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EXACT SOLUTION FOR THE LORENTZ MODEL OF A RAREFIED GAS. DYNAMIC EQUATIONS FOR MOMENTS OF THE DISTRIBUTION FUNCTION

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1. INTRODUCTION

The one-particle distribution function, f , or its Fourier-Laplace transform, Φ , provides the microscopic description of a rarefied gas of particles or quasiparticles. It obeys the Boltzmann equation, which for a class of simple model systems can be solved exactly. One of such models is the Lorentz gas[1,2]. The exact expression for Φ has been given by Hauge[1,2]. The asymptotics of the time dependence of f has been studies in ref.[3].

The crudest macroscopic description of rarefied gas is given in terms of the density of particles, n, or equivalently its Fourier-Laplace transform. These quantities are the angular averages of f or Φ , respectively. For the Lorentz gas it is possible to derive the dynamic equation for the density of particles. This can be done either using the Hauge solution[1,2], or with the help of the Zmanzig-Mori projection technique[4].

For the isotropic three-dimensional Lorentz gas the Fourier-Laplace transform Φ can be developed into a series of Legendre polynomials. The Fourier-Laplace transform of the particle density is the zeroth moment of Φ. Obviously, a more detailed description of the Lorentz gas will involve higher moments of Φ. A slightly modified version of the Zwanzig-Mori method allows us to derive a general set of exact equations for arbitrarily high moments of Φ. Such a generalization was proposed by Mori and collaborators[5]. It is the purpose of this paper to apply this approach to study the Lorentz model and to investigate consequences which follow from the obtained equations.

In Sect. 2 we give an outline of the derivation of the dynamic equations for the moments of Φ . The coefficients of these equations are functions of two dimensionless parameters $\kappa = kv\tau$ and $\zeta = z\tau$, where τ is the mean free time between collisions, z is the complex frequency, k is the modulus of the wave vector, and v is the particle velocity. It is also shown that for $\kappa \ll 1$ the ℓ-th moment depends mainly on its own initial value $(P_\ell, f(\kappa, 0))$. In Sect. 3 we study an explicit time dependence of a few lowest moments. In particular it is shown that $(P_0, f(\kappa, t))$ exhibits a short-time modified exponential decay and a long-time hydrodynamic behaviour. Other moments decay in time of the order of τ . The time dependence of the particular moments is specified by the form preexponential function.

2. EQUATIONS FOR MOMENTS OF THE DISTRIBUTION FUNCTION

Let us introduce the deviation of the distribution function from the equilibrium

$$\delta f(\vec{r}, \vec{v}, t) = f(\vec{r}, \vec{v}, t) - \frac{1}{4\pi} n_0 ,$$

where n_0 is the equilibrium density of particles. The Fourier-Laplace transform of $\delta f(\vec{r}, \vec{v}, t)$

$$\Phi(\kappa, \zeta, \mu) = \int_0^\infty dt e^{-zt} \int d^3r\, e^{-i\vec{k}\vec{r}}\, \delta f(\vec{r}, \vec{v}, t)$$

obeys the following integral equation

$$(z - ikv\mu)\Phi(\kappa, \zeta, \mu) = B\Phi(\kappa, \zeta, \mu) + h(\kappa, \mu). \tag{1}$$

Above $\mu = \cos\theta = \frac{\vec{k}\vec{v}}{|\vec{k}\vec{v}|}$. For the Lorentz model the collision operator B is a projector which project any function of θ onto subspace orthogonal to the zeroth Legendre polynomial

$$B = -\frac{1}{\tau} \mathcal{Q}_0 = -\frac{1}{\tau}(1 - \mathcal{P}_0). \tag{2}$$

Finally, the function $h(\kappa, \mu)$ is the Fourier transform of the initial value of the distribution function. The scalar product (A, B) is defined as the integral

$$(A, B) = \int_{-1}^{1} d\mu\, A^*(\mu)\, B(\mu). \tag{3}$$

We shall introduce the projection operator onto the ℓ-th Legendre polynomial P_ℓ

$$\mathcal{P}_\ell\, A(\mu) = P_\ell\,(\mu)(P_\ell\, , P_\ell\,)^{-1}\, (P_\ell\, , A), \qquad (\ell = 0, 1, \dots, n) \tag{4}$$

and the n-component vector $P(\mu) = \{ P_0, P_1, \dots, P_n\}$. The norm of this vector is related to the digamma function

$$(P, P) = \sum_{\ell=0}^{n} \frac{2}{2\ell + 1} = \{[\psi(n + \frac{2}{3}) - \psi(1)] + 2\ln 2\} .$$

