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INTEGRABLE REDUCTIONS OF MANYCOMPONENT MAGNETIC SYSTEMS IN (1,1) DIMENSIONS

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1. INTRODUCTION

Most of the magnetic crystals, as experimental studies show, have layered or manychained structures [1]. Moreover, for the majority of them the interlayer or interchain interactions have a considerable effect on the general dynamical behaviour of crystals. Some typical representatives of such systems are $CrCl_2$, $CuCl_2$, $RbNiCl_3$, $CsNiCl_3$ salts [1]. Similar structures may also be seen in organic compounds in the form of molecular chains. Theoretical description of many-layered structures is based on the manycomponent generalization of Heisenberg spin model [2,3]. The introduction of "colour" degrees of freedom for interacting spins in one-dimensional chains may also describe manylayered quasi two-dimensional magnetic systems with weak coupling. On the other hand, it is well known [4,5] that one-dimensional Hubbard model with a half-filled band corresponds to two-component Heisenberg spin chain with nontrivial intercomponent interactions. Manycomponent spin chain which corresponds consequently to some generalized Hubbard model may be used for describing collective excitations (and also their statistical properties) in the system with different sorts of spins [3,6,7]. It should be noted, however, that the situation is difficult for modelling mostly in the low temperature region, where the interlayer interactions become prominent [1]. The dynamical behaviour of a crystal in the said region is defined by connected states of magnons of different kinds, which in the quasi-classical approximation are described by particle-like solutions of nonlinear evolution equations, e.g., by solitons, bions, etc. A characteristic of solitons with immense physical interest is their stability range. This is in particular defined by the degree of approximation with which the given system coincides with some completely integrable one (i.e., by the time of phase mixing) [8,9]. In the investigation of nonlinear systems describing some physical process, it is therefore, very important to find all possible integrable reductions.

In the present work we investigate many-component spin models in (1+1) space-time dimensions. In the longwave limit we get a system of nonlinear equations describing a collective excitations of lattice and spin systems at low temperatures, which for a particular reduction has been shown to reduce to some generalization of a system found previously for Langmuir wave. The reduced system in the "ultrarelativistic" limit reduces

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further to give some colour generalization of the equations investigated through inverse scattering method (ISM)[10] and shown to be completely integrable. Other reductions lead to generalized vector nonlinear Schrödinger equation with $U(p,q)$ noncompact isotopic group. This system is also completely integrable, possesses a rich spectrum of soliton solution and may be investigated thoroughly using Inverse scattering method[11].

2. MANYCOMPONENT MAGNETIC CHAIN

Hamiltonian of the system, we consider, may be given in the following form

$$H = H_S + H_L , \tag{1}$$

where

$$H_S = -\frac{1}{2}\Big[\frac{1}{2}\sum_{i,j,\alpha,\beta} J_{ij}^{\alpha\beta}(S_i^{+\alpha}S_j^{-\beta} + S_i^{-\alpha}S_j^{+\beta}) + \sum_{\substack{i,j\\ \alpha,\beta,\gamma,\delta}} R_{ij}^{\alpha\beta\gamma\delta} S_i^{z\alpha\beta} S_j^{z\gamma\delta}\Big]$$

describes interaction of different sorts of spins and $H = T + U$ corresponds to lattice oscillations.

Neglecting interaction between the "colour" and "space" degrees of freedom in the exchange integrals and considering only nearest-neighbour interaction we get

$$\begin{aligned} J_{ij}^{\alpha\beta} &= J_{jj+\delta} K^{\alpha\beta} , \\ J_{ij}^{\alpha\beta\gamma\delta} &= \bar{J}_{jj+\delta} L_1^{\alpha\beta} L_2^{\gamma\delta} , \end{aligned} \tag{2}$$

where $J_{jj+\delta} \equiv J(|x_j - x_{j+\delta}|)$ are the exchange integrals and $\delta = \pm 1$. Using generalized Holstein-Primakov representation[12]

$$\begin{aligned} S_j^{+\alpha} &= (2s - a_j^{+\alpha} a_j^{\alpha})^{1/2} a_j^{\alpha} = (2s)^{1/2}\Big(a_j^{\alpha} - \frac{1}{4s} a_j^{+\alpha} a_j^{\alpha} a_j^{\alpha} - \frac{1}{32s^2}(a_j^{+\alpha} a_j^{\alpha})^2 a_j^{\alpha} + \\ &\qquad + \ldots\Big) , \\ S_j^{-\alpha} &= a_j^{+\alpha}(2s - a_j^{+\alpha} a_j^{\alpha})^{1/2} = (2s)^{1/2}\Big(a_j^{+\alpha} - \frac{1}{4s} a_j^{+\alpha} a_j^{+\alpha} a_j^{\alpha} - \frac{1}{32s^2} a_j^{+\alpha}(a_j^{+\alpha} a_j^{\alpha})^2 + \\ &\qquad + \ldots\Big) , \\ S_j^{z\alpha\beta} &= s\delta^{\alpha\beta} - a_j^{+\alpha} a_j^{\beta} \end{aligned} \tag{3}$$

with

$$[a_i^{\alpha}, a_j^{+\beta}] = \delta_{ij}\delta^{\alpha\beta}, \quad [a_i^{\alpha}, a_j^{\beta}] = [a_i^{+\alpha}, a_j^{+\beta}] = 0$$

