}_0 h)$ vanish, we obtain the identity

$$(P_0, \mathcal{R}\mathcal{Q}_0 h) = ikv(P_0, \mathcal{R}P_0)(P_0,P_0)^{-1}(P_1, \mathcal{R}_{\mathcal{Q}_0}\mathcal{Q}_0 h) = i\arctan\left(\frac{kv\tau}{1+\tau z}\right)(P_1, \mathcal{R}_{\mathcal{Q}_0}\mathcal{Q}_0 h)\,,$$

where we use the fact [2] that

$$\mathcal{P}_0\, \mathcal{R}(k, v, \mu, t) = (kv)^{-1} \arctan\left(\frac{kv\tau}{1+\tau z}\right).$$

The proof of equivalency of Eqs. (19) and (20) is thus complete.

The initial distribution function $h(k, v, \mu)$ can also be developed into a series of Legendre polynomials. The general method of calculating the nondiagonal matrix elements of the resolvent $(P_1, \mathcal{R}_{\mathcal{Q}_0}, P_\ell)(\ell=1,2,\dots)$ [16] shows that in the denominators they do not contain terms which would cancel the hydrodynamic pole. This means that the long time asymptotics of $(P_0, f(k,v,t))$ is dominated by the hydrodynamic eigenvalue Λ.

4. GENERALIZED DIFFUSION COEFFICIENT

Using Eqs. (18) and (10) we can write the hydrodynamic equation (19) in the form resembling the diffusion equation

$$[\zeta+\kappa^2 D(\kappa,\zeta)](P_0, \Phi'(\kappa,\zeta)) = i\tau\kappa(P_1, \mathcal{R}'_{\mathcal{Q}_0}(\kappa,\zeta)\mathcal{Q}_0 h'(\kappa)) + (P_0, h'(\kappa))\tau, \qquad (20b)$$

where

$$\mathcal{R}'_{\mathcal{Q}_0}(\kappa,\zeta,\mu) = (\zeta+1-i\kappa\mathcal{Q}_0 P_1(\mu)\mathcal{Q}_0)^{-1},\ \Phi'(\kappa,\zeta,\mu,t) \equiv \phi(\vec{k}, \vec{v}, t),\ h'(\kappa,\mu) = h(\vec{k}, \vec{v}).$$

We see that the generalized diffusion coefficient depends on two variables $\kappa = kv\tau$ and $\zeta = \tau z$

$$D(\kappa,\zeta) = \frac{1+\zeta}{\kappa^2}\left[\frac{\frac{\kappa}{1+\zeta}}{\arctan\left(\frac{\kappa}{1+\zeta}\right)} - 1\right]. \qquad (21)$$

As we henceforth work with quantities dependent on κ and ζ we will suppress primes for ease of notation.

The Chapman-Enskog solution given by Hauge yields a different κ-dependent diffusion coefficient $D_{CE}(\kappa)$. In order to obtain this coefficient one should assume the relation connecting ζ and κ

$$\zeta = [\kappa\cot\kappa - 1]. \qquad (22)$$

This gives

$$D_{CE}(\kappa) = \frac{1}{\kappa^2}(1-\kappa\cot\kappa).$$

For small κ one has

$$\kappa^2 D_{CE}(\kappa) = \frac{1}{3}\kappa^2 + \frac{1}{45}\kappa^4 + \frac{9}{945}\kappa^6 + \dots, \qquad (23)$$

which is in agreement with the generalized hydrodynamic equation (7).