The projection operator onto the subspace $\mathcal{H}_{\mathcal{P}}$ spanned by the Legendre polynomials $P_0, P_1, \dots, P_n$ is equal to the sum of projectors $\mathcal{P}_\ell$ $(\ell = 0, 1, \dots, n)$

$$\mathcal{P} = \sum_{\ell=0}^{n} \mathcal{P}_\ell\ . \tag{5a}$$

The operator $\mathcal{Q} = (1 - \mathcal{P})$ obeys the following identities

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$$\mathcal{Q} = 1-\mathcal{P} = \prod_{\ell=0}^{n}(1-\mathcal{P}_\ell) \equiv \prod_{\ell=0}^{n}\mathcal{Q}_\ell\,, \tag{5b}$$

where

$$\mathcal{Q}_\ell = 1-\mathcal{P}_\ell\,. \tag{5c}$$

It means that $\mathcal{Q}$ defines an orthogonal complement of $\mathcal{H}_\mathcal{P}$, which we denote by $\mathcal{H}_\mathcal{Q}$. Let us now derive an equation for $\mathcal{P}\Phi(\kappa,\zeta,\mu)$. Projecting both sides of Eq. (1) and eliminating the term relating the subspaces $\mathcal{H}_\mathcal{P}$ and $\mathcal{H}_\mathcal{Q}$, we obtain the following matrix equation:

$$[z + ikv(P,\mu P)(P,P)^{-1} + \frac{1}{\tau}(P,\mathcal{Q}_0 P) + (kv)^2(\mathcal{Q}\mu P,\mathcal{R}_\mathcal{Q}\mathcal{Q}\mu P)(P,P)^{-1}](P,\Phi) = (P,h) + ikv(\mathcal{Q}\mu P,\mathcal{R}_\mathcal{Q}\mathcal{Q}h), \tag{6}$$

where $\mathcal{R}_\mathcal{Q}$ is a resolvent operator acting in $\mathcal{H}_\mathcal{Q}$:

$$\mathcal{R}_\mathcal{Q}(\kappa,\zeta,\mu) = \tau[\zeta + 1 - i\kappa\mathcal{Q}\mu\mathcal{Q}]^{-1}\,. \tag{7}$$

Calculating the matrix elements which appear in the expression on the left-hand side of Eq. (6), we obtain a matrix $\Xi^{-1}(\kappa,\zeta)$ which determines the time evolution of $(P,\Phi(\kappa,\zeta))$

$$\Xi^{-1}(P,\Phi) = (P,h) + ikv(\mathcal{Q}\mu P,\mathcal{R}_\mathcal{Q}\mathcal{Q}h)\,,$$

$$\Xi^{-1} = \begin{bmatrix} z_0 & a_{01} & 0 & 0 & \cdots & 0 \\ a_{10} & z_1 & a_{12} & 0 & \cdots & 0 \\ 0 & a_{21} & z_2 & a_{23} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & z_{n-1} & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n-1} & z_n \end{bmatrix}, \tag{8}$$

where

$$z_0 = z,\quad z_1 = z_2 = \ldots z_{n-1} = z + \frac{1}{\tau},\quad z_n = z + \frac{1}{\tau} + \left(\frac{nkv}{2n+1}\right)^2(P_{n+1},\mathcal{R}_\mathcal{Q}P_{n+1}),$$

$$a_{j,j-1} = -\frac{j}{2j-1}\,\frac{(P_j,P_j)}{(P_{j-1},P_{j-1})}\,ikv,\quad a_{j,j+1} = -\frac{j+1}{2j+3}\,\frac{(P_j,P_j)}{(P_{j+1},P_{j+1})}\,ikv. \tag{9}$$