at low temperatures ($a_j^{+\alpha} a_j^{\alpha} = n_j^{\alpha} \ll 2s$) Hamiltonian (1) may be reduced to the pure boson form:

$$\begin{aligned} H = H_0 &- \frac{1}{2}\sum_{j\delta}\Big[J_{jj+\delta}\, s \sum_{\alpha\beta} K^{\alpha\beta}(a_j^{+\alpha} a_{j+\delta}^{\beta} + a_{j+\delta}^{+\beta} a_j^{\alpha}) - \\ &- \bar{J}_{jj+\delta}\Big\{ s \sum_{\alpha\beta}(\ell_2 L_1^{\alpha\beta} a_j^{+\alpha} a_j^{\beta} + \ell_1 L_2^{\alpha\beta} a_{j+\delta}^{+\alpha} a_{j+\delta}^{\beta}) + \\ &+ \sum_{\gamma\delta\alpha\beta} L_1^{\alpha\beta} L_2^{\gamma\delta} a_j^{+\alpha} a_j^{\beta} a_{j+\delta}^{+\delta} a_{j+\delta}^{\gamma}\Big\}\Big] + H_L , \end{aligned} \tag{4}$$

where

$$H_0 = (-\frac{s}{2} J(0) kN + s^2 \ell_1 \ell_2 \sum \bar{J}_{jj+\delta}, \quad k = \mathrm{Tr} K, \quad \ell_i = \mathrm{Tr} L_i ,$$

N is the total number of sites. In the longwave limit repeating the procedure developed in[13,14] we finally get the following system

$$\ddot{x} = v_0^2 x_{\xi\xi} + \frac{s}{m}\sum_{\alpha\beta} \bar{T}^{\alpha\beta}(\phi^{*\alpha}\phi^{\beta})_{\xi} , \tag{5a}$$

$$\begin{aligned} i\dot{\phi}^{\alpha} &= -b\sum_{\beta} K_{[\alpha\beta]}\phi^{\beta}_{\xi\xi} - s\sum_{\beta} T_{\alpha\beta}\phi^{\beta} + s\sum \bar{T}_{\alpha\beta}\phi^{\beta} x_{\xi} - \\ &\quad - \bar{J}(0)\Big\{\sum_{\gamma\delta} L_1^{\gamma\delta}\phi^{*\gamma}\phi^{\delta}\sum_{\beta} L_2^{\alpha\beta}\phi^{\beta} + \sum_{\gamma\delta} L_2^{\gamma\delta}\phi^{*\gamma}\phi^{\delta}\sum L_1^{\alpha\beta}\phi^{\beta}\Big\} , \end{aligned} \tag{5b}$$

where

$$b = J(0)\, s/2 ,$$

$$\bar{T}_{\alpha\beta} = J_1 K_{[\alpha\beta]} - \bar{J}_1(\ell_1 L_{2\alpha\beta} + \ell_2 L_{1\alpha\beta}) ,$$

$$T_{\alpha\beta} = J(0) K_{[\alpha\beta]} - \bar{J}(0)(\ell_1 L_{2\alpha\beta} + \ell_2 L_{1\alpha\beta})$$

and

$$J_{jj+\delta} \approx J(0) - J_1 \cdot |x_j - x_{j+\delta}| .$$

Here $x(\xi, t)$ is the lattice deviations and $\phi^{\alpha}(\xi, t)$ is the Schrödinger amplitude of spin distribution. In the "ultrarelativistic" limit ($v \to 1$) operator $\partial_t^2 - \partial_{\xi}^2$ may be replaced by $-2(\partial_t + \partial_{\xi})\partial_{\xi}$ and the first equation may be integrated by ξ once assuming trivial boundary conditions which give ultimately a "colour" generalization of the system due to Yajima and Oikawa[10] (the notation $\eta \equiv x_{\xi}$ is used):

$$\eta_t + v_0^2 \eta_{\xi} + \frac{s}{2m}\sum_{\alpha,\beta} \bar{T}^{\alpha\beta}(\phi^{*\alpha}\phi^{\beta})_{\xi} = 0$$

$$i\dot{\phi}^{\alpha} = -b\sum K_{[\alpha\beta]}\phi^{\beta}_{\xi\xi} - s\sum_{\beta} T_{\alpha\beta}\phi^{\beta} + s\sum \bar{T}_{\alpha\beta}\phi^{\beta}\cdot\eta, \tag{6}$$

where we have put

$$L_1^{\alpha\beta} = L_2^{\alpha\beta} \equiv 0 .$$

In the "quasistationary" limit /9a/ eq. (5b) may be reduced to

$$x_{\xi} = -\frac{s}{mv_0^2}\sum_{\alpha,\beta}\bar{T}^{\alpha\beta}(\phi^{*\alpha}\phi^{\beta}) + c, \tag{7}$$

c being an integration constant, which from (5b) gives the system of equations only for ϕ_a functions:

$$i\dot{\phi}^{\alpha} = -b\sum_{\beta} K_{[\alpha\beta]}\phi^{\beta}_{\xi\xi} - \sum_{\beta} R_{\alpha\beta}\phi^{\beta} - a\Big(\sum_{\gamma\delta}\bar{T}^{\gamma\delta}\phi^{*\alpha}\phi^{\delta}\Big)\times$$