For small κ and $\zeta \sim 1$ one can expand the generalized diffusion coefficient in terms of $\kappa/(1+\zeta)$

$$D(\kappa,\zeta) = \frac{1}{3}\,\frac{1}{1+\zeta} - \left(\frac{1}{5}-\frac{1}{9}\right)\frac{\kappa^2}{(1+\zeta)^3} + \left(\frac{1}{7}-\frac{2}{15}+\frac{1}{27}\right)\frac{\kappa^4}{(1+\zeta)^5} + \dots. \qquad (24)$$

For small ζ and κ relation (22) corresponds to double limit $\lim_{\kappa\to 0}\lim_{\zeta\to 0}$. We have

$$\lim_{\kappa\to 0} D_{CE}(\kappa) = \frac{1}{3}.$$

Moreover

$$\lim_{\kappa\to 0}\lim_{\zeta\to 0} D(\kappa,\zeta) = \lim_{\zeta\to 0}\lim_{\kappa\to 0} D(\kappa,\zeta) = \frac{1}{3}.$$

quite differently than in general case [16], where the order of limits is important.

5. THE ITERATIVE METHOD OF SOLUTION

Now we shall find the solution of the Boltzmann equation under a different boundary condition. We shall follow the approach formulated by Zubarev and Khonkin [5] (cf also [17]). Subtracting $\frac{\partial}{\partial t}(\mathcal{P}_0 f)$ from both sides of Eq. (1), we get

$$\frac{\partial}{\partial t}(\mathcal{Q}_0 f) - B\,\mathcal{Q}_0 f = -\frac{\partial(\mathcal{P}_0 f)}{\partial t} - \vec{v}\,\vec{\nabla} f.$$

The solution of this equation with the boundary condition

$$\mathcal{Q}_0 f = 0 \quad \text{for} \quad t \to -\infty$$

is

$$\mathcal{Q}_0 f(\vec{r}, \vec{v}, t) = -\int_{-\infty}^{t} dt_1\, e^{-(t-t_1)/\tau}\ \mathcal{Q}_0(\vec{v}\vec{\nabla} f(\vec{r}, \vec{v}, t_1)). \qquad (25a)$$

This means that

$$f(\vec{r},\vec{v},t) = \mathcal{P}_0\, f(\vec{r},\vec{v},t) - \int_{-\infty}^{t} dt_1\, e^{-(t-t_1)/\tau}\, \mathcal{Q}_0(\vec{v}\vec{\nabla} f(\vec{r},\vec{v},t_1)) . \tag{25b}$$

Obviously, our solution fulfils two important conditions, well known from the Chapman-Enskog theory (cf. Appendix)

$$n(\vec{r},t) \equiv 4\pi \mathcal{P}_0\, f(\vec{r},\vec{v},t) = 4\pi \mathcal{P}_0\, (\mathcal{P}_0\, f(\vec{r},\vec{v},t)),$$

$$j_a(\vec{r},t) \equiv 4\pi \mathcal{P}_0 v_a f(\vec{r},\vec{v},t) = 4\pi \mathcal{P}_0\, (v_a\, \mathcal{Q}_0\, f(\vec{r},\vec{v},t)).$$

Since,

$$\mathcal{Q}_0 (\vec{v}\vec{\nabla}) f = \mathcal{Q}_0 (\vec{v}\vec{\nabla})(\mathcal{P}_0 f + \mathcal{Q}_0 f)$$

both formulas (25a,b) are well suited for the iterative method of solution. In this way, we get the solution in the form of a series

$$\delta f \equiv \mathcal{Q}_0 f = \delta f^{(1)} + \delta f^{(2)} + \dots ,$$

where

$$\delta f^{(1)}(\vec{r},\vec{v},t) = -\int_{-\infty}^{t} dt_1\, e^{-(t-t_1)/\tau}\, \mathcal{Q}_0 \vec{v}\vec{\nabla}\mathcal{P}_0\, f(\vec{r},\vec{v},t_1) ,$$

$$\delta f^{(2)}(\vec{r},\vec{v},t) = \int_{-\infty}^{t} dt_1\, e^{-(t-t_1)/\tau} \times$$

$$\times \int_{-\infty}^{t_1} dt_2\, e^{-(t_1-t_2)/\tau}\, \mathcal{Q}_0(\vec{v}\vec{\nabla})\mathcal{Q}_0(\vec{v}\vec{\nabla})\, \mathcal{P}_0\, f(\vec{r},\vec{v},t_2), \dots .$$