Let us recall that the norm of P_ℓ is

$$(P_\ell,P_\ell) = \frac{2}{2\ell+1}\,.$$

In order to calculate (P,Φ) we have to find the inverse of $\Xi^{-1}(\kappa,\zeta)$. It can be done relatively easily by the following procedure used in the case of linear response theory by Karasudani et al.[5]. We shall, therefore, present the main steps of the calculations. Consider the following determinant:

$$D(i,j)[\kappa,\zeta] = \begin{vmatrix} z_i & a_{i,i+1} & 0 & \cdots & \cdots & 0 \\ a_{i+1,i} & z_{i+1} & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & a_{j-1,j} \\ 0 & \cdots & \cdots & 0 & a_{j,j-1} & z_j \end{vmatrix} \qquad 0 \le i \le j \le n$$

Expanding this determinant in terms of minors, we obtain a set of recursion relations

$$D(i,j)[\kappa,\zeta] = z_i D(i+1,j)[\kappa,\zeta] - a_{i,i+1}\,a_{i+1,i}D(i+2,j)[\kappa,\zeta], \tag{10a}$$

$$D(i,j)[\kappa,\zeta] = z_i D(i,j+1)[\kappa,\zeta] - a_{j-1,j}a_{j,j-1}D(i,j-2)[\kappa,\zeta]. \tag{10b}$$

We shall write next the determinant $D(0,n)[\kappa,\zeta]$ in terms of minors of the j-th row

$$D(0,n)[\kappa,\zeta] = -a_{j-1,j}a_{j,j-1}D(0,j-2)[\kappa,\zeta]D(j+1,n)[\kappa,\zeta] +$$
$$+ z_j D(0,j-1)[\kappa,\zeta]D(j+1,n)[\kappa,\zeta] -$$

$$- a_{j+1,j}\, a_{j,j+1} D(0,j-1)[\kappa,\zeta]\, D(j+2,n)[\kappa,\zeta]. \qquad (11)$$

Using Eqs. (10a, 10b) and (11), we get the diagonal elements of $\Xi[\kappa,\zeta]$

$$\Xi_{jj}(\kappa,\zeta) = \left[z_j - a_{j-1,j}\, a_{j,j-1} \frac{D(0,j-2)[\kappa,\zeta]}{D(0,j-1)[\kappa,\zeta]} - a_{j+1,j}\, a_{j,j+1} \frac{D(j+2,n)[\kappa,\zeta]}{D(j+1,n)[\kappa,\zeta]} \right]^{-1}. \qquad (12)$$

Recursion relations (10a,b) yield a continued fraction for the ratio of determinants

$$\frac{D(0,j+2)[\kappa,\zeta]}{D(0,j-1)[\kappa,\zeta]} = \frac{1}{z+\tau^{-1}+}\;\frac{(kv)^2 \frac{(j-1)^2}{(2j-1)(2j-3)}}{z+\tau^{-1}+}\;\frac{(kv)^2 \frac{(j-2)^2}{(2j-3)(2j-5)}}{z+\tau^{-1}+}\;\dots\frac{\frac{4}{15}(kv)^2}{z+\tau^{-1}+}\;\frac{\frac{1}{3}(kv)^2}{z}, \qquad (13a)$$

$$\frac{D(j+2,n)[\kappa,\zeta]}{D(j-1,n)[\kappa,\zeta]} = \frac{1}{z+\tau^{-1}+}\;\frac{(kv)^2 \frac{(j+2)^2}{(2j+3)(2j+5)}}{z+\tau^{-1}+}\;\dots\;\frac{(kv)^2 \frac{n^2}{(4n)^2-1}}{z+\tau^{-1}+(\frac{nkv}{2n+1})^2 (P_{n+1}, \mathfrak{R}_{\mathfrak{Q}} P_{n+1})}, \qquad (13b)$$

where we have used the standard notation for continued fractions[8]. The calculations for the nondiagonal elements of Ξ are somewhat more involved. We shall consider separately the cases: $i<j$ and $i>j$. In the first case we obtain

$$\Xi_{ij}(\kappa,\zeta) = (-1)^{i+j} \frac{\frac{D(0,i-1)[\kappa,\zeta]}{D(0,j-1)[\kappa,\zeta]} a_{i,i+1}\, a_{i+1,i+2} \dots a_{j-1,j}}{z_j - a_{j,j-1}\, a_{j-1,j} \frac{D(0,j-2)[\kappa,\zeta]}{D(0,j-1)[\kappa,\zeta]} - a_{j+1,j}\, a_{j,j+1} \frac{D(j+2,n)[\kappa,\zeta]}{D(j+1,n)[\kappa,\zeta]}}, \quad (i<j) \qquad (14a)$$