$$\times\sum_{\beta}\bar{T}^{\alpha\beta}\phi^{\beta} - \bar{J}(0)\Big\{\sum_{\gamma\delta} L^{\gamma\delta}\phi^{*\gamma}\phi^{\delta}\sum_{\beta} L^{\alpha\beta}\phi^{\beta} + \tag{8}$$

$$+\sum_{\gamma\delta} L_2^{\gamma\delta}\phi^{*\gamma}\phi^{\delta}\sum_{\beta} L_1^{\alpha\beta}\phi^{\beta}\Big\},$$

where $R_{\alpha\beta} = s(T_{\alpha\beta} - c\bar{T}_{\alpha\beta})$ and $a = s^2/mv_0^2$. This equations are in general not integrable. It is, therefore, desirable to consider some of its reductions

a) In the case when the exchange integrals related to colour degrees of freedom are proportional to each other, i.e.,

$$K_{[\alpha\beta]} = 2b_1 L_{1\alpha\beta} = 2b_2 L_{2\alpha\beta} \tag{9}$$

eqs. (8) may be obtained from the Hamiltonian

$$H = \int\{b(\phi^{+}_{\xi}K^{S}\phi_{\xi}) - \kappa(\phi^{+}K^{S}\phi)^2 - \tilde{\mu}(\phi^{+}K^{S}\phi)\}d\xi,$$

where,

$$\phi^{+}K^{S}\phi = \sum_{\alpha,\beta}\phi^{*\beta}K_{[\alpha,\beta]}\phi^{\beta}, \quad \kappa = a\nu^2 + \bar{J}(0)/(2b_1b_2),$$

$$\tilde{\mu} = s(\mu - c\nu), \quad \nu = \Big[J_1 - \frac{\bar{J}_1}{2}(b_1\ell_2 + b_2\ell_1)\Big], \tag{10}$$

and

$$\lambda = \Big[J(0) - \frac{\bar{J}(0)}{2}(b_1\ell_2 + b_2\ell_1)\Big].$$

Introducing $\tilde{\phi} = \phi^{+}K^{S}$ one gets the system of equations

$$\begin{aligned} i\dot{\phi} &= -b\phi - \kappa(\tilde{\phi}\cdot\phi)\phi - \mu\phi, \\ i\dot{\tilde{\phi}} &= -b\tilde{\phi} - \kappa(\tilde{\phi}^{*}\tilde{\phi})\tilde{\phi} - \mu\tilde{\phi}. \end{aligned} \tag{11}$$

Due to the Hermitian character of K^S the quadratic form $(\phi^{+}K^{S}\phi)$ may be reduced to the diagonal form:

$$\phi' = U\phi, \quad (\phi^{+}K^{S}\phi) = \phi'^{+}K_0\phi',$$

where $K_0 = UK^{S}U^{+}$ is a diagonal matrix with real elements $(K_0)_{ii} = \lambda_i$. After normalizing ϕ' we obtain

$$(\phi^{+}K^{S}\phi) = \psi^{+}\Gamma_0\psi = (\bar{\psi}\psi), \quad \Gamma_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad \bar{\psi} = \psi^{+}\Gamma_0,$$

i.e., U(p, q) internal product norm. Thus in this case, system (11) is equivalent to U(p,q) vector nonlinear Schrödinger equation (NLSE). Eqs. (6) with the application of analogical procedure reducés to

$$\begin{aligned} &\eta_t + v_0^2\eta_{\xi} + \frac{s}{2m}(\bar{\phi}\phi)_{\xi} = 0, \\ &i\dot{\phi} = -b\phi - \mu\phi + a\eta\phi, \end{aligned} \tag{12}$$

where

$$\bar{\phi} = \phi^{+}\Gamma_0,$$

which is integrable through inverse scattering method if one uses the following set of Lax operators $(\Phi = \phi\exp(i\frac{t}{2} - ix))$

$$\Psi_{\xi} = U\Psi, \quad \Psi_t = V\Psi,$$

$$U = \begin{pmatrix} 3i\zeta & 0 & 0 \\ 0 & i\zeta I_{NN} & 0 \\ 0 & 0 & -i\zeta \end{pmatrix} + \frac{1}{2i\zeta}\begin{pmatrix} \eta & -2i\zeta\bar{\Phi} & \eta \\ i\Phi & 0 & i\Phi \\ -\eta & 2i\zeta\bar{\Phi} & -\eta \end{pmatrix} \tag{13}$$

$$V = \frac{i}{2\zeta}\begin{pmatrix} \eta + \frac{\bar{\Phi}\Phi}{2} + 2\zeta^2(2\zeta/3 - 2), & -2i\zeta(-\zeta\bar{\Phi} + \frac{i}{2}\bar{\Phi}_{\xi} + \bar{\Phi}), & \eta + \frac{\bar{\Phi}\Phi}{2} \\ -i(\zeta\Phi + \frac{i}{2}\Phi_{\xi} - \Phi), & -8\zeta^3/3\cdot I_{NN}, & -i(-\zeta\Phi + \frac{i}{2}\Phi_{\xi} - \Phi) \\ -(\eta + \frac{(\bar{\Phi}\Phi)}{2}), & 2i\zeta(\zeta\bar{\Phi} + \frac{i}{2}\bar{\Phi}_{\xi} + \bar{\Phi}), & -(\eta + \frac{\bar{\Phi}\Phi}{2}) + 2\zeta^2(\frac{2\zeta}{3} + 2) \end{pmatrix}$$

The $U(p,q)$ NLSE has been studied thoroughly in ref.[11], where the authors have shown its integrability, found soliton solutions and discussed about its possible quasiclassical quantization.