The other terms have the same multiple integral structure. Using this series we can find the divergence of the particle current density

$$\nabla_a j_a(\vec{r},t) = -\int_{-\infty}^{t} dt_1\, e^{-(t-t_1)/\tau}\, \mathcal{P}_0 (\vec{v}\vec{\nabla})\mathcal{Q}_0(\vec{v}\vec{\nabla}) n(\vec{r},t_1) + \int_{-\infty}^{t} dt_1 e^{-(t-t_1)/\tau} \times$$

$$\times \int_{-\infty}^{t_1} dt_2 e^{-(t_1-t_2)/\tau} \int_{-\infty}^{t_2} dt_3 e^{-(t_2-t_3)/\tau}\, \mathcal{P}_0(\vec{v}\vec{\nabla})\mathcal{Q}_0(\vec{v}\vec{\nabla})\mathcal{Q}_0(\vec{v}\vec{\nabla})\mathcal{Q}_0(\vec{v}\vec{\nabla}) n(\vec{r},t_3) + \dots . \tag{26}$$

The Fourier-Laplace transform of the divergence of the current density gives the familiar series (24) together with the unhomogeneous term of Eq. (20b)

$$(P_1, \mathcal{R}_{\mathcal{Q}_0}(\kappa,\zeta)\, \mathcal{Q}_0 h) = \frac{1}{2\pi i (1+\zeta)\kappa} \sum_{n=0}^{\infty} \int_{-\infty}^{0} dt_1 e^{t_1/\tau} \int_{-\infty}^{t_1} dt_2 e^{-(t_1-t_2)/\tau} \dots$$

$$\times \int_{-\infty}^{t_{2n}} dt_{2n+1} e^{-(t_{2n}-t_{2n+1})/\tau} \int d^3r\, e^{i\vec{k}\vec{r}}\, \mathcal{P}_0 (\vec{v}\vec{\nabla}) (\mathcal{Q}_0 \vec{v}\vec{\nabla}\mathcal{Q}_0)^{2n+1} n(\vec{r}, t_{2n+1}) .$$

In order to find the coefficients of the Fourier-Laplace transform of the series in Eq. (26) one should calculate the angular averages $\mathcal{P}_0 (\hat{v}_a \mathcal{Q}_0 \hat{v}_{\mu_1} \mathcal{Q}_0 \hat{v}_{\mu_2} \mathcal{Q}_0 \dots \mathcal{Q}_0 \hat{v}_{\mu_n})$, where $\hat{v}_a = v_a/|\vec{v}|$. This can be done easily using the general formula for $\mathcal{P}_0(\hat{v}_a \hat{v}_{\mu_1} \dots \hat{v}_{\mu_n})$ given by Fedorov /18/. The same formula is used in the Appendix. Comparison of Eqs. (26) and (A8b,d,e) shows, that for $\zeta \ll 1$ the series generated using the Chapman-Enskog and Zubarev-Khonkin methods have only one common term, leading to the familiar diffusion equation. The same result can be obtained by putting $k=0$ and $z=0$ in the expression for memory kernel $(P_1, \mathcal{R}_{\mathcal{Q}_0} P_1)$.

6. CONCLUSION

We have shown that the Zwanzig-Mori projection method gives an exact formula for $\mathcal{P}_0\, \Phi(\kappa,\zeta,\mu)$. It means that this method can be used for solving other models of kinetics. The advantage of this technique is the possibility of using the well developed approximation schemes, e.g., the mode-mode coupling. The method applied here can be generalized along the "new continued fraction" approach proposed by Mori and collaborators, making it possible to derive the generalized "hydrodynamic" equations for higher moments of Φ /16/. Such equations have been derived for the Lorentz gas in stationary conditions /7/.

On the other hand, the Lorentz gas provides the model of diffusion coefficient exhibiting dispersion. Such models attract attention of many authors (cf. /18/).