in the latter one

$$\Xi_{ij}(\kappa,\zeta) = (-1)^{i+j} \frac{\frac{D(0,j-1)[\kappa,\zeta]}{D(0,i-1)[\kappa,\zeta]} a_{j+1,j}\, a_{j+2,j+1} \dots a_{i,i-1}}{z_i - a_{i-1,i}\, a_{i,i-1} \frac{D(0,i-2)[\kappa,\zeta]}{D(0,i-1)[\kappa,\zeta]} - a_{i+1,i}\, a_{i,i+1} \frac{D(i+1,n)[\kappa,\zeta]}{D(i+2,n)[\kappa,\zeta]}}, \quad (i>j) \qquad (14b)$$

The ratio of determinants in the denominators of (14a,b) can be also expressed in terms of continued fractions. This follows from Eq. (13a) and from the identity

$$\frac{D(0,j-1)}{D(0,i-1)} = \frac{D(0,j-1)}{D(0,j)}\,\frac{D(0,j)}{D(0,j+1)} \dots \frac{D(0,i-2)}{D(0,i-1)}, \qquad (i>j).$$

We can write now a formal solution of Eq. (8)

$$(P_i, \Phi(\kappa,\zeta)) = \sum_{0\le j<i} \Xi_{ij}(\kappa,\zeta)(P_j, h(\kappa)) + \Xi_{ii}(\kappa,\zeta)(P_i, h(\kappa)) +$$

$$+ \sum_{i<j\le n} \Xi_{ij}(\kappa,\zeta)[(P_i, h(\kappa)) + ikv(\mathfrak{Q}\mu P_j, \mathfrak{R}_{\mathfrak{Q}} \mathfrak{Q} h)\,\delta_{jn}], \quad (i = 0,1,2,\dots,n). \qquad (15a)$$

For $i=0$ this coincides with Eq. (19) of ref.[4], giving the particular representation for $(\mathfrak{Q}_0 \mu P_0, \mathfrak{R}_{\mathfrak{Q}}(\kappa,\zeta)\mathfrak{Q}_0 h(\kappa))$.

Let us estimate the matrix elements for small κ. Using the continued fraction expansions (13a,b), we can show that $D(0,j)[\kappa,\zeta] \sim \tau/(1+\zeta)$. Thus,

$$\Xi_{ij}(\kappa,\zeta) \sim \left(\frac{\kappa}{1+\zeta}\right)^{|i-j|} \Xi_{jj}(\kappa,\zeta), \qquad (\kappa \ll 1).$$

This means that for small κ we can limit ourselves to one term only on the right-hand side of Eq. (15a)

$$(P_i, \Phi(\kappa,\zeta)) \approx \Xi_{ii}(\kappa,\zeta)[(P_i, h(\kappa)) + ikv(\mathfrak{Q}\mu P_i, \mathfrak{R}_{\mathfrak{Q}}(\kappa,\zeta)\mathfrak{Q} h(\kappa))\delta_{i,n}]. \qquad (15b)$$

Let us now turn our attention to the analytic properties of the matrix elements $\Xi_{ij}(\kappa,\zeta)$. The familiar approximation consists in replacing the matrix element $(\mathfrak{Q}\mu P_n, \mathfrak{R}_{\mathfrak{Q}}\mathfrak{Q} h)$ by a number. In this approximation, the continued fractions appearing in the expressions for the matrix elements $\Xi_{ij}(\kappa,\zeta)$ are polynomials of the n-th degree. Thus, the only singularities of $(P_i, \Phi(\kappa,\zeta))$ are poles. Further, letting $n\to\infty$ we can relate matrix elements $\Xi_{ij}(\kappa,\zeta)$ to arctan $(\frac{\kappa}{1+\zeta})$, which has two branching points. In this limit $(\mathfrak{Q}\mu P_n, \mathfrak{R}_{\mathfrak{Q}}\mathfrak{Q} h)$ does not appear in our consideration.

In the next section we shall consider the corresponding expressions for Ξ_{00}, Ξ_{01}, Ξ_{11} and Ξ_{22}.