It is however, natural to ask which initial physical model corresponds to the $U(p,q)$ NLSE obtained. For answering this let us consider the following subreduction of (9):

$$\bar{J}_{jj+\delta} \equiv \rho J_{jj+\delta} \,, \qquad K^{\alpha\beta} = L_1^{\alpha\beta} = L_2^{\alpha\beta} = \epsilon_{(\alpha)} \delta^{\alpha\beta} \,, \tag{14}$$

where

$$\epsilon_{(\alpha)} = \begin{cases} +1, & \text{for} \quad \alpha = 1,2,\dots,p \\ -1, & \text{for} \quad \alpha = p+1,\, p+2,\dots,n\,. \end{cases}$$

In (14) there is no summation over α. The corresponding Hamiltonian of the initial system (1) describes manycomponent mixture of ferromagnetic and anti-ferromagnetic chains with negligible intercomponent interactions:

$$\begin{aligned} H = &- \frac{1}{2} \sum_{j,\delta} J_{jj+\delta} \Big\{ \frac{1}{2} \sum_{\alpha=1}^{p} (S_j^{+\alpha} S_{j+\delta}^{-\alpha} + S_j^{-\alpha} S_{j+\delta}^{+\alpha}) + \rho \sum_{\alpha=1}^{p} S_j^{z\alpha} \sum_{\rho=1}^{p} S_{j+\delta}^{z\beta} \Big\} + \\ &+ \frac{1}{2} \sum_{j,\delta} J_{jj+\delta} \Big\{ \frac{1}{2} \sum_{\gamma=p+1}^{n} (S_j^{+\alpha} S_{j+\delta}^{-\alpha} + S_j^{-\alpha} S_{j+\delta}^{+\alpha}) - \rho \sum_{\gamma=p+1}^{n} S_j^{z\gamma} \sum_{\sigma=p+1}^{n} S_{j+\delta}^{z\sigma} \Big\} + \\ &+ \frac{\rho}{2} \sum_{j,\delta} J_{jj+\delta} \Big(\sum_{\alpha=1}^{p} S_j^{z\alpha} \sum_{\sigma=p+1}^{n} S_{j+\delta}^{z\sigma} + \sum_{\sigma=p+1}^{n} S_j^{z\sigma} \sum_{\alpha=1}^{p} S_{j+\delta}^{z\alpha} \Big) , \end{aligned} \tag{15}$$

where the notation $S_j^{z\alpha} \equiv S_j^{z\alpha\alpha}$ has been used. Applying analogical procedure as before to Hamiltonian (15) one may find the following equation

$$i\dot{\phi}^{\alpha} = -2b\,\epsilon_{(\alpha)} \phi^{\alpha}_{\xi\xi} - \kappa \Big(\sum_{\beta} \epsilon_{(\beta)} |\phi^{\beta}|^2 \Big) \epsilon_{(\alpha)} \phi^{\alpha} - \mu \epsilon_{(\alpha)} \phi^{\alpha} \,, \tag{16}$$

where $q \equiv n - p$,

$$\kappa = 2\alpha J_1 (1 - \rho(p-q))^2 + \frac{\rho}{2} J(0), \quad \mu = 2s(J(0) - cJ_1)(1 - \rho(p-q))$$

which is again the $U(p,q)$ NLSE. In the limiting cases:

1) $p = n$, $q = 0$ one gets for the pure ferromagnetic system a $U(n,0)$ NLSE of attractive type

$$i\dot{\phi}^{\alpha} = -2b\phi^{\alpha}_{\xi\xi} - \kappa \Big(\sum_{\beta=1}^{n} |\phi^{\beta}|^2 \Big) \phi^{\alpha} - \mu\phi^{\alpha} \tag{17}$$

with

$$\kappa = 4\{\alpha J_1^2 (1 - \rho n)^2 + \rho \frac{J(0)}{2}\}, \quad \mu = 2s\{(J(0) - cJ_1)(1 + \rho n)\}$$

and

2) $p = 0$, $q = n$ for pure antiferromagnetic system we get a vector $U(0,n)$ NLSE of repulsive type

$$-i\dot{\phi} = -2b\phi^{\alpha}_{\xi\xi} + \kappa \Big(\sum_{\beta=1}^{n} |\phi^{\beta}|^2 \Big) \phi^{\alpha} - \mu\phi^{\alpha} \tag{18}$$

with

$$\kappa = 4\{\alpha J_1^2 (1 + \rho n)^2 + \rho\, J(0)/2\} \,.$$

For $n = 1$ eq. (17) reduces to a $U(1)$ NLSE investigated previously in ref.[14] for describing $CsNiF_3$ magnetic crystals.