The application of the Zubarev-Khonkin method to the Lorentz gas shows that it yields a quite new approximation scheme, so it can hardly be considered as a generalization of the Chapman-Enskog method. The comparison of the results of sect. 5 for $\zeta \ll 1$, with those of the Appendix shows how the Chapman-Enskog Ansatz (A4,A5) yields the additional terms, which renormalize all terms generated by simple perturbation appproach of Zubarev and Khonkin with the exception of the first term. Therefore, it seems interesting to extend the Balabanyan and Khonkin calculations for the real gas /17/ to the next order term in the distribution function.

The author is very indebted to V.B. Priezzhev for many useful discussions, to D.N.Zubarev for very fruitful suggestions, and to A.Guła for critical reading of the manuscript.

APPENDIX. Chapman-Enskog Method

Introducing a small parameter ϵ corresponding to the Knudsen number one can write the Boltzmann equation (1) as /6/

$$\epsilon\left(\frac{\partial f}{\partial t} + \vec v\,\vec\nabla f\right) = Bf. \qquad (A1)$$

Applying $\mathcal{P}_0$ to both sides of Eq. (A1), we obtain

$$\frac{\partial \mathcal{P}_0 f}{\partial t} + \vec\nabla\,\mathcal{P}_0\,\vec v\,\mathcal{Q}_0 f = 0 \qquad (A2)$$

Analogously, for $\mathcal{Q}_0$ one has

$$\epsilon\left(\frac{\partial \mathcal{Q}_0 f}{\partial t} + \vec v\,\vec\nabla\mathcal{P}_0 f + \vec\nabla\,\mathcal{Q}_0\,\vec v\,\mathcal{Q}_0 f\right) = B\mathcal{Q}_0 f. \qquad (A3)$$

We assume that $\mathcal{Q}_0 f$ can be written in the form of a series

$$\mathcal{Q}_0 f = \mathcal{Q}_0 f^{(0)} + \epsilon\,\mathcal{Q}_0 f^{(1)} + \epsilon^2\mathcal{Q}_0 f^{(2)} + \dots \qquad (A4)$$

Suppose additionally that $\frac{\partial \mathcal{P}_0 f}{\partial t}$ depends on time only via the angular average $\mathcal{P}_0 f$ and its space derivatives. Moreover, let us assume that $\frac{\partial \mathcal{P}_0 f}{\partial t}$ can also be developed into a series

$$\frac{\partial \mathcal{P}_0 f}{\partial t} \equiv \tilde\Phi(\vec r\,|\,\mathcal{P}_0 f, \nabla\mathcal{P}_0 f, \dots) = \tilde\Phi^{(0)} + \epsilon\tilde\Phi^{(1)} + \epsilon^2\tilde\Phi^{(2)} + \dots \qquad (A5)$$

Then, from Eq. (A2) we obtain the following expression for $\tilde\Phi^{(\ell)}$

$$\tilde\Phi^{(\ell)} = -v\,\nabla_\alpha(\mathcal{P}_0\hat v_\alpha\mathcal{Q}_0 f^{(\ell)}), \qquad (\ell = 0,1,2,\dots). \qquad (A6)$$

Equation (A3) is equivalent to

$$\sum_{m=0}^{\ell-1}\frac{\partial_m\mathcal{Q}_0 f^{(\ell-m-1)}}{\partial t} + v\vec\nabla\mathcal{Q}_0\hat{\vec v}\mathcal{Q}_0 f^{(\ell-1)} = B\mathcal{Q}_0 f^{(\ell)}, \qquad (A7a)$$

where

$$\frac{\partial_m}{\partial t} = \tilde\Phi^{(m)}\frac{\partial}{\partial(\mathcal{P}_0 f)} + \nabla_\alpha\tilde\Phi^{(m)}\frac{\partial}{\partial(\nabla_\alpha\mathcal{P}_0 f)} + \dots$$