3. DECAY LAW FOR THE LOWEST MOMENTS OF THE DISTRIBUTION FUNCTION

Now we can write a closed dynamical equation for $(P_i, \Phi(\kappa,\zeta))$

$$[z_i - \frac{a_{i-1,i}a_{i,i-1}}{z_{i-1}+}\cdots\frac{a_{01}a_{10}}{z_0} - \frac{a_{i+1,i}a_{i,i+1}}{z_{i+1}+}\cdots\frac{a_{n-1,n}a_{n,n-1}}{z_n+}\cdots](P_i,\Phi(\kappa,\zeta)) = (P_i, h(\kappa)).$$

The first (terminating) continued fraction is a polynomial in κ and ζ, the second one is an infinite continued fraction which can be related to arctan $[\kappa/(1+\zeta)]$. Taking the inverse Fourier-Laplace transform, we can derive a set of dynamic equations for the moments of $\delta f(\vec{r},\vec{v},t)$. We are, however, primarily interested in the time-dependence of these moments.

Let us consider some special cases. For $i=0$ and $\kappa \ll 1$, we get

$$[z + \frac{1}{3}(kv)^2 \frac{D(2,\infty)[\kappa,\zeta]}{D(1,\infty)[\kappa,\zeta]}](P_0, \Phi(\kappa,\zeta))_0 \approx (P_0, h(\kappa)). \tag{16a}$$

This gives us a new representation of the generalized diffusion coefficient introduced in our previous paper [4]

$$D(\kappa,\zeta) = \frac{1}{3r}\frac{D(2,\infty)[\kappa,\zeta]}{D(1,\infty)[\kappa,\zeta]}.$$

Using relation (13b) and the continued fraction representation of arctan $[\kappa/(1+\zeta)]$ [6], we obtain

$$(P_0, \Phi(\kappa,\zeta))_0 = \frac{r(P_0, h(\kappa))}{\dfrac{\kappa}{\arctan(\frac{\kappa}{1+\zeta})} - 1}. \tag{17a}$$

For $|\zeta| \to \infty$ one has: $(P_0, \Phi(\kappa,\zeta))_0 \to \zeta^{-1}$. The singularities of $(P_0, \Phi(\kappa,\zeta))_0$ are the following: a simple pole at

$$\zeta \equiv r\Lambda(\kappa) = (\kappa\cot\kappa - 1), \quad (\kappa \le \frac{\pi}{2}) \tag{18a}$$

a cut from

$$\zeta = -1 - i\kappa \quad \text{to} \quad \zeta = -1 + i\kappa. \tag{18b}$$

As we shall see below these singularities imply two different time scales, namely, the kinetic one for $t \sim r$, and the hydrodynamic one for $t \sim r/\kappa^2$.

Similarly, for $i=1$ we get

$$[z + r^{-1} + \frac{4}{15}(kv)^2 \frac{D(3,\infty)[\kappa,\zeta]}{D(2,\infty)[\kappa,\zeta]}](P_1, \Phi(\kappa,\zeta))_1 = (P_1, h(\kappa)). \tag{16b}$$

Above, the ratio of the determinants is also a function of arctan

$$\frac{D(3,\infty)[\kappa,\zeta]}{D(2,\infty)[\kappa,\zeta]} = \frac{a - \arctan a - \frac{1}{3}a^2 \arctan a}{\frac{4}{15}\frac{a\kappa}{r}(\arctan a - a)},$$

where $a = \kappa/(1+\zeta)$. Hence,

$$(P_1, \Phi(\kappa,\zeta))_1 \equiv \Xi_{11}(\kappa,\zeta)(P_1, h(\kappa)) = 3r(P_1, h(\kappa))[\frac{1}{\arctan a} - \frac{1}{a}]. \tag{17b}$$

For $|\zeta| \to \infty$ we have: $(P_1, \Phi(\kappa,\zeta)) \sim \zeta^{-1}$. The only singularity of $(P_1, \Phi(\kappa,\zeta))_1$ is the cut (18b).