3) For real crystals interactions between "colour" components are much weaker compared to interlattice interactions[1]. Hence, in the colour space also it is suggestive to consider interactions only among the nearest neighbours[2]. The corresponding reduction may be taken in the form

$$\begin{aligned} R_{ij}^{\alpha\beta\gamma\delta} &= -\rho\delta^{\alpha\beta}\delta^{\gamma\delta} J_{ij}^{\gamma\alpha} \,, \\ J_{ij}^{\alpha\beta} &= -(J_{jj+\delta} R^{\alpha\beta} + J' V_{ij}^{\alpha\beta}) \,, \\ R^{\alpha\beta} &= \delta^{\alpha\beta} + \epsilon\delta^{\beta,\alpha+\delta'} , \\ V_{ij}^{\alpha\beta} &= \delta_{ij}\, \delta^{\beta,\alpha+\delta'} \end{aligned} \tag{19}$$

with $J'/J \ll 1$, J and J' being respectively the intersite and interchain exchange integrals. The Hamiltonian (1) in such a case reduces to

$$\begin{aligned} H = &\frac{1}{2} \sum_{j,\sigma} J_{jj+\delta} \sum_{\alpha=1}^{n} \Big\{ \frac{1}{2} (S_j^{+\alpha} S_{j+\delta}^{-\alpha} + S_j^{-\alpha} S_{j+\delta}^{+\alpha}) + \rho S_j^{z\alpha} S_{j+\delta}^{z\alpha} \Big\} + \\ &+ \frac{1}{2} J' \sum_{\substack{\alpha=1 \\ \delta'=\pm 1}}^{n} \Big\{ \frac{1}{2} (S_j^{+\alpha} S_j^{-(\alpha+\delta')} + S_j^{-\alpha} S^{+(\alpha+\delta')}) + \rho S_j^{z\alpha} S_j^{z(\alpha+\delta')} \Big\} + \\ &+ \frac{1}{2} \epsilon \sum_{\delta,j} J_{jj+\delta} \sum_{\delta',\alpha} \Big\{ \frac{1}{2} (S_j^{+\alpha} S_{j+\delta}^{-(\alpha+\delta')} + S_j^{-\alpha} S_{j+\delta}^{+(\alpha+\delta')}) + \rho S_j^{z\alpha} S_{j+\delta}^{z(\alpha+\delta')} \Big\} . \end{aligned} \tag{20}$$

In the x-y limit, i.e., for $\rho \to 0$ one gets for the amplitude ϕ_α an $U(p,q)$ NLSE with broken "colour" symmetry:

$$i\dot{\phi}^{\alpha} = \frac{sJ(0)}{2}\sum_{\beta} R_{[\alpha\beta]}\phi^{\beta}_{\xi\xi} + \sum_{\beta} R'_{[\alpha\beta]}\phi^{\beta} - \alpha\Big(\sum_{\gamma\delta} R_{[\gamma\delta]}\phi^{*\gamma}\phi^{\delta}\Big)\sum_{\beta} R_{[\alpha\beta]}\phi^{\beta}, \tag{21}$$

where

$$R'_{[\alpha\beta]} = s(J(0) - cJ_1)R_{[\alpha\beta]} + J's\,\delta^{\beta,(\alpha+\delta)} .$$

The intrinsic $U(p,q)$ symmetry, however, is recovered in the case $J' \to 0$.

3. GENERALIZED MANY-COMPONENT PEIERLS-HUBBARD MODEL

As it has been shown in refs./4,5/ a one-dimensional one-band Hubbard model is equivalent to a two-component Heisenberg chain. The reduction of generalized Heisenberg chain (1) to the "colour" generalization of Peierls-Hubbard model may be given by (2) with the following additional assumptions

$$J_{jj+\delta} = -J^{\circ}_{jj+\delta}, \quad J^{\circ}(0) = 2t, \quad J^{\circ}_1 = 2I, \quad s = 1/2, \tag{22}$$

and

$$K_{\alpha\beta} = (I \otimes \sigma_1)_{\alpha\beta}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_{ab} = \delta_{ab}, \quad a, b = \overline{1, n}.$$

We also denote further

$$\sigma = \begin{cases} k, & \text{for } k \in D^+, \qquad D^+ = [1, 2, \ldots, n/2] \\ -(n+1-k) & \text{for } k \in D^-, \qquad D^- = [n/2+1, \ldots, n], \end{cases}$$

where

$$\alpha, \beta = \begin{cases} 2k-1, & \text{for } \alpha, \beta \text{ odd} \\ 2k, & \text{for } \alpha, \beta \text{ even} \end{cases} \tag{23}$$

and put

$$S_{2k-1} = \begin{cases} S^A_{\sigma} & \text{for } k \in D^+ \\ S^A_{-\sigma} & \text{for } k \in D^- \end{cases} \tag{24a}$$

$$S_{2k} = \begin{cases} S^B_{\sigma} & \text{for } k \in D^+ \\ S^B_{-\sigma} & \text{for } k \in D^- . \end{cases} \tag{24b}$$

We introduce the following reduction for matrix

$$R^{\alpha\beta\delta\gamma}_{ij} \Rightarrow -(U_0\delta_{ij}\, P^{aa}_{1\gamma\gamma} + \mathcal{J}_{ij} P^{aa}_{2\gamma\gamma}), \tag{25}$$

where δ_{ij} is Cronecker symbol.