and

$$B\mathcal{Q}_0 f^{(0)} = 0. \qquad (A7b)$$

In the zeroth approximation

$$B\mathcal{Q}_0 f^{(0)} = 0, \qquad \tilde\Phi^{(0)} = 0, \qquad (A8a)$$

hence $\frac{\partial_0}{\partial t} \equiv 0$. The first approximation gives /4/

$$\mathcal{Q}_0 f^{(1)} = -\lambda\mathcal{Q}_0\hat{\vec v}\,\vec\nabla(\mathcal{P}_0 f),$$

$$\tilde\Phi^{(1)} = \frac{\lambda^2}{\tau}\mathcal{P}_0(\hat v_{\alpha_1}\mathcal{Q}_0\hat v_{\alpha_2})\nabla_{\alpha_1}\nabla_{\alpha_2}(\mathcal{P}_0 f) = \frac{B_1}{2!}2^2\frac{\lambda^2}{\tau}\nabla^2\mathcal{P}_0 f = D\nabla^2\mathcal{P}_0 f. \qquad (A8b)$$

In the second step we obtain

$$\tilde\Phi^{(2)} = 0, \qquad \frac{\partial_2}{\partial t} \equiv 0. \qquad (A8c)$$

In this approximation

$$\mathcal{Q}_0 f^{(2)} = \lambda^2\mathcal{Q}_0\hat v_{\alpha_1}\mathcal{Q}_0\hat v_{\alpha_2}\nabla_{\alpha_1}\nabla_{\alpha_2}\mathcal{P}_0 f.$$

The next order expressions become

$$\tilde\Phi^{(3)} = \frac{\lambda^4}{\tau}\{-\mathcal{P}_0(\hat v_{\alpha_1}\hat v_{\alpha_2})\frac{1}{3}\nabla_{\alpha_1}\nabla_{\alpha_2}\Delta(\mathcal{P}_0 f) + \mathcal{P}_0(\hat v_{\alpha_4}\mathcal{Q}_0\hat v_{\alpha_3}\mathcal{Q}_0\hat v_{\alpha_2}\mathcal{Q}_0\hat v_{\alpha_1})\times$$

$$\times\nabla_{\alpha_1}\nabla_{\alpha_2}\nabla_{\alpha_3}\nabla_{\alpha_4}\mathcal{P}_0 f\} = -\frac{1}{45}\frac{\lambda^4}{\tau}\Delta^2\mathcal{P}_0 f = -\frac{B_2}{4!}2^4\frac{\lambda^4}{\tau}(\nabla^2)^2(\mathcal{P}_0 f), \qquad (A8d)$$

$$\mathcal{Q}_0 f^{(3)} = \frac{1}{3}\lambda^3\mathcal{Q}_0\hat v_\alpha\nabla_\alpha\Delta\mathcal{P}_0 f - \lambda^3\mathcal{Q}_0\hat v_{\alpha_1}\mathcal{Q}_0\hat v_{\alpha_2}\mathcal{Q}_0\hat v_{\alpha_3}\nabla_{\alpha_1}\nabla_{\alpha_2}\nabla_{\alpha_3}(\mathcal{P}_0 f).$$

In the fourth approximation

$$\tilde{\Phi}^{(4)} = 0 , \quad \frac{\partial_4}{\partial t} \equiv 0 ,$$

and $\mathcal{Q}_0 f^{(4)}$ contains two terms

$$\mathcal{Q}_0 f^{(4)} = \lambda^4 \{ \mathcal{Q}_0 \hat{v}_{a_1} \mathcal{Q}_0 \hat{v}_{a_2} \mathcal{Q}_0 \hat{v}_{a_3} \mathcal{Q}_0 \hat{v}_{a_4} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} - \frac{2}{3} \mathcal{Q}_0 \hat{v}_{a_1} \mathcal{Q}_0 \hat{v}_{a_2} \nabla_{a_1} \nabla_{a_2} \} \mathcal{P}_0 f .$$