The equation for $(P_2, \Phi(\kappa,\zeta))_2$ is more complicated. Since,

$$[z + r^{-1} + \frac{4}{15}(kv)^2 \frac{z}{z(z+\frac{1}{r}) + \frac{1}{3}(kv)^2} + \frac{9}{35}(kv)^2 \frac{D(4,\infty)[\kappa,\zeta]}{D(3,\infty)[\kappa,\zeta]}](P_2, \Phi(\kappa,\zeta))_2 = (P_2, h(\kappa)) \tag{16c}$$

and

$$\frac{D(4,\infty)[\kappa,\zeta]}{D(3,\infty)[\kappa,\zeta]} = \frac{a(1+\frac{4}{15}a^2) - (1-\frac{3}{5}a^2)\arctan a}{\frac{9}{35}\frac{\kappa a}{r}[(1+\frac{1}{3}a^2)\arctan a - a]}$$

we have

$$(P_2, \Phi(\kappa,\zeta))_2 = \Xi_{22}(\kappa,\zeta)(P_2, h(\kappa)) =$$

$$= \frac{(P_2, h(\kappa))}{\zeta + 1 + \dfrac{\frac{4}{15}\kappa^2\zeta}{\zeta(\zeta+1) + \frac{1}{3}\kappa^2} + \kappa\dfrac{a(1+\frac{4}{15}a^2) - (1+\frac{3}{5}a^2)\arctan a}{a[(1+\frac{1}{3}a^2)\arctan a - a]}}. \tag{17c}$$

To get some feeling of the off-diagonal terms, let us consider the contribution of $\Xi_{01}(P_1, h(\kappa))$ to $(P_0, \Phi(\kappa,\zeta))$. We find

$$(P_0, \Phi(\kappa,\zeta))_1 = 3i\tau(P_1, h(\kappa)) \frac{1 - \frac{1}{\alpha}\arctan\alpha}{\kappa - \arctan\alpha} . \tag{17d}$$

For $|\zeta| \to \infty$

$$(P_0, \Phi(\kappa,\zeta))_1 \to \zeta^{-2} .$$

Let us calculate $(P_i, \delta f(\kappa, t))_i$ for $i = 0, 1$. Since the moments of $\Phi(\kappa,\zeta)$ are defined for $\mathrm{Re}\,\zeta > 0$, we shall continue them analytically to the negative half-plane and then use the methods of theory of complex variable. For $(P_0, \delta f(\kappa, t))_0$ the standard formula for the inverse of the Laplace transform gives

$$(P_0, \delta f(\kappa, t)) = \frac{1}{2\pi i}\int_{\tau a - i\infty}^{\tau a + i\infty} d\zeta\, e^{\zeta t/\tau} \frac{(P_0, h(\kappa))}{\dfrac{\kappa}{\arctan\left(\frac{\kappa}{1+\zeta}\right)} - 1} \quad (\mathrm{Re}\,\zeta > 0). \tag{19}$$

The number τa is chosen to be greater than $\Lambda(\kappa)$ defined in Eq. (18a). The contour in Eq. (19) can be chosen as shown in the Figure. The contribution of circles C_R, C_ρ, C'_ρ (of radius R and ρ, respectively) vanishes for $R \to \infty$, $\rho \to 0$. We choose the branch of logarithm for which

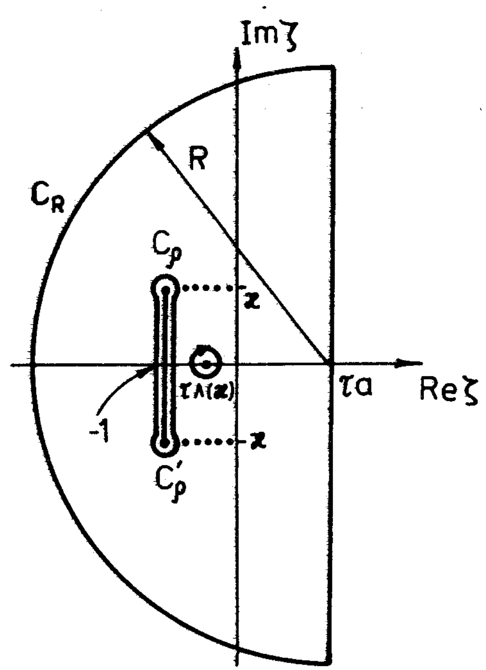
$$\lim_{|\zeta| \to \infty} \ln\left(\frac{1 + \zeta + i\kappa}{1 + \zeta - i\kappa}\right) = 0 .$$