$$\mathcal{J}_{ij} = \begin{cases} \mathcal{J}, & \text{for } i,j \text{ nearest neighbours} \\ 0, & \text{otherwise} \end{cases}$$

and

$$P^{aa}_{1\gamma\gamma} = L^{+aa}_A L^{-\gamma\gamma}_A + L^{+aa}_B L^{-\gamma\gamma}_B ,$$

$$P^{aa}_{2\gamma\gamma} = (L^{+aa}_A L^{+\gamma\gamma}_B + L^{-aa}_A L^{-\gamma\gamma}_B) + (L^{+aa}_B L^{+\gamma\gamma}_A + L^{-\gamma\gamma}_B L^{-aa}_A)$$

with

$$L^{+aa}_{A(B)} = \begin{cases} (I \otimes e_{A(B)})_{aa} & \text{for } k \in D^+ , \\ 0 & \text{for } k \in D^- \end{cases} \tag{26}$$

$$L^{-\gamma\gamma}_{A(B)} = \begin{cases} 0 & \text{for } k \subset D^+ \\ (I \otimes e_{A(B)})_{\gamma\gamma} & \text{for } k \in D^- \end{cases}$$

$$e_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The spin system (1) with the above notations and assumptions takes the form

$$H = \frac{1}{4}\sum_{j,\delta}\sum_{\sigma=1}^{n} J^{\circ}_{jj+\delta}(S^{-A}_{j,\sigma}S^{+B}_{j+\delta,\sigma} + S^{-B}_{j,\sigma}S^{+A}_{j+\delta,\sigma} + S^{+A}_{j,\sigma}S^{-B}_{j+\delta,\sigma} + S^{+B}_{j,\sigma}S^{-A}_{j+\delta,\sigma}) + \frac{U_0}{2}\sum_{j\in A\cup B}\sum_{\sigma=1}^{n/2} S^z_{j,\sigma}\sum_{\sigma=1}^{n/2} S^z_{j,-\sigma} + $$

$$+\frac{\mathcal{J}}{2}\Big(\sum_{\substack{j\in A\\ j+\delta\in B,\\ \&\ \text{reverse}}}\sum_{\sigma=1}^{n/2}S^{zA}_{j,\sigma}\sum_{\sigma=1}^{n/2}S^{zB}_{j+\delta,\sigma}+\sum_{\sigma=1}^{n/2}S^{zA}_{j,-\sigma}\sum_{\sigma=1}^{n/2}S^{zB}_{j+\delta,\sigma}\Big)+T+U. \tag{27}$$

Now with the application of generalized Jordan-Wigner trick

$$S_j^{+\alpha}=\exp\Big[-i\pi\Big(\sum_{\beta=1}^{\alpha-1}\sum_{k=1}^{j-1}c_k^{+\beta}c_k^{\beta}\Big]c_j^{\alpha},$$

$$S_j^{-\alpha}=c_j^{+\alpha}\exp\Big[i\pi\Big(\sum_{\beta=1}^{\alpha-1}\sum_{k=1}^{j-1}c_k^{+\beta}c_k^{\beta}\Big)\Big], \tag{28}$$

$$S_j^{z\alpha\beta}=\frac{1}{2}\delta^{\alpha\beta}-c_j^{+\alpha}c_j^{\beta}$$

with

$$[c_i^{\alpha},c_j^{+\beta}]_+=\delta_{ij}\,\delta_{\alpha\beta},\quad [c_i^{\alpha},c_j^{\beta}]_+=[c_i^{+\alpha},c_j^{+\beta}]_+=0,$$

one comes to a many-component generalized Peierls-Hubbard model in the near-neighbour limit with lattice excitations considered as classical fields:

$$H=\frac{1}{4}\sum_{\sigma=1}^{n/2}\sum_j J^{o}_{jj+1}(c^{+A}_{j\sigma}c^{B}_{j+1\sigma}+c^{+B}_{j+1}c^{A}_{j\sigma}+c^{+B}_{j\sigma}c^{A}_{-j+1\sigma}+c^{+A}_{j+1\sigma}c^{B}_{j\sigma}+\text{H.C.})+$$

$$+\frac{U}{2}\sum_{j\in A\cup B}\sum_{\sigma=1}^{n/2}n_{j\sigma}\sum_{\sigma=1}^{n/2}n_{j,-\sigma}-\mu\sum_{j\in A\cup B}\sum_{\sigma=1}^{n}n_{j\sigma}+ \tag{29}$$

$$+\frac{\mathcal{J}}{2}\sum_{\substack{j\in A\\ j+\delta\in B\\ \&\ \text{reverse}}}\Big(\sum_{\sigma=1}^{n/2}n_{j\sigma}\sum_{\sigma=1}^{n/2}n_{j+\delta,\sigma}+\sum_{\sigma=1}^{n/2}n_{j,-\sigma}\sum_{\sigma=1}^{n/2}n_{j+\delta,-\sigma}\Big)+$$

$$+H_0+T+U,$$

where

$$H_0=[(\mathcal{J}+U_0)n^2N]/8,\qquad \mu=[(3\mathcal{J}+U_0)n]/8,$$

$$n_{j\sigma}=c^{+}_{j\sigma}c_{j\sigma}.$$

We may easily check that for $\sigma=\pm1$ (29) reduces to a generalized Peierls-Hubbard model [7] (PHM)

$$H=\frac{1}{4}\sum_{\substack{\sigma=\pm1\\ j\in A,\ j+1\in B,\\ \&\ \text{reverse}}}J^{o}(|x_{j+1}-x_j|)(c^{+}_{j\sigma}c_{j+1\sigma}+c^{+}_{j+1\sigma}c_{j\sigma}+\text{H.C.})- \tag{30}$$