Since the calculations become increasingly complicated we confine ourselves to the fifth step

$$\mathcal{Q}_0 f^{(5)} = -\lambda^5 \mathcal{Q}_0 \hat{v}_{a_1} \mathcal{Q}_0 \hat{v}_{a_2} \mathcal{Q}_0 \hat{v}_{a_3} \mathcal{Q}_0 \hat{v}_{a_4} \mathcal{Q}_0 \hat{v}_{a_5} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \mathcal{P}_0 f +$$

$$+ \lambda^5 \mathcal{Q}_0 \hat{v}_{a_1} \mathcal{Q}_0 \hat{v}_{a_2} \mathcal{Q}_0 \hat{v}_{a_3} \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \Delta \mathcal{P}_0 f - \frac{6}{45} \mathcal{Q}_0 \hat{v}_a \nabla_a \Delta^2 \mathcal{P}_0 f .$$

For $\tilde{\Phi}^{(5)}$ this expression yields

$$\tilde{\Phi}^{(5)} = \frac{\lambda^5}{\tau} \{ \mathcal{P}_0 (v_{a_1} \mathcal{Q}_0 v_{a_2} \mathcal{Q}_0 v_{a_3} \mathcal{Q}_0 v_{a_4} \mathcal{Q}_0 v_{a_5} \mathcal{Q}_0 v_{a_6}) \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \nabla_{a_5} \nabla_{a_6} \mathcal{P}_0 f -$$

$$- \mathcal{P}_0 (\hat{v}_{a_1} \mathcal{Q}_0 \hat{v}_{a_2} \mathcal{Q}_0 \hat{v}_{a_3} \mathcal{Q}_0 \hat{v}_{a_4}) \nabla_{a_1} \nabla_{a_2} \nabla_{a_3} \nabla_{a_4} \Delta \mathcal{P}_0 f +$$

$$+ \frac{6}{45} (\mathcal{P}_0 (\hat{v}_{a_1} \mathcal{Q}_0 \hat{v}_{a_2}) \nabla_{a_1} \nabla_{a_2} \Delta^2 \mathcal{P}_0 f = \frac{B_3}{6!} 2^6 (\nabla^2)^3 \mathcal{P}_0 f . \qquad \text{(A8e)}$$

Collecting all terms (A8a,b,c,d, e) we obtain

$$\frac{\partial n}{\partial t} = \frac{B_1}{2} 2^2 \frac{\lambda^2}{\tau} \nabla^2 n - \frac{B_2}{4!} 2^4 \frac{\lambda^4}{\tau} (\nabla^2)^2 n + \frac{B_3}{6!} 2^6 \frac{\lambda^6}{\tau} (\nabla^2)^3 n + \ldots$$

which is in agreement with the Hauge result (7).

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Received by Publishing Department
on December 16, 1982.

Пашкевич Т. E17-82-875

Точное и приближенное уравнения диффузии для газа Лоренца

Модель Лоренца разреженного газа используется для проверки разных методов решения кинетического уравнения Больцмана. Показано, что проекционный метод Цванцига-Мори дает обобщенное уравнение диффузии, которое совпадает с результатом точного решения Хауге. Метод решения кинетического уравнения, предложенный Зубаревым и Хонькиным, приводит к разложению точного обобщенного коэффициента диффузии в ряд. Этот метод сравнивается с методом Чепмена-Энскога.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1982

Paszkiewicz T. E17-82-875

Exact and Approximate Generalized Diffusion Equation for the Lorentz Gas

The Lorentz model of rarefied gas is used for testing two different methods of solving the Boltzmann kinetic equation. It is shown that the Zwanzig-Mori method gives the generalized diffusion equation which agrees with the exact Hauge solution. The Zubarev-Khonkin approach gives a series expansion of the exact generalized diffusion coefficient. Their method is compared with the Chapman-Enskog method.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1982