Then, along the cut one has

$$\zeta + 1 = i\sigma \qquad (-\kappa \le \sigma \le \kappa)$$

and the analytic continuation of the logarithm from the interval (κ, ∞) to the cut is $\ln\left(\frac{\kappa+\sigma}{\kappa-\sigma}\right) - i\pi$. After evaluating the residuum and the integrals around the cut, we determine that

$$\frac{(P_0, \delta f(\kappa, t))_0}{\kappa (P_0, h(\kappa))} = e^{\tau\Lambda(\kappa) t/\tau}\, \kappa(1 + \cot^2\kappa) + e^{-t/\tau}\, \frac{1}{\pi}\int_0^1 dx \,\times$$



$$\times\left[\frac{\cos(\lambda x)\ln^2\left(\frac{1+x}{1-x}\right) + 3\pi(3\pi + 2\kappa)\cos(\lambda x) - 2\kappa\sin(\lambda x)\ln\left(\frac{1+x}{1-x}\right)}{(3\pi + 2\kappa) + \ln\left(\frac{1+x}{1-x}\right)} - \right. \tag{20a}$$

$$\left. - \frac{\cos(\lambda x)\ln^2\left(\frac{1+x}{1-x}\right) + \pi(\pi + 2\kappa)\cos(\lambda x) - 2\kappa\sin(\lambda x)\ln\left(\frac{1+x}{1-x}\right)}{(\pi + 2\kappa)^2 + \ln^2\left(\frac{1+x}{1-x}\right)}\right] \equiv A(t,\kappa) + B(t/\tau, \kappa),$$

where $\lambda = \kappa t/\tau$. For $t \sim \tau$ the oscillating functions of the integrand are almost constant. Hence, in the kinetic region $(t = \tau\tilde{t})$

$$\frac{(P_0, \delta f(\kappa, t))_0}{\kappa (P_0, h(\kappa))} \sim e^{\tau\Lambda(\kappa)\tilde{t}}\, \kappa(1 + \cot^2\kappa) + e^{-\tilde{t}}\,[C\cos(\kappa\tilde{t}) + \mathcal{O}(\kappa)] ,$$

where C is a number. For $t \sim \frac{\tau}{\kappa^2}$, the periodic functions oscillate rapidly and the integral is very small. Taking $t = \frac{\tau}{\kappa^2}\tilde{t}$ we have

$$\frac{(P_0, \delta f(\kappa,t))_0}{\kappa (P_0, h(\kappa))} \sim e^{\frac{r\Lambda(\kappa)}{\kappa^2}\tilde{t}} + e^{-\frac{\tilde{t}}{\kappa^2}}[a\cos(\frac{\tilde{t}}{\kappa}) + \mathcal{O}(\kappa)],$$

where

$$a = 4\pi[\frac{1}{3}\mathrm{Re}E\ (1+i\frac{3\pi}{2}) + \mathrm{Re}E\ (1+i\frac{\pi}{2})] - \frac{\pi}{2}[3\mathrm{Re}E\ (1+i\frac{3\pi}{2}) + \mathrm{Re}E\ (1+i\frac{\pi}{2})],$$

$E_1(z)$ is the exponential integral of complex argument [6]. Since the only singularity of $(P_1, \Phi(\kappa,\zeta))_1$ is the cut (18b), for the first moment we obtain

$$\pi \frac{(P_1, \delta f(\kappa, t))_1}{6\kappa (P_1, h(\kappa))} = e^{-t/r} \int_0^1 dx\, [\frac{3\pi\cos(\lambda x) + \sin(\lambda x)\ln(\frac{1+x}{1-x})}{\ln^2(\frac{1+x}{1-x}) + (3\pi)^2} - \tag{20b}$$

$$- \frac{\pi\cos(\lambda x) + \sin(\lambda x)\ln(\frac{1+x}{1-x})}{\ln^2(\frac{1+x}{1-x}) + \pi^2}].$$

Hence, in the kinetic region

$$\pi \frac{(P_1, \delta f(\kappa,t))_1}{6\kappa (P_1, h(\kappa))} \sim e^{-\tilde{t}} \cos(\kappa\tilde{t}).$$

In the hydrodynamic region

$$\frac{\pi (P_1, \delta f(\kappa, t))_1}{6\kappa (P_1, h(\kappa))} \sim e^{-\tilde{t}/\kappa^2}\,[a_1\cos(\frac{\tilde{t}}{\kappa}) - b\sin(\frac{\tilde{t}}{\kappa})],$$

where

$$a_1 = \mathrm{Re}E_1(1 + i\frac{3\pi}{2}) + \mathrm{Re}E_1(1 + i\frac{\pi}{2}),$$

$$b = \mathrm{Im}E_1(1 + i\frac{3\pi}{2}) + \mathrm{Im}E_1(1 + i\frac{\pi}{2}).$$

The values of the function $E_1(z)$ can be found in tables [7]. The higher moments of $\delta f(\kappa, \mu, t)$ are also vanishingly small in the hydrodynamic region.