$$-\mu\sum_{\substack{\sigma=\pm1\\ j\in A\cup B}}n_{j\sigma}+\frac{U}{2}\sum_{j\in A\cup B}n_{j\uparrow}n_{j\downarrow}+\frac{\mathcal{J}}{2}\sum_{\substack{\sigma=\pm1\\ j\in A,\ j+\delta\in B,\\ \&\ \text{reverse}}}n_{j\sigma}n_{j+\delta,\sigma}+H_0+T+U$$

with

$$H_0=(\mathcal{J}+U)N/8,\qquad \mu=(3\mathcal{J}+U_0)/8.$$

For this reduction the field equations in the discrete case take the forms

$$m\ddot{x}_j=m\omega_0^2\Delta^2x_j-I\sum_{\sigma,j}^{n}(\phi_j^{*A}\Delta\phi_j^{B\sigma}+\phi_j^{A\sigma}\Delta\phi_j^{*B\sigma}+ \tag{31a}$$
$$+\phi_j^{*B\sigma}\Delta\phi_j^{A\sigma}+\phi_j^{B\sigma}\Delta\phi_j^{*A\sigma})$$

and

$$i\dot{\phi}_j^{A\sigma}=t(\phi_{j+1}^{B\sigma}+\phi_{j-1}^{B\sigma})+2I(x_j\Delta\phi_j^{B\sigma}-\Delta(x_j\phi_j^{B\sigma}))-\mu\phi_j^{A\sigma}+ \tag{31b}$$
$$+\frac{U}{2}\Big(\sum_{\sigma,j}^{n/2}|\phi_j^{A-\sigma}|^2\phi^{A\sigma}\Big)+\frac{\mathcal{J}}{2}\sum_{\sigma}^{h}(|\phi_{j+1}^{B\sigma}|^2+|\phi_{j-1}^{B\sigma}|^2)\phi_j^{A\sigma},$$

where $2\Delta\phi_j\equiv\phi_{j+1}-\phi_{j-1}$. It is well known that the ground state in a Hubbard model may be an anti-ferromagnetic state. But since for a colour generalization of Hubbard model the situation is not known, we demonstrate here the cases, when the ground state is an antiferromagnetic as well as a ferromagnetic one. Usually for an antiferromagnetic state we assume $\langle n^{A}_{j\sigma}\rangle=\langle n^{B}_{j\pm1,\sigma}\rangle$, which cancels the term proportional to I in (31a). Therefore, we consider only the long-wave limit and assume

$$\phi^{B\sigma}_{j\pm1}=\phi^{A-\sigma}_j\pm\Delta\phi^{A-\sigma}_j+\frac{1}{2}\Delta^2\phi^{A-\sigma}_j, \tag{32a}$$

for an antiferromagnetic ground state and

$$\phi^{B\sigma}_{j\pm1}=\phi^{A\sigma}_j\pm\Delta\phi^{A\sigma}_j+\frac{1}{2}\Delta^2\phi^{A\sigma}_j, \tag{32b}$$

for a ferromagnetic ground state with $\phi_j^{B}=0$ for $i\in A$ and analogously for B.

Therefore, in the long-wave and quasi-stationary approximation we get

$$x_\xi=\frac{I}{m\omega_0^2}\sum_{\sigma}^{n}(\phi^{*\sigma}\phi^{-\sigma}+\phi^{*-\sigma}\phi^{\sigma})+c, \tag{33a}$$

for an antiferromagnetic and

$$x_\xi=\frac{2I}{m\omega_0^2}\sum_{\sigma}^{n}\phi^{*\sigma}\phi^{\sigma}+c, \tag{33b}$$

for a ferromagnetic case. Consequently, one gets finally the field equation in the form

$$i\dot{\phi}^{\sigma} = t\phi^{-\sigma}_{\xi\xi} + 2(t - Tc)\phi^{-\sigma} - \frac{(2I)^2}{m\omega_0^2}\sum_{\sigma}^{n/2}(\phi^{*\sigma}\phi^{-\sigma} + \phi^{*-\sigma}\phi^{\sigma})\phi^{\sigma} -$$

$$-\mu\phi^{\sigma} + \tilde{U}\Big(\sum_{\sigma}^{n/2}|\phi^{-\sigma}|^2\Big)\phi^{\sigma}, \tag{34a}$$

with $\tilde{U} = \frac{U_0}{2} + \mathcal{J}$, where the terms like $\sum_{\sigma}^{n/2}|\phi^{-\sigma}|^2_{\xi\xi}\phi^{\sigma}$ has been neglected. (similarly for $\phi^{-\sigma}$). For the ferromagnetic case the analogous equation is

$$i\dot{\phi}^{\sigma} = t\phi^{\sigma}_{\xi\xi} + 2(t - Ic - \frac{\mu}{2})\phi^{\sigma} - \frac{(2I)^2}{m\omega_0^2}\Big(\sum_{\sigma}^{n/2}|\phi^{\sigma}|^2 +$$

$$+ |\phi^{-\sigma}|^2\Big)\phi^{\sigma} + \tilde{U}\sum_{\sigma}^{n}|\phi^{-\sigma}|^2\phi^{\sigma}. \tag{34b}$$

It has been discussed by several authors [5a,b] that Hubbard model describes different physical systems at different limitations, e.g., in the limit $U\to\infty$, i.e., when the Coulomb repulsion plays the leading role, the model may be used to describe organic charge transfer salts of TCNQ (Tetracyanoquino-dimethan) [5] and in the opposite limit, $U\to 0$, i.e., when the hopping integral plays the central role, it describes mixed valency planer compounds of transition metals (MVPC). The first limit corresponds to an Ising model and the second to a XY model. The equations (34a,b) which correspond to a Generalized Peierls-Hubbard model are not integrable in general. But in the limit μ, $\tilde{U}\to 0$, the above equations become completely integrable. Introducing new functions

$$C^{\sigma}_{+} = \phi^{\sigma} + \phi^{-\sigma}, \quad C^{\sigma}_{-} = \phi^{\sigma} - \phi^{-\sigma} \tag{35}$$

in the integrable limit one gets

$$i\dot{C}^{\sigma}_{+} = tC^{\sigma}_{+\xi\xi} + 2(t - Ic)C^{\sigma}_{+} - a\sum_{\sigma}^{n/2}(|C^{\sigma}_{+}|^2 - |C^{\sigma}_{-}|^2)C^{\sigma}_{+}, \tag{36}$$

and

$$i\dot{C}^{*\sigma}_{-} = tC^{*\sigma}_{-\xi\xi} + 2(t - Ic)C^{*\sigma}_{-} - a\sum_{\sigma}^{n/2}(|C^{\sigma}_{+}|^2 - |C^{\sigma}_{-}|^2)C^{*\sigma}_{-}$$

for antiferromagnetic case and

$$\begin{aligned} i\dot{C}^{\sigma}_{+} &= tC^{\sigma}_{+\xi\xi} + 2(t - Ic)C^{\sigma}_{+} - a\Big(\sum_{\sigma}^{n/2}|C^{\sigma}_{+}|^2 + |C^{\sigma}_{-}|^2\Big)C^{\sigma}_{+}, \\ i\dot{C}^{\sigma}_{-} &= tC^{\sigma}_{-\xi\xi} + 2(t - Ic)C^{\sigma}_{-} - a\Big(\sum_{\sigma}^{n/2}|C^{\sigma}_{+}|^2 + |C^{\sigma}_{-}|^2\Big)C^{\sigma}_{-} \end{aligned} \tag{37}$$

for ferromagnetic case, where $a = 2I^2/m\omega_0^2$. Introducing vector functions $\Psi^{a} = \begin{pmatrix} C^{\sigma}_{+} \\ C^{*\sigma}_{-} \end{pmatrix}$ in (36) and $\Psi^{a} = \begin{pmatrix} C^{\sigma}_{+} \\ C^{\sigma}_{-} \end{pmatrix}$ in (37) we observe that Hubbard model gives NLSE with $U(\frac{n}{2},\frac{n}{2})$ symmetry in the antiferromagnetic case, and NLSE with $U(n,0)$ in the ferromagnetic case.

4. CONCLUSION

We have investigated manycomponent spin system and shown that under certain assumptions (e.g., longwave limit, low temperature limit, etc.) it may be associated with various field models with internal ("colour") symmetries. Some of these models, such as models described by the nonlinear Schrödinger equation with $U(p,q)$ colour symmetry (obtained in the quasistationary limit) and by the colour generalized Yajuma-Oikawa equations (obtained at the nearsound limit) are completely integrable systems. Other nonintegrable reductions may also in some sense (see for example refs. [8,9]) be considered as systems close to integrable ones.

All the equations obtained, besides linear phonon and magnon solutions admit also nonlinear soliton solutions *.

The Hamiltonian of the system in case of integrable equations may, in principle, be factorized, i.e., may be represented as the sum of the contributions of independent excitation modes. In the simplest $U(1)$ NLSE case such modes are only two, e.g., magnon and soliton modes. These components from the point of their statistical property act as noninteracting ideal gases [15]. Using this conception we may calculate the dynamical structural factors for the models considered here (in the case of nonintegrable reductions the result is accurate upto the pre-exponential factor) and consider separately the contributions of magnon and soliton excitation modes **.

In the conclusion we note that the consideration of phonon anharmonism leads to a system of coupled Schrödinger-Boussinesq

* We do not present here these solutions since they were thoroughly discussed in ref. [6-12, 14-17].

** The results of such investigation are supposed to be published by the authors.

equations or to NLSE with saturable nonlinearity [18]. This gives the hope that results obtained here may also be generalized to the manydimensional systems.

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Кунду А., Маханьков В.Г., Пашаев О.К. E17-82-677

Интегрируемые редукции в многокомпонентных магнитных системах в /1,1/ пространстве-времени

Предложена многокомпонентная модель, обобщающая спиновую цепочку Гейзенберга. Рассмотрены некоторые ее редукции, приводящие к моделям различных магнитных систем. Показано, что динамика перечисленных моделей определяется их близостью к некоторым интегрируемым системам и совпадает с ними при определенных предположениях.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1982

Kundu A., Makhankov V.G., Pashaev O.K. E17-82-677

Integrable Reductions of Manycomponent Magnetic Systems in (1,1) Dimensions

A generalized manycomponent Heisenberg spin chain with phonon interaction is proposed. Some reductions of the proposed model leading to different real magnetic systems such as manychained magnetic crystals with nontrivial interchain couplings, a mixture of manychained ferro- and antiferromagnets, a "colour" generalized Pierls-Hubbard model, etc., are studied. It has been shown that the dynamics of all the above real models is close to some integrable systems and coincides with them in certain limits. Such integrable systems are the generalized coupled system of Yajima and Oikawa and U(p, q) nonlinear Schrödinger equation, already well studied.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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