The order of magnitude and the time dependence of the zeroth (ZM) (20a) and the first moment (FM) (20b) of the distribution function are illustrated in the Table. For comparison the contri-

Table

The time dependence of the zeroth (ZM) and first moments (FM) of the distribution function. For comparison the contribution of hydrodynamic term (A) and kinetic term (B) to ZM are listed separately.

t/τ \ κ		π	0.9	0.1	0.01	0.001
1	A	0	0.1102E+0 1	1.0000E+0 1	0.1000E +0 3	0.1000E +0 4
	B	-0.1680E-0 2	0.1557E-0 1	0.2454E-0 2	0.2519E -0 3	0.2525E -0 4
	ZM	-0.1680E-0 2	0.1118E +0 1	0.1000E+0 2	0.1000E +0 3	0.1000E +0 4
	FM	0.1339E-0 1	-0.6883E -0 1	-0.6038E-0 1	-0.5876E -0 1	-0.5860E -0 1
10	A	0	0.8416E -0 1	0.9704E+0 1	0.9997E +0 2	1.0000E +0 3
	B	-0.2832E-0 6	0.3187E -0 6	0.3835E-0 6	0.8760E -0 7	0.3129E -0 8
	ZM	-0.2832E-0 6	0.8416E -0 1	0.9704E+0 1	0.9997E +0 2	1.0000E +0 3
	FM	0.8399E-0 6	-0.1001E -0 5	-0.8520E-0 5	-0.2020E -0 4	-0.7252E -0 5
10^2	A	0	0.5676E -12	0.7187E+0 1	0.9967E+ 0 2	1.0000E +0 3
	B	-	0	0	0	0
	ZM	-	0.5676E- 12	0.7187E+0 1	0.9967E+ 0 2	1.0000E +0 3
	FM	-	0	0	0	0
10^3	A	0	0	0.3571E+0 0	0.7167E +0 2	0.9986E +0 3
	B	-	0	0	0	0
	ZM	-	0	0.3571E+0 0	0.7167E +0 2	0.9986E +0 3
	FM	-	0	0	0	0

bution of the hydrodynamic term, A, and the kinetic term, B, are listed separately. For $\kappa \ll 1$ the kinetic term of the zeroth moment is much smaller and vanishes much faster than the hydrodynamic one. Since for the particular initial condition $h(\kappa,\mu) \sim P_0(\mu)$ the solution (20a) is exact for any κ, we include also the results for $\kappa > \pi/2$. From the Table it is seen that in this case the magnitudes of both moments are comparable.

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Пашкевич Т. E17-82-874

Точное решение уравнения Больцмана для разреженного газа Лоренца. Динамические уравнения для моментов функции распределения

Изучается зависимость от времени функции распределения для трехмерной, изотропной модели Лоренца разреженного газа. Преобразование Фурье-Лапласа функции распределения раскладывается в ряд по полиномам Лежандра. С помощью нового разложения в цепную дробь,предложенного Мори и сотрудниками, получен набор уравнений для коэффициентов этого ряда. Получена явная зависимость от времени нескольких первых моментов в пределе малых волновых векторов. За исключением нулевого момента все они затухают экспоненциально с характерным временем, равным времени свободного пробега.

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Paszkiewicz T. E17-82-874

Exact Solution for the Lorentz Model of a Rarefied Gas. Dynamic Equations for Moments of the Distribution Function

Time dependence of the distribution function of the three-dimensional isotropic Lorentz gas is studied. The Fourier-Laplace transform of the distribution function is expanded in the series of Legendre polynomials. A set of exact equations for the coefficients of this series is derived using the "new continued fraction" representation of Mori and collaborators. The explicit time dependence of a few moments is found in the limit of small wave vectors. With the exception of the zeroth moment, their decay is characterized by the mean free time between collisions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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